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LIE ALGEBRAS WITH PRESCRIBED $\mathfrak{sl}_3$ DECOMPOSITION

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In memory of Hyo Chul Myung

ABSTRACT. In this work, we consider Lie algebras $\mathcal{L}$ containing a subalgebra isomorphic to $\mathfrak{sl}_3$ and such that $\mathcal{L}$ decomposes as a module for that $\mathfrak{sl}_3$ subalgebra into copies of the adjoint module, the natural three-dimensional module and its dual, and the trivial one-dimensional module. We determine the multiplication in $\mathcal{L}$ and establish connections with structurable algebras by exploiting symmetry relative to the symmetric group S_4.

1. Introduction

The Lie algebra $\mathfrak{gl}_{n+k}$ of $(n+k)\times(n+k)$ matrices over a field $\mathbb{F}$ of characteristic 0 under the commutator product $[x,y]=xy-yx$, when viewed as a module for the copy of $\mathfrak{gl}_n$ in its northwest corner, decomposes into k copies of the natural n-dimensional $\mathfrak{gl}_n$-module $\mathsf{V}=\mathbb{F}^n$, k copies of the dual module $\mathsf{V}^*=\mathsf{Hom}(\mathsf{V},\mathbb{F})$, a copy of the Lie algebra $\mathfrak{gl}_k$ in its southeast corner, and the copy of $\mathfrak{gl}_n$:

$$\mathfrak{gl}_{n+k}=\mathfrak{gl}_n\oplus\mathsf{V}^{\oplus k}\oplus(\mathsf{V}^*)^{\oplus k}\oplus\mathfrak{gl}_k.$$

As a result, we may write

$$\mathfrak{gl}_{n+k}\cong\mathfrak{gl}_n\oplus(\mathsf{V}\otimes\mathsf{B})\oplus(\mathsf{V}^*\otimes\mathsf{C})\oplus\mathfrak{gl}_k,$$

where $\mathsf{B}=\mathsf{C}=\mathbb{F}^k$. This second expression reflects the decomposition of $\mathfrak{gl}_{n+k}$ as a module for $\mathfrak{gl}_n\oplus\mathfrak{gl}_k$. When restricted to $\mathfrak{sl}_n$, the $\mathfrak{gl}_n$-modules V and V^* remain irreducible, while $\mathfrak{gl}_n$ decomposes into a copy of the adjoint module and a trivial $\mathfrak{sl}_n$-module spanned by the identity matrix $\mathfrak{gl}_n=\mathfrak{sl}_n\oplus\mathbb{F}\mathsf{I}_\mathsf{n}$. Thus, we have the $\mathfrak{sl}_n$ decomposition of $\mathfrak{gl}_{n+k}$,

$$\mathfrak{gl}_{n+k}\cong\mathfrak{sl}_n\oplus(\mathsf{V}\otimes\mathsf{B})\oplus(\mathsf{V}^*\otimes\mathsf{C})\oplus\big(\mathfrak{gl}_k\oplus\mathbb{F}\mathsf{I}_n\big), \tag{1.1}$$

where $\mathfrak{gl}_k\oplus\mathbb{F}\mathsf{I}_n$ is the sum of the trivial $\mathfrak{sl}_n$-modules in $\mathfrak{gl}_{n+k}$. Decompositions such as (1.1) also arise in the study of direct limits of simple Lie algebras and give insight into their structure.

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Indeed, suppose we have a chain of homomorphisms,

$$\mathfrak{g}^{(1)} \stackrel{\varphi_1}{\to} \mathfrak{g}^{(2)} \stackrel{\varphi_2}{\to} \cdots \to \mathfrak{g}^{(i)} \stackrel{\varphi_i}{\to} \mathfrak{g}^{(i+1)} \to \cdots, \tag{1.2}$$

where $\mathfrak{g}^{(i)} = \mathfrak{sl}(\mathsf{V}^{(i)})$. Assume that $\mathfrak{sl}(\mathsf{V})$ is a fixed term in the chain for some $\mathsf{V} = \mathsf{V}^{(j)}$, and $\dim \mathsf{V} = n$. We identify $\mathfrak{sl}(\mathsf{V})$ with $\mathfrak{sl}_n$ by choosing a basis for V and assume $\mathsf{V}^{(i)} = \mathsf{V}^{\oplus k_i} \oplus \mathbb{F}^{\oplus z_i}$ as a module for $\mathfrak{sl}_n$ for $i \geq j$. Then the limit Lie algebra $\mathcal{L} = \varinjlim \mathfrak{g}^{(i)}$ admits a decomposition relative to $\mathfrak{sl}_n$,

$$\mathcal{L} \cong (\mathfrak{sl}_n \otimes \mathsf{A}) \oplus (\mathsf{V} \otimes \mathsf{B}) \oplus (\mathsf{V}^* \otimes \mathsf{C}) \oplus \mathfrak{s}, \tag{1.3}$$

where $\mathfrak{s}$ is the sum of the trivial $\mathfrak{sl}_n$-modules (see [3, Sec. 5]). Bahturin and Benkart in [3, Sec. 4] study Lie algebras having such a decomposition and describe the multiplication in $\mathcal{L}$ and the possibilities for $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathfrak{s}$ when $\dim \mathsf{V} \geq 4$. When $\dim \mathsf{V} = 2$, then V^* is isomorphic to V as a module for $\mathfrak{sl}_2 = \mathfrak{sl}(\mathsf{V})$. In this case, a Lie algebra having a decomposition, $\mathcal{L} = (\mathfrak{sl}_2 \otimes \mathsf{A}) \oplus (\mathsf{V} \otimes \mathsf{B}) \oplus \mathfrak{s}$, is graded by the root system BC_1, and its structure has been described in [4].

In this paper, we investigate the missing case when $\dim \mathsf{V} = 3$, which presents very distinctive features. For direct limit Lie algebras of the type considered above, we could, of course, choose a larger space $\mathsf{V}^{(j)}$ having $\dim \mathsf{V}^{(j)} \geq 4$ and apply the results of [3]. However, there are many examples of Lie algebras which admit very interesting decompositions as in (1.3) for $n = 3$. The exceptional simple Lie algebras provide examples of this phenomenon.

Example 1.1. Each exceptional simple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic 0 has an automorphism ψ of order 3 that corresponds to a certain node in the Dynkin diagram of the associated affine Lie algebra. The node is marked with a "3" in [10, Table Aff 1]. Removing that node gives the Dynkin diagram of a finite-dimensional semisimple Lie algebra $\mathfrak{sl}_3 \oplus \mathfrak{s}$, which is the subalgebra of fixed points of the automorphism ψ. The Lie algebra $\mathfrak{s}$ is the centralizer of $\mathfrak{sl}_3$ in $\mathcal{L}$; hence, it is the sum of trivial $\mathfrak{sl}_3$-modules under the adjoint action. In the table below we display the Lie algebra $\mathfrak{s}$:

(1.4)

$\mathcal{L}$	G_2	F_4	E_6	E_7	E_8
$\mathfrak{s}$	0	$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_3$	$\mathfrak{sl}_6$	E_6

For the Lie algebra G_2 we have the well-known decomposition (see [9, Prop. 3])

$$\mathsf{G}_2 \cong \mathfrak{sl}_3 \oplus \mathsf{V} \oplus \mathsf{V}^*$$

relative to $\mathfrak{sl}_3$ (where $\mathfrak{sl}_3$ corresponds to the long roots of G_2 and $\mathsf{V} = \mathbb{F}^3$). This decomposition can be viewed as the decomposition into eigenspaces relative to ψ, where V corresponds to the eigenvalue ω (a primitive cube root of 1), V^* to the eigenvalue ω^2, and $\mathfrak{sl}_3$ to the eigenvalue 1.

For the other exceptional Lie algebras,

$$\mathcal{L} \cong \mathfrak{sl}_3 \oplus (\mathsf{V} \otimes \mathsf{B}) \oplus (\mathsf{V}^* \otimes \mathsf{C}) \oplus \mathfrak{s}, \tag{1.5}$$

where B and C can be identified with $\mathsf{H}_3(\mathcal{C})$, the algebra of 3×3 hermitian matrices over a composition algebra $\mathcal{C}$ under the product $h \circ h' = 1/2(hh' + h'h)$. Thus, elements of B have the form

$$h = \begin{bmatrix} \alpha & a & b \\ \bar{a} & \beta & c \\ \bar{b} & \bar{c} & \gamma \end{bmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{F}$, $a, b, c \in \mathcal{C}$, and "$\bar{\ }$" is the standard involution in $\mathcal{C}$. The composition algebra $\mathcal{C}$ is displayed in the table below, where K is the algebra $\mathbb{F} \times \mathbb{F}$, Q the algebra of quaternions, and O the algebra of octonions:

(1.6)

$\mathcal{L}$	F_4	E_6	E_7	E_8
$\mathcal{C}$	$\mathbb{F}$	K	Q	O

The algebra $\mathfrak{s}$ can be identified with the structure Lie algebra of $\mathsf{B} = \mathsf{H}_3(\mathcal{C})$, $\mathfrak{s} = \mathsf{Der}(\mathsf{B}) \oplus \mathsf{L}_{\mathsf{B}_0}$, consisting of the derivations and multiplication maps $\mathsf{L}_h(h') = h \circ h'$ for $h \in \mathsf{B}_0$ (the matrices in B of trace 0). Here $\mathsf{V} \otimes \mathsf{B}$ is the ω-eigenspace of ψ, $\mathsf{V}^* \otimes \mathsf{C}$ the ω^2-eigenspace, and $\mathfrak{sl}_3 \oplus \mathfrak{s}$ the 1-eigenspace.

For example, when $\mathcal{C} = \mathsf{O}$, it is well known that $\mathsf{B} = \mathsf{H}_3(\mathsf{O})$ is the 27-dimensional exceptional simple Jordan algebra, and its structure algebra $\mathfrak{s}$ is a simple Lie algebra of type E_6 (see for example, [11, Chap. IV, Sec. 4]). As a module for E_6, B is irreducible, and relative to a certain Cartan subalgebra, it has as highest weight the first fundamental weight. The module C is an irreducible E_6-module (the dual module of B) which has as highest weight the last fundamental weight. Thus,

$$\mathsf{E}_8 = \mathfrak{sl}_3 \oplus (\mathsf{V} \otimes \mathsf{B}) \oplus (\mathsf{V}^* \otimes \mathsf{C}) \oplus \mathsf{E}_6.$$

Reading right to left, we see the decomposition of E_8 as a module for the subalgebra of type E_6, and reading left to right, its decomposition as an $\mathfrak{sl}_3$-module. □

Recently, Lie algebras with a decomposition (1.5) have been considered by Faulkner [8, Lem. 22] in connection with his classification of structurable superalgebras of classical type. (Structurable algebras, which were introduced and studied in [1], form a certain variety of algebras generalizing associative algebras with involution and Jordan algebras.)

In this work, we examine Lie algebras $\mathcal{L}$ such that $\mathcal{L}$ has a subalgebra $\mathfrak{sl}_3$ and such that $\mathcal{L}$ admits a decomposition as in (1.3) into copies of $\mathfrak{sl}_3$, $\mathsf{V} = \mathbb{F}^3$, V^*, and trivial modules relative to the action of $\mathfrak{sl}_3$. Applying results in [3] and [5], we determine that A is an alternative algebra, B is a left A-module, and C is a right A-module, and we describe $\mathfrak{s}$ and the multiplication in $\mathcal{L}$.

Using the fact that V can be given the structure of a module for the symmetric group S_4, we obtain an action of S_4 by automorphisms on $\mathcal{L}$. The elements $\tau_1 = (1\,2)(3\,4)$ and $\tau_2 = (1\,4)(2\,3)$ generate a normal subgroup of S_4 which is a Klein 4-subgroup. Results of Elduque and Okubo [7] enable us to deduce that $\mathcal{L}_0 = \{X \in \mathcal{L} \mid \tau_1 X = X,\ \tau_2 X = -X\}$ is a structurable algebra under a certain multiplication. We identify the structurable algebra $\mathcal{L}_0$ with the space of 2×2 matrices

$$\mathcal{A} = \begin{bmatrix} \mathsf{A} & \mathsf{C} \\ \mathsf{B} & \mathsf{A} \end{bmatrix}$$

under a suitable multiplication. When $\mathcal{L}$ is the exceptional Lie algebra E_8, then $\mathcal{A} = \begin{bmatrix} \mathbb{F} & \mathsf{C} \\ \mathsf{B} & \mathbb{F} \end{bmatrix}$ where $\mathsf{B} = \mathsf{C} = \mathsf{H}_3(\mathsf{O})$. This is a simple structurable algebra (see [1, Secs. 8 and 9]).

2. Lie algebras with prescribed $\mathfrak{sl}_3$ decomposition

Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2, 3$ (this assumption on the underlying field will be kept throughout) which contains a subalgebra isomorphic to $\mathfrak{sl}(\mathsf{V})$, for a vector space V of dimension 3, so that $\mathcal{L}$ decomposes, as a module for $\mathfrak{sl}(\mathsf{V})$ into a direct sum of copies of the adjoint module, the natural module V, its dual V^*, and the trivial one-dimensional module. Thus, we write as in (1.3):

$$\mathcal{L} = \big(\mathfrak{sl}(\mathsf{V}) \otimes \mathsf{A}\big) \oplus \big(\mathsf{V} \otimes \mathsf{B}\big) \oplus \big(\mathsf{V}^* \otimes \mathsf{C}\big) \oplus \mathfrak{s} \tag{2.1}$$

for suitable vector spaces $\mathsf{A}, \mathsf{B}, \mathsf{C}$, and for a Lie subalgebra $\mathfrak{s}$, which is the subalgebra of elements of $\mathcal{L}$ annihilated by the elements in $\mathfrak{sl}(\mathsf{V})$. The vector space A contains a distinguished element $1 \in \mathsf{A}$ such that $\mathfrak{sl}(\mathsf{V}) \otimes 1$ is the subalgebra isomorphic to $\mathfrak{sl}(\mathsf{V})$ we started with.

Fix a nonzero linear map $\det : \bigwedge^3 \mathsf{V} \to \mathbb{F}$. This determines another such form $\det : \bigwedge^3 \mathsf{V}^* \to \mathbb{F}$ such that $\det(f_1 \wedge f_2 \wedge f_3)\det(v_1 \wedge v_2 \wedge v_3) = \mathbf{det}\Big(f_i(v_j)\Big)$ for any $f_1, f_2, f_3 \in \mathsf{V}^*$ and $v_1, v_2, v_3 \in \mathsf{V}$. (The symbol "**det**" denotes the usual determinant.) This allows us to identify $\bigwedge^2 \mathsf{V}$ with V^*: $u_1 \wedge u_2 \leftrightarrow \det(u_1 \wedge u_2 \wedge \_)$ and, in the same vein, $\bigwedge^2 \mathsf{V}^*$ with V.

The invariance of the bracket in $\mathcal{L}$ relative to the subalgebra $\mathfrak{sl}(\mathsf{V})$ gives equations as in [3, (19)]:

$$\begin{aligned}
[x \otimes a, y \otimes a'] &= [x, y] \otimes \frac{1}{2} a \circ a' \,+\, x \circ y \otimes \frac{1}{2}[a, a'] + (x|y) D_{a,a'},\\
[x \otimes a, u \otimes b] &= xu \otimes ab,\\
[v^* \otimes c, x \otimes a] &= v^* x \otimes ca,\\
[u \otimes b, v^* \otimes c] &= \big(uv^* - \frac{1}{3}(v^* u)\mathsf{I}_3\big) \otimes T(b, c) + \frac{1}{3}(v^* u) D_{b,c},\\
[u_1 \otimes b_1, u_2 \otimes b_2] &= (u_1 \wedge u_2) \otimes (b_1 \times b_2),\\
[v_1^* \otimes c_1, v_2^* \otimes c_2] &= (v_1^* \wedge v_2^*) \otimes (c_1 \times c_2),\\
[d, x \otimes a] &= x \otimes da,\\
[d, u \otimes b] &= u \otimes db,\\
[d, v^* \otimes c] &= v^* \otimes dc,
\end{aligned} \tag{2.2}$$

for any $x, y \in \mathfrak{sl}(\mathsf{V})$, $u, u_1, u_2 \in \mathsf{V}$, $v^*, v_1^*, v_2^* \in \mathsf{V}^*$, $d \in \mathfrak{s}$, $a, a' \in \mathsf{A}$, $b, b_1, b_2 \in \mathsf{B}$, and $c, c_1, c_2 \in \mathsf{C}$, and for bilinear maps:

$$\begin{aligned}
&\mathsf{A} \times \mathsf{A} \to \mathsf{A} : (a, a') \mapsto a \circ a' \quad \text{commutative},\\
&\mathsf{A} \times \mathsf{A} \to \mathsf{A} : (a, a') \mapsto [a, a'] \quad \text{anticommutative},\\
&\mathsf{A} \times \mathsf{A} \to \mathfrak{s}, (a, a') \mapsto D_{a,a'} \quad \text{skew-symmetric},\\
&\mathsf{A} \times \mathsf{B} \to \mathsf{B} : (a, b) \mapsto ab,\\
&\mathsf{C} \times \mathsf{A} \to \mathsf{C} : (c, a) \mapsto ca,\\
&\mathsf{B} \times \mathsf{C} \to \mathsf{A} : (b, c) \mapsto T(b, c),\\
&\mathsf{B} \times \mathsf{C} \to \mathfrak{s} : (b, c) \mapsto D_{b,c},\\
&\mathsf{B} \times \mathsf{B} \to \mathsf{C} : (b_1, b_2) \mapsto b_1 \times b_2 \quad \text{symmetric},\\
&\mathsf{C} \times \mathsf{C} \to \mathsf{B} : (c_1, c_2) \mapsto c_1 \times c_2 \quad \text{symmetric},
\end{aligned} \tag{2.3}$$

and representations $\mathfrak{s} \to \mathfrak{gl}(\mathsf{A}), \mathfrak{gl}(\mathsf{B}), \mathfrak{gl}(\mathsf{C})$, whose action is denoted by da, db, and dc for $d \in \mathfrak{s}$ and $a \in \mathsf{A}$, $b \in \mathsf{B}$ and $c \in \mathsf{C}$, where, as in [3, (17)],

$$\begin{aligned} x \circ y &= xy + yx - \frac{2}{3}\,\mathrm{tr}(xy)\mathsf{I}_3, \\ (x|y) &= \frac{1}{3}\,\mathrm{tr}(xy), \end{aligned} \tag{2.4}$$

for $x, y \in \mathfrak{sl}(\mathsf{V})$, and I_3 denotes the identity map. The difference with [3, (19)] lies in the appearance of the symmetric maps $b_1 \times b_2$ and $c_1 \times c_2$ when V has dimension 3. This slight difference has a huge impact.

The distinguished element $1 \in \mathsf{A}$ satisfies $1 \circ a = a$, $[1, a] = 0$, $D_{1,a} = 0$, $1b = b$, $c1 = c$, and $d1 = 0$ for any $a \in \mathsf{A}$, $b \in \mathsf{B}$, $c \in \mathsf{C}$ and $d \in \mathfrak{s}$.

Theorem 2.1. *Let $\mathcal{L}$ be a vector space as in (2.1) and define an anticommutative bracket in $\mathcal{L}$ by (2.2) for bilinear maps as in (2.3). Then $\mathcal{L}$ is a Lie algebra if and only if the following conditions are satisfied:*

(0) *$\mathfrak{s}$ is a Lie subalgebra of $\mathcal{L}$, A, B, C are modules for $\mathfrak{s}$ relative to the given actions, and the bilinear maps in (2.3) are $\mathfrak{s}$-invariant.*

(1) *A is an alternative algebra relative to the multiplication*

$$aa' = \frac{1}{2} a \circ a' + \frac{1}{2}[a, a'],$$

and the map $A \times A \to A : (a, a') \mapsto D_{a,a'}$ satisfies the conditions

$$\sum_{\text{cyclic}} D_{a_1, a_2 a_3} = 0,$$

$$D_{a_1,a_2} a_3 = [[a_1, a_2], a_3] + 3\big((a_1 a_3)a_2 - a_1(a_3 a_2)\big),$$

for any $a_1, a_2, a_3 \in \mathsf{A}$.

(2) *For any $a_1, a_2 \in \mathsf{A}$, $b \in \mathsf{B}$ and $c \in \mathsf{C}$,*

$$a_1(a_2 b) = (a_1 a_2)b,$$
$$(ca_1)a_2 = c(a_1 a_2),$$

so that B (respectively C) is a left associative module (resp. right associative module) for A, and

$$D_{a_1,a_2} b = [a_1, a_2]b,$$
$$D_{a_1,a_2} c = c[a_2, a_1].$$

(3) *For any $a \in \mathsf{A}$, $b \in \mathsf{B}$ and $c \in \mathsf{C}$,*

$$aT(b, c) = T(ab, c), \quad T(b, c)a = T(b, ca),$$
$$D_{a,T(b,c)} = D_{ab,c} - D_{b,ca},$$
$$D_{b,c} a = [T(b, c), a].$$

(4) *For any $a \in \mathsf{A}$, $b_1, b_2 \in \mathsf{B}$ and $c_1, c_2 \in \mathsf{C}$,*

$$(b_1 \times b_2)a = (ab_1) \times b_2 = b_1 \times (ab_2),$$
$$a(c_1 \times c_2) = (c_1 a) \times c_2 = c_1 \times (c_2 a).$$

(5) *$D_{b,b\times b} = 0$ for any $b \in \mathsf{B}$ and $D_{c\times c,c} = 0$ for any $c \in \mathsf{C}$. In addition, the trilinear maps $\mathsf{B}\times\mathsf{B}\times\mathsf{B} \to \mathsf{B} : (b_1, b_2, b_3) \mapsto T(b_1, b_2 \times b_3)$ and $\mathsf{C}\times\mathsf{C}\times\mathsf{C} \to \mathsf{C} : (c_1, c_2, c_3) \mapsto T(c_1 \times c_2, c_3)$ are symmetric.*

(6) *For any* $b, b_1, b_2 \in \mathsf{B}$ *and* $c, c_1, c_2 \in \mathsf{C}$,

$$(b_1 \times b_2) \times c = -\frac{1}{3}T(b_1, c)b_2 + \frac{1}{3}D_{b_1,c}b_2 - T(b_2, c)b_1,$$
$$b \times (c_1 \times c_2) = -c_2T(b, c_1) - \frac{1}{3}c_1T(b, c_2) - \frac{1}{3}D_{b,c_2}c_1.$$

Proof. If $\mathcal{L}$ is a Lie algebra under the bracket defined in (2.2), then it is clear that $\mathfrak{s}$ is a Lie subalgebra and all the conditions in item (0) are satisfied. Moreover, $\big(\mathfrak{sl}(\mathsf{V}) \otimes \mathsf{A}\big) \oplus \mathfrak{s}$ is a Lie subalgebra, and the arguments in [5, Sec. 3] show that A is an alternative algebra, and the conditions in item (1) are satisfied.

The arguments in Propositions 4.3 and 4.4 and equations (25) and (27) in [3] work here and give the conditions in item (2). (Note that there is a minus sign missing in [3, (27)].)

Now, equations (30)–(33) in [3] establish the identitites in (3). The Jacobi identity applied to elements $x \otimes a$, $u_1 \otimes b_1$ and $u_2 \otimes b_2$, for $x \in \mathfrak{sl}(\mathsf{V})$, $u_1, u_2 \in \mathsf{V}$, $a \in \mathsf{A}$ and $b_1, b_2 \in \mathsf{B}$, give the first equation in item (4), the second one being similar.

The Jacobi identity for elements $u_i \otimes b_i$, $i = 1, 2, 3$, for $u_i \in \mathsf{V}$ and $b_i \in \mathsf{B}$ gives

$$\sum_{\text{cyclic}} D_{b_1,b_2\times b_3} = 0, \quad T(b_1, b_2 \times b_3) = T(b_2, b_3 \times b_1),$$

which, in view of the symmetry of the bilinear map $b_1 \times b_2$, proves half of the assertions in item (5), the other half being implied by the Jacobi identity for elements $v_i^* \otimes c_i$, $i = 1, 2, 3$, for $v_i^* \in \mathsf{V}^*$ and $c_i \in \mathsf{C}$.

Finally, for elements $u_1, u_2 \in \mathsf{V}$, $v^* \in \mathsf{V}^*$, $b_1, b_2 \in \mathsf{B}$, $c \in \mathsf{C}$,

$$[[u_1 \otimes b_1, u_2 \otimes b_2], v^* \otimes c] = [(u_1 \wedge u_2) \otimes (b_1 \times b_2), v^* \otimes c]$$
$$= (u_1 \wedge u_2) \wedge v^* \otimes (b_1 \times b_2) \times c = \big((v^*u_1)u_2 - (v^*u_2)u_1\big) \otimes (b_1 \times b_2) \times c,$$

while

$$[[u_1 \otimes b_1, v^* \otimes c], u_2 \otimes b_2]$$
$$= [\big(u_1v^* - \frac{1}{3}(v^*u_1)\mathsf{I}_3\big) \otimes T(b_1, c) + \frac{1}{3}v^*u_1D_{b_1,c}, u_2 \otimes b_2]$$
$$= \big((v^*u_2)u_1 - \frac{1}{3}(v^*u_1)u_2\big) \otimes T(b_1, c)b_2 + \frac{1}{3}(v^*u_1)u_2 \otimes D_{b_1,c}b_2,$$

$$[u_1 \otimes b_1, [u_2 \otimes b_2, v^* \otimes c]]$$
$$[u_1 \otimes b_1, \big(u_2v^* - \frac{1}{3}(v^*u_2)\mathsf{I}_3\big) \otimes T(b_2, c) + \frac{1}{3}v^*u_2D_{b_2,c}]$$
$$= -\big((v^*u_1)u_2 - \frac{1}{3}(v^*u_2)u_1\big) \otimes T(b_2, c)b_1 - \frac{1}{3}(v^*u_2)u_1 \otimes D_{b_2,c}b_1.$$

Hence the Jacobi identity here is equivalent to the first condition in item (6); the second condition can be proven in a similar way.

The converse follows from straightforward computations. □

Given an alternative algebra A, the ideal $\mathsf{E}(\mathsf{A})$ generated by the associators $(a_1, a_2, a_3) = (a_1a_2)a_3 - a_1(a_2a_3)$ is $\mathsf{E}(\mathsf{A}) = (\mathsf{A}, \mathsf{A}, \mathsf{A}) + (\mathsf{A}, \mathsf{A}, \mathsf{A})\mathsf{A} = (\mathsf{A}, \mathsf{A}, \mathsf{A}) + \mathsf{A}(\mathsf{A}, \mathsf{A}, \mathsf{A})$. The associative nucleus of A is $\mathsf{N}(\mathsf{A}) := \{a \in \mathsf{A} \mid (a, \mathsf{A}, \mathsf{A}) = 0\}$, while the center is $\mathsf{Z}(\mathsf{A}) = \{a \in \mathsf{N}(\mathsf{A}) \mid aa' = a'a, \ \ \forall\, a' \in \mathsf{A}\}$.

Corollary 2.2. *Let* $\mathcal{L}$ *be a Lie algebra which contains a subalgebra isomorphic to* $\mathfrak{sl}(\mathsf{V})$ *for a vector space* V *of dimension* 3 *so that* $\mathcal{L}$ *decomposes, as a module for*

$\mathfrak{sl}(\mathsf{V})$, *as in* (2.1). *Then, with the notation used so far, the alternative algebra* A *is unital (the distinguished element* 1 *being its unit element), with* 1 *acting as the identity on both* B *and* C*, and the following conditions hold:*

- $\mathsf{E}(\mathsf{A})\mathsf{B} = 0 = \mathsf{C}\mathsf{E}(\mathsf{A})$, *so that* B *(respectively* C*) is a left (resp. right) module for the associative algebra* $\mathsf{A}/\mathsf{E}(\mathsf{A})$.
- $T(\mathsf{B},\mathsf{C})$ *is an ideal of* A *contained in its associative nucleus* $\mathsf{N}(\mathsf{A})$*, and* $T(\mathsf{B},\mathsf{B}\times\mathsf{B})$ *and* $T(\mathsf{C}\times\mathsf{C},\mathsf{C})$ *are ideals of* A *contained in* $\mathsf{Z}(\mathsf{A})$.
- *For any* $b, b_1, b_2 \in \mathsf{B}$ *and any* $c, c_1, c_2 \in \mathsf{C}$*, the following conditions hold:*
$$D_{b_1,c}b_2 - D_{b_2,c}b_1 = 2\big(T(b_2,c)b_1 - T(b_1,c)b_2\big),$$
$$D_{b,c_2}c_1 - D_{b,c_1}c_2 = 2\big(c_1T(b,c_2) - c_2T(b,c_1)\big).$$
- *If the Lie algebra* $\mathcal{L}$ *is simple, then either the algebra* A *is associative or else* $\mathsf{A} = \mathsf{E}(\mathsf{A})$ *and* $\mathsf{B} = \mathsf{C} = 0$. *Moreover, if* $\mathsf{B} \neq 0$*, then* C *coincides with* $\mathsf{B}\times\mathsf{B}$*, and* A *coincides with* $T(\mathsf{B},\mathsf{B}\times\mathsf{B})$*, and* A *is a commutative and associative algebra.*

Proof. For any $a_1, a_2, a_3 \in \mathsf{A}$ and $b \in \mathsf{B}$,
$$\begin{aligned}(a_1,a_2,a_3)b &= (a_1a_2)(a_3b) - a_1((a_2a_3)b)\\ &= a_1(a_2(a_3b)) - a_1(a_2(a_3b)) = 0,\end{aligned}$$
because of Theorem 2.1, item (2). Also, this result shows that $\mathsf{ann}_{\mathsf{A}}(\mathsf{B}) = \{a \in \mathsf{A} \mid a\mathsf{B} = 0\}$ is an ideal of A. Hence, $\mathsf{E}(\mathsf{A})\mathsf{B} = 0$, as $\mathsf{E}(\mathsf{A})$ is the ideal generated by $(\mathsf{A},\mathsf{A},\mathsf{A})$. In a similar manner, one proves $\mathsf{C}\mathsf{E}(\mathsf{A}) = 0$.

For any $b \in \mathsf{B}$ and $c \in \mathsf{C}$, $T(b,c)$ is an element of A, and $\mathsf{ad}_{T(b,c)} : a \mapsto [T(b,c),a] = D_{b,c}a$ is a derivation of A by the previous theorem. Since A is alternative, this shows that $T(b,c)$ is in the associative nucleus $\mathsf{N}(\mathsf{A})$. Now for $b_1, b_2, b_3 \in \mathsf{B}$,
$$\begin{aligned}aT(b_1, b_2\times b_3) &= T(ab_1, b_2\times b_3) = T(b_2, b_3\times(ab_1))\\ &= T(b_2,(b_3\times b_1)a) = T(b_2, b_3\times b_1)a = T(b_1, b_2\times b_3)a,\end{aligned}$$
which proves that $T(\mathsf{B},\mathsf{B}\times\mathsf{B})$ is an ideal of A contained in the center $\mathsf{Z}(\mathsf{A})$. By similar arguments, $T(\mathsf{C}\times\mathsf{C},\mathsf{C})$ is shown to be contained in $\mathsf{Z}(\mathsf{A})$ too.

For any $b_1, b_2 \in \mathsf{B}$ and $c \in \mathsf{C}$, the previous theorem gives
$$(b_1\times b_2)\times c = -\frac{1}{3}T(b_1,c)b_2 + \frac{1}{3}D_{b_1,c}b_2 - T(b_2,c)b_1.$$
We permute b_1 and b_2 and use the fact that $\times$ is symmetric to get
$$D_{b_1,c}b_2 - D_{b_2,c}b_1 = 2\big(T(b_2,c)b_1 - T(b_1,c)b_2\big).$$
With the same arguments we prove
$$D_{b,c_2}c_1 - D_{b,c_1}c_2 = 2\big(c_1T(b,c_2) - c_2T(b,c_1)\big)$$
for any $b \in \mathsf{B}$, and $c_1, c_2 \in \mathsf{C}$.

Finally, since the ideal $\mathsf{E}(\mathsf{A})$ of the alternative algebra A is invariant under derivations, the subspace $\big(\mathfrak{sl}(\mathsf{V})\otimes\mathsf{E}(\mathsf{A})\big)\oplus D_{\mathsf{E}(\mathsf{A}),\mathsf{A}}$ is an ideal of the Lie algebra $\mathcal{L}$. In particular, if $\mathcal{L}$ is simple, then either $\mathsf{A} = \mathsf{E}(\mathsf{A})$ and $\mathsf{B} = \mathsf{C} = 0$ or $\mathsf{E}(\mathsf{A}) = 0$ and A is associative. Moreover, if B is nonzero, the ideal of $\mathcal{L}$ generated by $\mathsf{V}\otimes\mathsf{B}$ is $\big(\mathfrak{sl}(\mathsf{V})\otimes T(\mathsf{B},\mathsf{C})\big)\oplus\big(\mathsf{V}\otimes\mathsf{B}\big)\oplus\big(\mathsf{V}^*\otimes(\mathsf{B}\times\mathsf{B})\big)\oplus D_{\mathsf{B},\mathsf{C}}$. Hence if $\mathcal{L}$ is simple, we obtain $\mathsf{C} = \mathsf{B}\times\mathsf{B}$ and $\mathsf{A} = T(\mathsf{B},\mathsf{C}) = T(\mathsf{B},\mathsf{B}\times\mathsf{B})$, which is commutative and associative. □

3. Structurable algebras

This section is devoted to establishing a relationship between the Lie algebras with prescribed $\mathfrak{sl}_3$ decomposition considered above with a class of structurable algebras. This will be done by exploiting the action of a subgroup of the group of automorphisms of the Lie algebra isomorphic to the symmetric group S_4.

Theorem 3.1. *Let $\mathcal{L}$ be a Lie algebra which contains a subalgebra isomorphic to $\mathfrak{sl}(\mathsf{V})$ for a vector space V of dimension 3 so that $\mathcal{L}$ decomposes, as a module for $\mathfrak{sl}(\mathsf{V})$, as in (2.1). Then, with the notation used so far, the vector space*

$$\mathcal{A} = \begin{pmatrix} \mathsf{A} & \mathsf{C} \\ \mathsf{B} & \mathsf{A} \end{pmatrix} \tag{3.1}$$

with the multiplication

$$\begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix} \cdot \begin{pmatrix} a_1' & c' \\ b' & a_2' \end{pmatrix} = \begin{pmatrix} a_1 a_1' - T(b',c) & c'a_1 + ca_2' + b \times b' \\ a_1'b + a_2 b' + c \times c' & a_2' a_2 - T(b,c') \end{pmatrix} \tag{3.2}$$

and the involution

$$\overline{\begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix}} = \begin{pmatrix} a_2 & c \\ b & a_1 \end{pmatrix} \tag{3.3}$$

is a structurable algebra.

Proof. Take a basis $\{e_1, e_2, e_3\}$ of V with $\det(e_1 \wedge e_2 \wedge e_3) = 1$ and its dual basis $\{e_1^*, e_2^*, e_3^*\}$ in V^*.

The symmetric group S_4 acts on V as follows [7, (7.1)]:

$$\begin{aligned} \tau_1 = (12)(34) :&\quad e_1 \mapsto e_1,\ e_2 \mapsto -e_2,\ e_3 \mapsto -e_3, \\ \tau_2 = (23)(14) :&\quad e_1 \mapsto -e_1,\ e_2 \mapsto e_2,\ e_3 \mapsto -e_3, \\ \varphi = (123) :&\quad e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1, \\ \tau = (12) :&\quad e_1 \mapsto -e_1,\ e_2 \mapsto -e_3,\ e_3 \mapsto -e_2. \end{aligned}$$

(Thus V is the tensor product of the sign module and the standard irreducible three-dimensional module for S_4, and in this way, S_4 embeds in the special linear group $\mathsf{SL}(\mathsf{V})$.)

The inner product given by $(e_i|e_j) = \delta_{ij}$ for any $i, j \in \{1,2,3\}$ is invariant under the action of S_4, so V is self-dual as an S_4-module, and the action of S_4 on V^* (where $\sigma v^* = v^* \sigma^{-1}$) is given by the "same formulas":

$$\begin{aligned} \tau_1 = (12)(34) :&\quad e_1^* \mapsto e_1^*,\ e_2^* \mapsto -e_2^*,\ e_3^* \mapsto -e_3^*, \\ \tau_2 = (23)(14) :&\quad e_1^* \mapsto -e_1^*,\ e_2^* \mapsto e_2^*,\ e_3^* \mapsto -e_3^*, \\ \varphi = (123) :&\quad e_1^* \mapsto e_2^* \mapsto e_3^* \mapsto e_1^*, \\ \tau = (12) :&\quad e_1^* \mapsto -e_1^*,\ e_2^* \mapsto -e_3^*,\ e_3^* \mapsto -e_2^*. \end{aligned}$$

Since S_4 acts by elements in $\mathsf{SL}(\mathsf{V})$, this action of S_4 on V and on V^* extends to an action by automorphisms on the whole algebra $\mathcal{L}$. Then the subspace

$$\mathcal{L}_0 = \{X \in \mathcal{L} \mid \tau_1 X = X,\ \tau_2 X = -X\}$$

becomes a structurable algebra [7, Thm. 7.5] with involution and multiplication given by the formulas

$$\begin{aligned} \bar{X} &= -\tau X, \\ X \cdot Y &= -\tau\big([\varphi X, \varphi^2 Y]\big), \end{aligned}$$

for any $X, Y \in \mathcal{L}_0$. But we easily deduce that

$$\mathcal{L}_0 = (e_2 e_3^* \otimes \mathsf{A}) \oplus (e_3 e_2^* \otimes \mathsf{A}) \oplus (e_1 \otimes \mathsf{B}) \oplus (e_1^* \otimes \mathsf{C}).$$

Identifying $\mathcal{L}_0$ with the 2×2 matrices $\mathcal{A} = \begin{pmatrix} \mathsf{A} & \mathsf{C} \\ \mathsf{B} & \mathsf{A} \end{pmatrix}$ by means of

$$-e_2 e_3^* \otimes a_1 + e_3 e_2^* \otimes a_2 + e_1 \otimes b + e_1^* \otimes c \leftrightarrow \begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix},$$

we determine that the structurable product and the involution become

$$\begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix} \cdot \begin{pmatrix} a_1' & c' \\ b' & a_2' \end{pmatrix} = \begin{pmatrix} a_1 a_1' - T(b', c) & c' a_1 + c a_2' + b \times b' \\ a_1' b + a_2 b' + c \times c' & a_2' a_2 - T(b, c') \end{pmatrix},$$
$$\overline{\begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix}} = \begin{pmatrix} a_2 & c \\ b & a_1 \end{pmatrix},$$

as required. □

Items (5) and (6) of Theorem 2.1 show that for any $b \in \mathsf{B}$ and $c \in \mathsf{C}$,

$$(3.4)\qquad \begin{aligned} (b \times b) \times (b \times b) &= -\frac{4}{3} T(b, b \times b) b, \\ (c \times c) \times (c \times c) &= -\frac{4}{3} c T(c \times c, c). \end{aligned}$$

Also, using Theorem 2.1 and Corollary 2.2, we compute that

$$\begin{aligned} (c \times (b \times b)) \times b &= -\frac{4}{3}(T(b,c)b) \times b + \frac{1}{3}(D_{b,c} b) \times b \\ &= -\frac{4}{3}(b \times b) T(b,c) + \frac{1}{6} D_{b,c}(b \times b) \quad \text{(as the product} \\ &\qquad\qquad b_1 \times b_2 \text{ from } \mathsf{B} \times \mathsf{B} \text{ into } \mathsf{C} \text{ is } \mathfrak{s}\text{-invariant)} \\ &= -\frac{4}{3}(b \times b) T(b,c) + \frac{1}{3}\big((b \times b) T(b,c) - c T(b, b \times b)\big) \\ &= -(b \times b) T(b,c) - \frac{1}{3} c T(b, b \times b), \end{aligned}$$

and an analogous result with the roles of b and c interchanged. So we conclude that the equations

$$(3.5)\qquad \begin{aligned} (c \times (b \times b)) \times b &= -(b \times b) T(b,c) - \frac{1}{3} c T(b, b \times b), \\ (b \times (c \times c)) \times c &= -T(b,c)(c \times c) - \frac{1}{3} T(c \times c, c) b, \end{aligned}$$

hold for any $b \in \mathsf{B}$ and $c \in \mathsf{C}$.

Equations (3.4) and (3.5) are precisely the ones that appear in [2, Ex. 6.4] and are needed to ensure that the algebra defined there, which coincides with our $\mathcal{A}$ but with the added restrictions of A being commutative and associative, is structurable. (Note that the bilinear form $T(.,.)$ considered in [2, Ex. 6.4] equals our $-T(.,.)$.)

However some of the previous arguments show that if our structurable algebra $\mathcal{A}$ is simple and $\mathsf{A} \neq 0$, then A is simple, and since $T(\mathsf{B}, \mathsf{B} \times \mathsf{B})$ and $T(\mathsf{C} \times \mathsf{C}, \mathsf{C})$ are ideals of A contained in the center $\mathsf{Z}(\mathsf{A})$, either A is commutative and associative or else $T(\mathsf{B}, \mathsf{B} \times \mathsf{B}) = 0 = T(\mathsf{C} \times \mathsf{C}, \mathsf{C})$. But in this case, the subspace $\begin{pmatrix} 0 & \mathsf{B} \times \mathsf{B} \\ \mathsf{C} \times \mathsf{C} & 0 \end{pmatrix}$ becomes an ideal, so if $\mathcal{A}$ is simple, either A is commutative and associative or else $\mathsf{B} \times \mathsf{B} = 0 = \mathsf{C} \times \mathsf{C}$.

Therefore, when considering simple algebras, we are dealing exactly with the situation considered by Allison and Faulkner in [2].

Theorem 3.1 shows that the restrictions on the bilinear maps involved are sufficient to ensure that the algebra $\mathcal{A}$ in (3.1), with multiplication (3.2) and involution (3.3) is a structurable algebra.

A natural question to ask is whether these conditions are also necessary. More precisely, does any structurable algebra of the form $\mathcal{A}$ as in (3.1) with multiplication (3.2) and involution (3.3), constructed from a unital alternative algebra A, left and right unital "associative" modules B and C, and bilinear maps $T(b,c)$, $b_1 \times b_2$, and $c_1 \times c_2$ coordinatize a Lie algebra $\mathcal{L}$ with a subalgebra isomorphic to $\mathfrak{sl}(\mathsf{V})$ for a vector space V of dimension 3 and with decomposition as in (2.1)? (We do not impose any further conditions on these bilinear maps besides requiring that the resulting algebra $\mathcal{A}$ be structurable.)

Our last result answers this question in the affirmative.

Theorem 3.2. *Let* A *be a unital alternative algebra; let* B *(respectively* C*) be a left (respectively right) unital associative module for* A*; and let* $\mathsf{B} \times \mathsf{C} \to \mathsf{A} : (b,c) \mapsto T(b,c)$*,* $\mathsf{B} \times \mathsf{B} \to \mathsf{C} : (b_1, b_2) \mapsto b_1 \times b_2$*, and* $\mathsf{C} \times \mathsf{C} \to \mathsf{B} : (c_1, c_2) \mapsto c_1 \times c_2$ *be bilinear maps which make the vector space* $\mathcal{A}$ *in* (3.1) *with the multiplication* (3.2) *and involution in* (3.3) *into a structurable algebra. Then there is a Lie algebra* $\mathcal{L}$ *containing a subalgebra isomorphic to* $\mathfrak{sl}(\mathsf{V})$*, for a vector space* V *of dimension* 3*, such that* $\mathcal{L}$ *decomposes as in* (2.1) *for a suitable vector space* $\mathfrak{s}$ *such that the Lie bracket on* $\mathcal{L}$ *is given by* (2.2) *for some bilinear maps* $\mathsf{A} \times \mathsf{A} \to \mathfrak{s} : (a, a') \mapsto D_{a,a'}$*,* $\mathsf{B} \times \mathsf{C} \to \mathfrak{s} : (b,c) \mapsto D_{b,c}$*,* $\mathfrak{s} \times \mathsf{A} \to \mathsf{A} : (d,a) \mapsto da$*,* $\mathfrak{s} \times \mathsf{B} \to \mathsf{B} : (d,b) \mapsto db$*, and* $\mathfrak{s} \times \mathsf{C} \to \mathsf{C} : (d,c) \mapsto dc$*.*

Proof. Consider the Lie algebra $\mathcal{L} = \mathcal{K}(\mathcal{A}, -, \gamma, \mathcal{V})$ in [2, Sec. 4] attached to the structurable algebra $(\mathcal{A}, -)$, the triple $\gamma = (1,1,1)$, and the Lie subalgebra $\mathcal{V} = \mathcal{T}_I$. This Lie algebra $\mathcal{L}$, which coincides with the Lie algebra $\mathfrak{g}(\mathcal{A}, \cdot, -)$ in [6, Ex. 3.1], is the direct sum

$$\mathcal{L} = \mathcal{T}_I \oplus \mathcal{A}[12] \oplus \mathcal{A}[23] \oplus \mathcal{A}[31],$$

where $\mathcal{T}_I$ is the span of the triples $T = (T_1, T_2, T_3)$ with

$$\begin{aligned} T_i &= L_{\bar{x}} L_y - L_{\bar{y}} L_x, \\ T_j &= R_{\bar{x}} R_y - R_{\bar{y}} R_x, \\ T_k &= R_{\bar{x}y - \bar{y}x} + L_y L_{\bar{x}} - L_x L_{\bar{y}}, \end{aligned} \tag{3.6}$$

for $x, y \in \mathcal{A}$ and (i,j,k) a cyclic permutation of $(1,2,3)$. Here $L_x y = xy = R_y x$. The subspace $\mathcal{T}_I$ is a Lie algebra with componentwise bracket, and the Lie bracket in $\mathcal{L}$ is given by extending the bracket in $\mathcal{T}_I$ by setting $x[ij] = -\bar{x}[ji]$ for any $x \in \mathcal{A}$ and

$$\begin{aligned} [x[ij], y[jk]] &= -[x[jk], y[ij]] = (xy)[ik], \\ [T, x[ij]] &= -[x[ij], T] = T_k(x)[ij], \\ [x[ij], y[ij]] &= T, \end{aligned} \tag{3.7}$$

for $x, y \in \mathcal{A}$, where (i,j,k) is a cyclic permutation of $(1,2,3)$, and $T = (T_1, T_2, T_3)$ is as in (3.6). Theorems 4.1 and 5.5 in [2] show that $\mathcal{L}$ is indeed a Lie algebra. Since we are assuming that the characteristic of the field is $\neq 2, 3$, Corollary 3.5 of [2] shows that $\mathcal{T}_I = \{(D,D,D) \mid D \in \mathsf{Der}(\mathcal{A}, \cdot, -)\} \oplus \{(L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2}) \mid$

$s_i \in \mathcal{A},\ \bar{s}_i = -s_i,\ s_1 + s_2 + s_3 = 0\}$. Here $\mathsf{Der}(\mathcal{A}, \cdot, -)$ is the space of derivations relative to the product "$\cdot$" which commute with the involution "$-$".

For any $a \in \mathsf{A}$, consider the linear span $\mathfrak{sl}_3[a]$ of the elements $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)[ij]$ for $i \neq j$ and the triples $(L_{\alpha_2 s} - R_{\alpha_3 s}, L_{\alpha_3 s} - R_{\alpha_1 s}, L_{\alpha_1 s} - R_{\alpha_2 s})$ for $\alpha_i \in \mathbb{F}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $s = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$. (Note that $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)[ij] = -\left(\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}\right)[ji]$.)

Also for any $b \in \mathsf{B}$, consider the linear span $\mathsf{V}[b]$ of the elements $\left(\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right)[ij]$, and for any $c \in \mathsf{C}$ the linear span $\mathsf{V}^*[c]$ of the elements $\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)[ij]$.

Straightforward computations using (3.7) imply that

- $\mathfrak{sl}_3[1]$ is a Lie subalgebra of $\mathcal{L}$ isomorphic to $\mathfrak{sl}(\mathsf{V})$ ($\dim \mathsf{V} = 3$),
- $\mathfrak{sl}_3[a]$ is an adjoint module for $\mathfrak{sl}_3[1]$ for any $a \in \mathsf{A}$,
- $\mathsf{V}[b]$ is the natural module for $\mathfrak{sl}_3[1]$ for any $b \in \mathsf{B}$,
- $\mathsf{V}^*[c]$ is the dual module for $\mathfrak{sl}_3[1]$ for any $c \in \mathsf{C}$, and
- $\mathfrak{s} = \{(D, D, D) \mid D \in \mathsf{Der}(\mathcal{A}, \cdot, -)\}$ is a Lie subalgebra which commutes with $\mathfrak{sl}_3[1]$.

Actually, if we fix a basis $\{e_1, e_2, e_3\}$ of V as before with $\det(e_1 \wedge e_2 \wedge e_3) = 1$ and the dual basis $\{e_1^*, e_2^*, e_3^*\}$ in V^*, we may identify $\mathfrak{sl}_3[a]$ with $\mathfrak{sl}(\mathsf{V}) \otimes a$ for $a \in \mathsf{A}$ by means of

$$e_i e_j^* \otimes a \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}[ij], \qquad \text{for} \quad i \neq j$$

$$\sum_{i=1}^{3} \alpha_i e_i e_i^* \otimes a \leftrightarrow (L_{\alpha_2 s} - R_{\alpha_3 s}, L_{\alpha_3 s} - R_{\alpha_1 s}, L_{\alpha_1 s} - R_{\alpha_2 s}),$$

for $\alpha_i \in \mathbb{F}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $s = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$. Also for any $b \in \mathsf{B}$ and $c \in \mathsf{C}$, we may identify $\mathsf{V}[b]$ with $\mathsf{V} \otimes b$ and $\mathsf{V}^*[c]$ with $\mathsf{V}^* \otimes c$ via

$$e_i \otimes b \leftrightarrow \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}[jk], \qquad e_i^* \otimes c \leftrightarrow \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}[jk],$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

In this way we recover the decomposition in (1.3) with bracket as in (2.2) for suitable maps $D_{\cdot,\cdot}$, as required. □

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