GENERALIZED LUCAS-LEHMER TESTS USING PELL CONICS

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Abstract. Pell conics are used to write a Proth-Riesel twin-primality test. We discuss easy-to-find primality certificates for integers of the form \( m^n h \pm 1 \). The known primality test for \( 3^n h \pm 1 \) is associated with \( X^2 + 3Y^2 = 4 \).

1. Introduction

Like elliptic curves, there is a group law on the Pell conics \([4]\). These are affine curves of genus 0 of the form \( C : X^2 - \Delta Y^2 = 4 \) where \( \Delta \) is a fundamental discriminant. Geometrically, points \( P \) and \( Q \) on Pell conics are added by taking the line parallel to \( PQ \), passing through the point \( O = (2, 0) \), and intersecting with \( C \) at \( P + Q \). Algebraically this is

\[
(x_1, y_1) + (x_2, y_2) = \left( \frac{x_1 x_2 + \Delta y_1 y_2}{2}, \frac{x_1 y_2 + x_2 y_1}{2} \right).
\]

Multiplication by 2 and 3 is given by

\[
2(x, y) = (x^2 - 2, xy),
\]
\[
3(x, y) = (x^3 - 3x, (x^2 - 1)y).
\]

Lemmermeyer \([4]\) considered the arithmetic of Pell conics indicating many interesting similarities with elliptic curves. The following theorem appears in \([4]\) in a more general form. We use \( \left( \frac{\Delta}{p} \right) \) to mean the Legendre symbol.

Theorem 1.1. Let \( C : X^2 - \Delta Y^2 = 4 \), \( \Delta \) be a fundamental discriminant, \( p \) be prime and \( N \) be an integer. Then:

- \( C(\mathbb{Z}) \) is an abelian group with identity \( O = (2, 0) \), and point \( T = (-2, 0) \) of order 2. No other points have \( y = 0 \) or \( x = \pm 2 \). The inverse of \( (x, y) \) is \( (x, -y) \).
- \( C(\mathbb{Z}/N) \) is an abelian group with identity \( O \).
- \( C(\mathbb{F}_p) \) is a cyclic group of order \( p - \left( \frac{\Delta}{p} \right) \).

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For multiplication by \( m \geq 1 \), including points of Equation (2), we define polynomials
\[
(3) \quad f_0 = 2; f_1 = x; f_{i+1} = xf_i - f_{i-1},
\]
\[
(4) \quad g_0 = 0; g_1 = 1; g_{i+1} = xg_i - g_{i-1},
\]
\[
(5) \quad F_1 = 1; F_3 = x + 1; F_{2i+3} = xF_{2i+1} - F_{2i-1}.
\]

**Lemma 1.2.** Let \( C \) be a Pell conic and \( P = (x, y) \in C(\mathbb{Z}) \). Then for \( m \geq 1 \), \( mP = (f_m(x), y \cdot g_m(x)) \), and for odd \( m \geq 1 \),
\[
(6) \quad f_m(x) = (x - 2)F_m(x)^2 + 2.
\]

**Proof.** \( mP = (f_m(x), y \cdot g_m(x)) \) may be proved by induction using Equations (1), (3), and (4). To prove Equation (6), induction shows that
\[
(7) \quad F_{2i-1}^2 - x \cdot F_{2i-1}F_{2i+1} + F_{2i+1}^2 = x + 2,
\]
and using Equation (7) in another induction proves Equation (6). \( \square \)

Corollary 1.3 is a direct result of Lemma 1.2. \( C(\mathbb{F}_p)[m] \) is used to denote the set of points \( P \) of \( C(\mathbb{F}_p) \) for which \( mP = 0 \).

**Corollary 1.3.** Let \( C \) be a Pell conic, \( P = (x, y) \in C(\mathbb{F}_p) \), and \( p \) prime. Then for odd \( m \geq 1 \), \( P \in C(\mathbb{F}_p)[m] \backslash \emptyset \) if and only if \( F_m(x) \equiv 0 \) (mod \( p \)).

Lemmermeyer [4] discussed primality proving using Pell conics, giving Theorem 1.5 as an analogue of Lucas’ theorem. It is assumed that the integers \( N \) are greater than 1 and coprime to 6.

**Theorem 1.4** (Lucas). If \( a^{N-1} \equiv 1 \) (mod \( N \)) but \( a^{\frac{N-1}{q}} \not\equiv 1 \) (mod \( N \)) for every prime factor \( q \) of \( N - 1 \), then \( N \) is prime.

**Theorem 1.5.** Let \( N \geq 5 \) and \( C : X^2 - \Delta Y^2 = 4 \) be a Pell conic defined over \( \mathbb{Z}/N \) with \( \left( \frac{\Delta}{N} \right) = -1 \). Then \( N \) is prime if and only if for some point \( P \in C(\mathbb{Z}/N) \), 
\( (N + 1)P = \emptyset \), but \( \frac{N+1}{q}P \not\equiv \emptyset \) for every prime factor \( q \) of \( N + 1 \).

Lemmermeyer [4] remarked that there are Proth versions in which only part of \( N \pm 1 \) needs to be factored and commented that in the same way Gross [2] gave an elliptic curve ‘Lucas-Lehmer’ test, the Lucas-Lehmer test itself may be proved with the Pell conic \( C : X^2 - 12Y^2 = 4 \) and the point \((4,1)\). This method is extended here to more general primality proving.

2. **Twin primes of the form \( 2^n h \pm 1 \)**

The theory of Pell conics is applied to a test similar to Riesel’s [6] generalization of the Lucas-Lehmer test to \( N = 2^n h - 1 \). Our test, however, also includes Proth’s Theorem. One advantage in their combination is a single primality certificate for a pair of twin primes. The uppercase letter \( \mathbb{Q} \) will henceforth be used exclusively for a fixed generator of \( C(\mathbb{F}_p) \).

**Lemma 2.1.** Let \( p \) be an odd prime and let \( C : X^2 - \Delta Y^2 = 4 \) be a Pell conic such that \( p \equiv \left( \frac{\Delta}{p} \right) \) (mod 4). There is an exact sequence of group homomorphisms
\[
(8) \quad 0 \longrightarrow 2C(\mathbb{F}_p) \longrightarrow C(\mathbb{F}_p) \overset{\theta}{\longrightarrow} \{ \pm 1 \}^X \longrightarrow 0,
\]
where \( \theta : (x, y) \mapsto \left( \frac{x+2}{p} \right) \).
Proof. Let \( P_1, P_2 \in C(F_p) \). We must prove that \( \theta(P_1) \cdot \theta(P_2) = \theta(P_1 + P_2) \). We will use the fact that \( \theta(P) = 1 \) if and only if \( P \in 2C(F_p) \), but this also establishes that \( \ker \theta = 2C(F_p) \). By Theorem 1, \( C(F_p) \) is cyclic, generated by \( Q \), so there exist integers \( \ell_1 \) and \( \ell_2 \) such that \( P_1 = \ell_1 Q \) and \( P_2 = \ell_2 Q \). If \( \theta(P_1) = \theta(P_2) \) then \( \ell_1 \equiv \ell_2 \) (mod 2) so \( \ell_1 + \ell_2 \) is even and \( P_1 + P_2 = (\ell_1 + \ell_2)Q \in 2C(F_p) \) so that \( \theta(P_1 + P_2) = 1 \). If \( \theta(P_1) \neq \theta(P_2) \) then \( \ell_1 \neq \ell_2 \) (mod 2) so \( \ell_1 + \ell_2 \) is odd and \( P_1 + P_2 = (\ell_1 + \ell_2)Q \notin 2C(F_p) \) so that \( \theta(P_1 + P_2) = -1 \). It follows that \( \theta \) is a group homomorphism. \( \theta(Q) = -1 \) since \( \#C(F_p) = p - (\frac{\Delta}{p}) \equiv 0 \) (mod 4) and \( \theta(2Q) = 1 \), so \( \theta \) is surjective. \( \square \)

It follows that if \( p = 2^n h + 1 \) is prime, where \( h \) is odd and \( n \geq 2 \), and the Pell conic \( C : X^2 - \Delta Y^2 = 4 \) is chosen such that \( p \equiv (\frac{\Delta}{p}) \) (mod 4), then:

- If \( \left(\frac{x(p)+2}{p}\right) = 1 \), then \( P \in 2C(F_p) \) and \( 2^{n-1}hP = 0 \).
- If \( \left(\frac{x(p)+2}{p}\right) = -1 \), then \( P \notin 2C(F_p) \) and \( 2^{n-1}hP = \mathcal{T} \).

Observe that the order of \( C(F_p) \) is equal to \( 2^n h \), so \( 2^{n-1}hP = \mathcal{O} \) or \( \mathcal{T} \). Now \( P = \ell Q \), so \( 2^{n-1}hP = 2^{n-1}h\ell Q = 0 \) if and only if \( 2^n h \equiv 2^{n-1}h \ell \equiv 0 \), because \( Q \) is the generator, if and only if \( \ell \) is even. If \( \left(\frac{x(p)+2}{p}\right) = -1 \), then \( 2^{n-1}hP = \mathcal{T} \).

Lemma 2.2. Let \( N = 2^n h + 1 \), where \( h \) is odd. Let \( C : X^2 - \Delta Y^2 = 4 \) be a Pell conic with \( \Delta \nmid N \). If there exists a \( P \in C(\mathbb{Z}/N) \) such that \( 2^{n-1}hP = \mathcal{T} \), then every prime factor \( q \) of \( N \) satisfies \( q \equiv (\frac{\Delta}{q}) \) (mod \( 2^n \)).

Proof. Suppose \( 2^{n-1}hP = \mathcal{T} \) in \( C(\mathbb{Z}/N) \). Let \( q \) be a prime divisor of \( N \). Let \( o_q(P) \) denote the order of the point \( P \) on \( C : X^2 - \Delta Y^2 = 4 \) over \( F_q \). Clearly we also have \( 2^{n-1}hP = \mathcal{T} \) in \( C(F_q) \). Therefore \( o_q(P) \mid 2^n h \), but \( o_q(P) \nmid 2^{n-1}h \). Since \( h \) is odd, \( 2^n \mid o_q(P) \). Now \( C(F_q) \) is cyclic of order \( q - (\frac{\Delta}{q}) \), so \( o_q(P) \mid q - (\frac{\Delta}{q}) \). That is, \( 2^n \mid q - (\frac{\Delta}{q}) \) or simply \( q \equiv (\frac{\Delta}{q}) \) (mod \( 2^n \)). \( \square \)

For the following, \( (\cdot) \) will mean the Jacobi symbol.

Theorem 2.3. Let \( N = 2^n h + 1 \) where \( 0 < h < 2^n \), \( h \) is odd and \( n \geq 2 \). Let \( C : X^2 - \Delta Y^2 = 4 \) be a Pell conic satisfying \( (\frac{\Delta}{N}) \equiv N \) (mod 4), and let \( P = (x,y) \in C(\mathbb{Z}/N) \) be a point such that \( (\frac{x+y}{2}) = -1 \). Then \( N \) is prime if and only if \( f_{2^{n-1}h}(x) \equiv -2 \) (mod \( N \)), where \( f_{2^{n-1}h}(x) \) is the \( 2^{n-1}h \)-th polynomial satisfying Equation (3).

Proof. If \( N \) is prime, then by Lemma 2.1 \( 2^{n-1}hP = \mathcal{T} \). Conversely, suppose \( 2^{n-1}hP = \mathcal{T} \) while \( N \) is composite. Then by Lemma 2.2 every prime factor \( q \) of \( N \), and hence every factor, satisfies \( q \equiv (\frac{\Delta}{q}) \) (mod \( 2^n \)). If \( N \equiv 1 \) (mod 4), then \( N \) may factor as \( N = (2^n h_1 + 1)(2^n h_2 + 1) \) or \( N = (2^n h_1 - 1)(2^n h_2 - 1) \), so \( h = \pm(h_1 + h_2) + h_1 h_2 2^n = (h_1 + h_2 - 1)(2^n + 1) \pm 1 + (h_1 - 1)(h_2 - 1)2^n \geq 2^n \) since \( h_1 \) and \( h_2 \) cannot both be 1 because \( h \) is odd, a contradiction. If \( N \equiv -1 \) (mod 4), then \( N \) may factor as \( N = (2^n h_1 + 1)(2^n h_2 - 1) \), and we find that \( h = h_2 - h_1 + h_1 h_2 2^n = h_2 + h_1(h_2 2^n - 1) \geq 2^n \), a contradiction. Therefore \( N \) is prime. \( \square \)
We will assume from here on that \( f, g, \) and \( F \) always refer to the polynomials defined in Equations (3), (4), and (5) respectively. We give an application of Theorem 2.3 to pairs of twin primes.

**Algorithm 2.4.** To certify the primality of a pair of twin primes of the form \( 2^n h \pm 1 \):

1. Choose an integer \( n \geq 2 \) and a positive odd integer \( h < 2^n \), and set \( r = 2^n h \).
2. Find a fundamental discriminant \( \Delta \) such that \( \Delta = -1 \) and \( \Delta \equiv 1 \mod 2 \).
3. Find an integer \( x \) such that \( \left( \frac{x+2}{r+1} \right) = \left( \frac{x-2}{r-1} \right) = -1 \).
4. Compute \( f_{r/2}(x) \mod r-1 \) and \( r+1 \). If both \( f_{r/2}(x) \equiv -2 \mod r \pm 1 \), then the points \( (x, \cdot) \) and Pell conic \( C : X^2 - \Delta Y^2 = 4 \) certify that \( r \pm 1 \) is a pair of twin primes.

**Proof.** If \( r \pm 1 \) is composite, then by Theorem 2.3, \( f_{r/2}(x) \not\equiv -2 \mod r \pm 1 \), so we must prove that if \( r \pm 1 \) are prime and \( \left( \frac{x+2}{r+1} \right) = \left( \frac{x-2}{r-1} \right) = -1 \) then there exist points \( P \in C(F_{r \pm 1}) \) such that \( x = x(P) \) and \( P \notin 2C(F_{r \pm 1}) \). Clearly, there exist points \( P \in C(F_{r \pm 1}) \) such that \( x = x(P) \) if and only if \( \left( \frac{x^2 - 4}{r \pm 1} \right) = \pm 1 \) if and only if \( \left( \frac{x^2 - 4}{r \pm 1} \right) = 1 \). Part (4) then follows from Theorem 2.3. \( \square \)

In Step (4) of Algorithm 2.4, it is unnecessary to evaluate, at the integer \( x \) chosen in Step (3), each of the polynomials \( f_i \) preceding \( f_{r/2} \), as illustrated in the following example.

**Example 2.5.** The Pell conic \( C : X^2 - 28 Y^2 = 4 \) and points \((17, y) \in C(Z/r \pm 1)\) certify the twin primes 191 and 193:

1. \( r = 2^6 \cdot 3 \).
2. \( \left( \frac{28}{191} \right) = -1 \) and \( \left( \frac{28}{193} \right) = 1 \), so \( C : X^2 - 28 Y^2 = 4 \) is suitable.
3. \( \left( \frac{16}{197} \right) = \left( \frac{13}{197} \right) = -1 \), so the points \((17, 50) \in C(Z/191)\) and \((17, 47) \in C(Z/193)\) have the required properties.
4. To compute \( f_{2^6 \cdot 3}(17) \mod 191 \) and 193, write \( h = 3 \) in binary as \( b = 11 \).

The \( k \)-th term \( t_k \) of a sequence \( B \) is obtained by taking the first \( k \) digits of \( b \) from left to right, \( B = \{1, 11\} \). To compute \( f_{k}(17) \mod 191 \) and \( f_{k}(17) \mod 193 \), for each \( t_k \in B \), we wish to compute a sequence of pairs \((f_{k}, f_{k} + f_{k})(17))\) evaluated modulo 191 and 193, where

\[
(f_{k+1}, f_{1+t_k}) = \begin{cases} 
(f_{k}^2 - 2, f_{k} \cdot f_{1+t_k} - x) & \text{if } t_{k+1} \text{ is even}, \\
(f_{k} \cdot f_{1+t_k} - x, f_{k}^2 + f_{k}) & \text{if } t_{k+1} \text{ is odd}
\end{cases}
\]

and \( f_{1}(x) = x \) and \( f_{2}(x) = x^2 - 2 \). We have \( f_{3}(17) \equiv 87 \mod 191 \) and \( f_{3}(17) \equiv 37 \mod 193 \). Using \( f_{2^n - 1} = f_{2^n - 1}(f_k) \), we iterate terms of the sequences determined by repeated doubling, \( n - 1 = 5 \) times, modulo 191 and 193: \( \{87, 118, 170, 57, 0, 189\} \mod 191 \) and \( \{37, 16, 61, 52, 0, 191\} \mod 193 \).

Equation (9) has been used by Williams [7] in the same way to efficiently evaluate Lucas sequences.

3. **Easy-to-find primality certificates for** \( m^n h \pm 1 \)

The main result of this section differs from the previous since we use solved Pell conics over integers to certify primes of the form \( m^n h \pm 1 \). The smallest non-trivial
point of \( C(\mathbb{Z}) \) with \( x, y > 0 \), the fundamental solution, is usually a generator of 
\( C(\mathbb{F}_p) \). We begin with the main result which builds on a result of Williams \[7\].

The lemmas supporting Theorem 3.4 are included below. Remark 3.1 allows a comparison with Lemma 2.1.

**Remark 3.1.** Let \( p \) be an odd prime, \( m \) be an odd integer and \( C : X^2 - \Delta Y^2 = 4 \) be
a Pell conic such that \( p \equiv \left( \frac{\Delta}{p} \right) \pmod{m} \). Let \( \mu_m \) denote the multiplicative group
of \( m \)-th roots of unity, generated by \( \omega \). There is an exact sequence

\[
0 \to mC(\mathbb{F}_p) \to C(\mathbb{F}_p) \overset{\phi}{\to} \mu_m \to 0,
\]

where \( \phi : \mathcal{P} = \ell \Omega \to \omega^\ell \).

**Proof.** Let \( \mathcal{P}_1, \mathcal{P}_2 \in C(\mathbb{F}_p) \). Writing \( \mathcal{P}_1 = \ell_1 \Omega \) and \( \mathcal{P}_2 = \ell_2 \Omega \), \( \phi(\mathcal{P}_1) \cdot \phi(\mathcal{P}_2) = \omega^\ell_1 \omega^\ell_2 = \omega^{\ell_1+\ell_2} = \phi(\mathcal{P}_1 + \mathcal{P}_2) \). Again letting \( \mathcal{P} = \ell \Omega \), \( \phi(\mathcal{P}) = 1 \) if and only if
\( \omega^\ell = 1 \), if and only if \( m \mid \ell \), if and only if \( \mathcal{P} \in mC(\mathbb{F}_p) \). So \( \ker(\phi) = mC(\mathbb{F}_p) \). \( \square \)

**Lemma 3.2.** Let \( p = m^a h + 1 \) be prime, where \( m \) is odd, \( h \) is even, not divisible
by \( m \) and \( n \geq 2 \). Let \( C : X^2 - \Delta Y^2 = 4 \) be a Pell conic where \( \left( \frac{\Delta}{p} \right) \equiv p \pmod{m} \).

If \( \mathcal{P} = (x, y) \in C(\mathbb{F}_p) \) but \( \mathcal{P} \not\in mC(\mathbb{F}_p) \), then \( s \equiv f_m(x) \pmod{p} \) satisfies
\( F_m(s) \equiv 0 \pmod{p} \). If \( \mathcal{P} \in mC(\mathbb{F}_p) \), then \( s \equiv 2 \pmod{p} \).

**Proof.** If \( \mathcal{P} \not\in mC(\mathbb{F}_p) \), then \( m^{n-1} h \mathcal{P} = m^n h \Omega = \emptyset \) so that \( s \equiv 2 \pmod{p} \). Points
\( \mathcal{P} \in C(\mathbb{F}_p) \) satisfy \( m^{n-1} h \mathcal{P} \in C(\mathbb{F}_p)[m] \). Assuming \( \mathcal{P} \in C(\mathbb{F}_p) \setminus mC(\mathbb{F}_p) \) we must show that
\( m^{n-1} h \mathcal{P} \not\in \emptyset \). Now \( \mathcal{P} = \ell \Omega \) for some positive integer \( \ell \), so \( m^{n-1} h \mathcal{P} = m^n h \Omega = \emptyset \) if and only if
\( m^h | m^{n-1} h \ell \), because \( \Omega \) is a generator, if and only if \( m \mid \ell \). Since \( \mathcal{P} \not\in mC(\mathbb{F}_p) \), \( m \not| \ell \), so \( m^{n-1} h \mathcal{P} \in C(\mathbb{F}_p)[m] \setminus 0 \), and it follows that
\( s \equiv f_m(x) \pmod{p} \) satisfies \( F_m(s) \equiv 0 \pmod{p} \) by Corollary 1.3 \( \square \)

The proof of Lemma 3.3 is similar to the proof of Lemma 2.2.

**Lemma 3.3.** Let \( N = m^n h + 1 \), where \( m \) is odd, \( h \) is even, not divisible by \( m \) and
\( n \geq 2 \). Let \( C : X^2 - \Delta Y^2 = 4 \) be a Pell conic satisfying \( \left( \frac{\Delta}{N} \right) \equiv N \pmod{m} \). If
there exists a \( \mathcal{P} \in C(\mathbb{Z}/N) \) such that \( s \equiv f_m(x(\mathcal{P})) \pmod{N} \) satisfies \( F_m(s) \equiv 0 \pmod{N} \),
then every prime factor \( q \) of \( N \) satisfies \( q \equiv \left( \frac{\Delta}{q} \right) \pmod{m^n} \).

For the purpose of certifying primes of the form \( m^n h + 1 \) in the case where it is
not checked whether the point \( \mathcal{P} \) belongs to \( mC(\mathbb{Z}/N) \), it should be assumed that
the Pell conic satisfies \( \Delta > 0 \) and \( \mathcal{P} \) is the fundamental solution of \( C(\mathbb{Z}) \),
reduced modulo \( N \) to an element of \( C(\mathbb{Z}/N) \), in order to increase the chance that
\( \mathcal{P} \not\in mC(\mathbb{Z}/N) \).

**Theorem 3.4.** Let \( N = m^n h + 1 \), where \( m \) is odd, \( h \) is even, not divisible by \( m \),
\( 0 < h < m^n \) and \( n \geq 2 \). Let \( C \) be a Pell conic satisfying \( \left( \frac{\Delta}{N} \right) \equiv N \pmod{m} \). Let
\( \mathcal{P} = (x, y) \in C(\mathbb{Z}/N) \) and let \( s \equiv f_m(x) \pmod{N} \). Then:

(1) If \( F_m(s) \equiv 0 \pmod{N} \), then \( N \) is prime.

(2) If \( s \not\equiv 2 \pmod{N} \) and \( F_m(s) \not\equiv 0 \pmod{N} \), then \( N \) is composite.

(3) If \( s \equiv 2 \pmod{N} \), then \( N \) is either prime or a Lucas pseudoprime. Another
Pell conic is required.

(4) If it is known that \( \mathcal{P} \not\in mC(\mathbb{Z}/N) \), then \( N \) is prime if and only if \( F_m(s) \equiv 0 \pmod{N} \).
Proof: If $N$ is prime, then by Lemma 3.2 $F_m(s) \equiv 0 \pmod{N}$ or $s \equiv 2 \pmod{N}$. Conversely, suppose $F_m(s) \equiv 0 \pmod{N}$ while $N$ is composite. By Lemma 3.3 every prime factor $q$ of $N$, and hence every factor, satisfies $q \equiv \pm 1 \pmod{m^n}$. If $N \equiv 1 \pmod{m}$, then $N$ may factor as $N = (m^n h_1 + 1)(m^n h_2 - 1)$. These correspond to $h = \pm (h_1 + h_2) + h_1 h_2 m^n = h_1 (m^n \pm 1) + h_2 (m^n \pm 1) - m^n (h_1 - 1)(h_2 - 1)m^n$. Now $h_1$ and $h_2$ must be even since $h$ is even so that $h > m^n$, a contradiction, and $N$ must be prime. If $N \equiv -1 \pmod{m}$, then $N$ may factor as $N = (m^n h_1 + 1)(m^n h_2 - 1)$. This corresponds to $h = h_2 - h_1 + h_1 h_2 m^n$. If $h_1 = h_2 = 2$, then $h = 4 \cdot m^n$ so that $m \mid h$, a contradiction. Writing $h = h_2 + h_1 (m^n - 1) + h_1 (h_2 - 1)m^n$ and noting that $h_2$ must be even, $h > m^n$, a contradiction, so $N$ must be prime. This completes the proof of case (1). If the conditions of case (2) are satisfied, then $m^n h \not\equiv 0 \pmod{N}$, so $N$ is composite by Theorem 1.1. Case (3) follows from Lemma 3.2, noting that $N$ may be a Lucas pseudoprime since the $f_j$ are terms of a Lucas sequence. If $N$ is prime and $\mathcal{P} \not\in mC(\mathbb{F}_N)$, then by Lemma 3.2 $F_m(s) \equiv 0 \pmod{N}$. This, together with case (1), completes the proof of case (4).

Given that $N$ is prime and the Pell conic $C$, with $\Delta > 0$ and $\left(\frac{\Delta}{N}\right) = N \pmod{m}$, is randomly chosen while always using the fundamental solution of $C(\mathbb{Z})$ reduced modulo $N$ to a point of $C(\mathbb{Z}/N)$ as the point $\mathcal{P}$ of Theorem 3.4 this result leads to a primality certificate with a probability of $1 - \frac{1}{2^m}$. This attests to the occurrence of item (3) of Theorem 3.4 being unlikely. The $f_m(x)$ are in fact the Dickson polynomials of the first kind $D_m(x, a)$, with $a = 1$. The following comes from [5].

**Theorem 3.5** (Dickson). Let $a \in \mathbb{F}_q^*$, where $q$ is the power of a prime. The Dickson polynomial $D_m(x, a)$ permutes the field $\mathbb{F}_q$ if and only if $\gcd(m, q^2 - 1) = 1$.

If $p = m^n h \equiv 1 \pm 1$ is prime and $m$ is odd while $h$ is even, then $\gcd(m, p^2 - 1) = m$ and $f_m(x) = D_m(x, 1)$ cannot permute $\mathbb{F}_p$. That is, there exists a $\mathcal{P} \in C(\mathbb{F}_p)$ such that $\mathcal{P} \not\in mC(\mathbb{F}_p)$, where $C : X^2 - \Delta Y^2 = 4$ is a Pell conic with the required property $\left(\frac{\Delta}{p}\right) \equiv p \pmod{m}$.

**Corollary 3.6.** Let $p = m^n h \equiv 1$ be prime where $m$ is odd. Let $C : X^2 - \Delta Y^2 = 4$ be a Pell conic satisfying $\left(\frac{\Delta}{p}\right) \equiv p \pmod{m}$. Then the proportion $\frac{1}{m}$ of the points $\mathcal{P} \in C(\mathbb{F}_p)$ satisfy $\mathcal{P} \in mC(\mathbb{F}_p)$.

**Proof.** Let $\Omega$ be the generator of the cyclic group $C(\mathbb{F}_p)$. Suppose $\Omega = m\mathcal{P}$. Then all of the elements of $C(\mathbb{F}_p)$ are divisible by $m$, contradicting Theorem 3.5. Thus $\Omega$ is not divisible by $m$. Under the isomorphism $C(\mathbb{F}_p) \simeq \mathbb{Z}/m^n h$, the proportion $\frac{1}{m}$ of the points of $C(\mathbb{F}_p)$ are divisible by $m$.

**Remark 3.7.** Let $h$ be an integer. Neglecting the computation of the binary sequence $B$ of Example 2.5 and modular additions, the evaluation of $f_h$ modulo an integer $N$ requires at most $2 \log_2(h)$ modular multiplications [5].

Induction may be used with Equations (5) and (7) to establish the identities

$$ F_{4j-1} = F_{2j+1} \cdot F_{2j-1} - F_{2j-1}^2 + 1, $$

$$ F_{4j+1} = F_{2j+1}^2 - F_{2j+1} \cdot F_{2j-1} - 1, $$

$$ F_{4j+3} = (x - 1) F_{2j+1}^2 - F_{2j+1} F_{2j-1} + 1. $$

We give an example showing how the $F_m$ may be computed using Equation (10).
Example 3.8. We show how \( F_{795}(x) \) may be evaluated. The binary representation of \( m = 795 \) is \( m_2 = 1100011011 \). The \( k \)-th term \( u_k \) of a sequence \( U = \{u_k\} \) is obtained by taking the first \( k + 1 \) digits of \( a \) from left to right from \( k = 1 \) up to, but not including \( m_2 \), to \( k = \lfloor \log_2(m) \rfloor - 1 \):

\[
U = \{11, 110, 1100, 11000, 110001, 1100011, 11000110, 110001101\}.
\]

Define a new sequence \( V = \{v_k\} \) by \( v_k = 2[u_k/2] \):

\[
V = \{10, 110, 1100, 11000, 110000, 1100000, 11000000, 110000000\} \quad (\text{in base } 10).
\]

The pairs \( (F_{-1+v_k}, F_{1+v_k}) \) may be evaluated recursively by using

\[
\begin{align*}
F_{-1+v_k+1} &= \begin{cases} F_{1+v_k} F_{-1+v_k} - F_2^{1+v_k - 1} & \text{if } v_{k+1} \equiv 0 \pmod{4}, \\
F_1^{2+v_k} - F_1^{2+v_k} F_{-1+v_k} & \text{if } v_{k+1} \equiv 2 \pmod{4}, 
\end{cases} \\
F_{1+v_k+1} &= \begin{cases} F_2^{1+v_k} - F_1^{1+v_k} F_{-1+v_k} - 1 & \text{if } v_{k+1} \equiv 0 \pmod{4}, \\
(x-1) F_1^{2+v_k} - F_1^{2+v_k} F_{-1+v_k} + 1 & \text{if } v_{k+1} \equiv 2 \pmod{4}, 
\end{cases}
\end{align*}
\]

which follow from Equations (11), until the value of \((F_{397}, F_{399})\) is known. Finally \( F_{795} = F_{399} \cdot F_{397} - F_{397}^2 + 1 \).

Example 3.9. We certify the primality of \( N = 795^5 \cdot 1588 - 1 \). Now \( (\frac{13}{N}) = -1 \equiv N \pmod{795} \), so the Pell conic \( C : X^2 - 13Y^2 = 4 \) is suitable for applying Theorem 3.3. The point \((11,3)\) is the fundamental solution of \( C(\mathbb{Z}) \). We must evaluate \( s \equiv f_{795^4,1588}(11) \pmod{N} \). The binary representation of \( 795^4 \cdot 1588 = 1001000000111011000111010100011011101100 \) in base 10. As in Example 2.5 we recursively evaluate modulo \( N \), the sequence of pairs

\[
\{(f_1(11), f_2(11)), (f_2(11), f_3(11)), (f_3(11), f_5(11)), (f_5(11), f_{10}(11)), \ldots, (f_{795^4,1584}(11), f_{795^4,1584+1}(11))\} \pmod{N} \quad \text{using Equation (9)}
\]

\[
\equiv \{(119, 1298), (14159, 154451), (2186871698, 23855111399), \ldots, (363731798219995454, 244122496081218988)\} \pmod{N},
\]

\[
s \equiv 363731798219995454^2 - 2 \equiv 38801172344360862 \pmod{N}.
\]

To evaluate \( F_{795}(s) \pmod{N} \), we proceed as in Example 3.8 using Equations (11) and (12):

\[
\{(F_1(s), F_3(s)), (F_3(s), F_7(s)), (F_7(s), F_{11}(s)), \ldots, (F_{397}(s), F_{399}(s))\} \pmod{N} \equiv \{(1, 38801172344360863), (329280233123969461, 62155946453030219), \ldots, (139048125143085063, 364686195778250318)\} \pmod{N}.
\]

Now \( F_{795}(s) = F_{399} \cdot F_{397} - F_{397}^2(s) + 1 \equiv 0 \pmod{N} \), so \( N \) is prime.

4. The primality test for \( 3^p h \pm 1 \) using \( X^2 + 3Y^2 = 4 \)

The recursions of both the classical Lucas-Lehmer test and that of the primality test \[ \text{[1]} \] for \( 3^p h \pm 1 \) coincide with repeated duplication and repeated multiplication by 3 respectively. We have a geometric variation of the test of Berribezaita and Berry \[ \text{[1]} \].

Theorem [12] relies on the cubic reciprocity law in the unique factorization domain \( \mathbb{Z}[\omega] \) where \( \omega \) is a primitive cube root of unity. See \[ \text{[3]} \] for the following and for the other various identities of the cubic residue symbol \( \left( \frac{\cdot}{3} \right) \).
Theorem 4.1 (Eisenstein). If \( \alpha \) and \( \pi \) are primes of \( \mathbb{Z}[\omega] \) which are congruent to \( \pm 1 \) (mod 3), then \( \left( \frac{\alpha}{\pi} \right)_3 = \left( \frac{\pi}{\alpha} \right)_3 \).

Below we simply associate the primality test \( \mathbb{I} \) for \( 3^n h \pm 1 \) with the curve \( X^2 + 3Y^2 = 4 \). This avoids case (3) of Theorem 3.4 altogether, since one may not wish to accept 2 chances in 3 for obtaining a primality certificate.

Theorem 4.2. Let \( N = 3^n h + \epsilon \), where \( \epsilon = \pm 1 \), \( h \) is even, not divisible by 3, \( 0 < h < 3^n \) and \( n \geq 2 \). Let \( \alpha \in \mathbb{Z}[\omega] \) be a prime satisfying \( \alpha \equiv \pm 1 \) (mod 3) and \( \left( \frac{N}{\alpha} \right)_3 \neq 1 \). Let \( \beta = (\alpha/\alpha')^2 = \beta_0 + \beta_1 \omega, \mathbb{P}_\beta = (2\beta_0 - \beta_1, \beta_1), \) and \( C : X^2 + 3Y^2 = 4 \). Let \( s \equiv f_{3^{n-1}h}(x(\mathbb{P}_\beta)) \) (mod \( N \)). Then \( N \) is prime if and only if \( s \equiv -1 \) (mod \( N \)).

Proof. Since the norm of \( \beta \) is equal to 1, \( \mathbb{P}_\beta \in \mathbb{C}(\mathbb{Z}/N) \). We must show that \( \left( \frac{N}{\alpha} \right)_3 \neq 1 \) implies that \( \mathbb{P}_\beta \not\in \mathbb{C}(\mathbb{Z}/N) \). Assuming that \( \mathbb{P}_\beta \in \mathbb{C}(\mathbb{Z}/N) \), \( x^3 - 3x - (2\beta_0 - \beta_1) \) is reducible modulo \( N \) so that \( \beta \) is a cubic residue modulo \( N \), and hence modulo every primary prime divisor \( \pi \) of \( N \) meaning \( \left( \frac{\beta}{\pi} \right)_3 = 1 \). Now we show that this implies that \( \left( \frac{\pi}{\alpha} \right)_3 = 1 \) so that \( \left( \frac{N}{\alpha} \right)_3 = 1 \), a contradiction. If \( \epsilon = 1 \), then

\[
\left( \frac{\alpha}{\pi} \right)_3 = \left( \frac{\alpha'}{\pi} \right)_3 \left( \frac{\alpha'}{\pi} \right)^2 = \left( \frac{\pi}{\alpha} \right)_3 \left( \frac{\pi}{\alpha} \right)_3 = \left( \frac{\pi}{\alpha} \right)_3 = 1
\]

so that \( \left( \frac{N}{\alpha} \right)_3 = 1 \). If \( \epsilon = -1 \), then by the above calculation, \( \left( \frac{\pi}{\alpha} \right)_3 = 1 \) so that again \( \left( \frac{N}{\alpha} \right)_3 = 1 \). Thereferce it must follow that \( \mathbb{P}_\beta \not\in \mathbb{C}(\mathbb{F}_N) \). Noting that \( \left( \frac{N}{\alpha} \right)_3 \equiv N \) (mod 3), the result follows from case (4) of Theorem 3.3. Note that \( F_3(-1) \equiv 0 \) (mod \( N \)) and only if \( s \equiv -1 \) (mod \( N \)). \( \square \)

The first term of the recursion \( T_{k+1} = T_k^3 - 3T_k \) in the statement of Theorem 4.2 according to \( \mathbb{I} \) is the trace \( \text{Tr}(\beta^k) \) which is \( f_k(x(\mathbb{P}_\beta)) \) so that the computation of \( s \equiv f_{3^{n-1}h}(x(\mathbb{P}_\beta)) \equiv f_{3^{n-1}}(f_h(x(\mathbb{P}_\beta))) \) (mod \( N \)) requires repeated multiplication by 3 on \( X^3 + 3Y^2 = 4 \) over \( \mathbb{Z}/N \).

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