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STABILITY OF COMPLEX FOLIATIONS TRANSVERSE TO FIBRATIONS

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ABSTRACT. We prove that a holomorphic foliation of codimension k which is transverse to the fibers of a fibration and has a compact leaf with finite holonomy group is a Seifert fibration, i.e., has all leaves compact with finite holonomy. This is the case for C^1-small deformations of a foliation where the original foliation exhibits a compact leaf and the base B of the fibration satisfies $H^1(B,\mathbb{R}) = 0$ and $H^1(B,\mathrm{GL}(k,\mathbb{R})) = 0$.

1. INTRODUCTION

Let $\eta = (E, \pi, B, F)$ be a (locally trivial) fibration with total space E, fiber F, base B and projection $\pi\colon E \to B$. A foliation $\mathcal{F}$ on E is *transverse to* η if: (1) for each $p \in E$, the leaf L_p of $\mathcal{F}$ with $p \in L_p$ is transverse to the fiber $\pi^{-1}(q)$, $q = \pi(p)$; (2) $\dim(\mathcal{F}) + \dim(F) = \dim(E)$; (3) for each leaf L of $\mathcal{F}$, the restriction $\pi|_L : L \to B$ is a covering map. According to a theorem of Ehresmann ([2, Ch. V], [3]) if the fiber F is compact, then conditions (1) and (2) together already imply (3). Such foliations are conjugate to suspensions and are characterized by their *global holonomy* [2, Theorem 3, p. 103] and [3, Theorem 6.1, page 59].

In [6] we study the case where the foliation has codimension one, and in [7] we study the case where the ambient manifold is a hyperbolic complex manifold. In the present work we regard the general codimension $k \geq 1$ case but from the viewpoint of stability theory. More precisely, we study under which conditions it is possible to prove that the foliation is given by the fibers of a fibration. A foliation $\mathcal{F}$ on M is called a *Seifert fibration* if all leaves are compact with finite holonomy groups.

Our main results are:

Theorem 1.1. *Let $\mathcal{F}$ be a holomorphic foliation transverse to a fibration $\pi\colon E \xrightarrow{F} B$ with fiber F. If $\mathcal{F}$ has a compact leaf with finite holonomy group, then $\mathcal{F}$ is a Seifert fibration. This occurs if $\mathcal{F}$ is of codimension k, has a compact leaf and the base B satisfies*

$$H^1(B,\mathbb{R}) = 0,\ H^1(B,\mathrm{GL}(k,\mathbb{C})) = 0.$$

Such a foliation admits a meromorphic first integral $\varphi\colon E \to \mathbf{CP}^k$ if, and only if, the fiber F admits a non-constant function $f\colon F \to \mathbf{CP}^k$.

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We point out that the total space E is not assumed to be compact in Theorem 1.1. Nevertheless this hypothesis is necessary in the next consequence of it.

Corollary 1.2. *Let $\mathcal{F}$ be a holomorphic codimension k foliation transverse to a fibration $\pi\colon E \xrightarrow{F} B$ with fiber F and compact total space E. Assume that $\mathcal{F}$ has a compact leaf $L \in \mathcal{F}$ such that either the fundamental group $\pi_1(L)$ is finite or the conditions below are satisfied:*

(i) $H^1(L,\mathbb{R}) = 0$,
(ii) $\mathrm{Hom}(\pi_1(L), \mathrm{GL}(k,\mathbb{R})) = \mathrm{Id}$.

Then any codimension k holomorphic foliation in E close enough to $\mathcal{F}$, with respect to the C^1 topology, is a Seifert fibration.

The proof of the main part of Theorem 1.1 can be outlined as follows. Since $\mathcal{F}$ is conjugate to a suspension of a group of diffeomorphisms of the fiber F, we can rely on the global holonomy of the foliation. As a general fact that holds also for smooth foliations, if the global holonomy group is finite, then $\mathcal{F}$ is a Seifert fibration (cf. Lemma 2.2). Also from the smooth framework we have that the existence of a compact leaf gives a common fixed point for the elements of the global holonomy (cf. Lemma 2.3). From this point and further, we begin to use techniques from complex variables. More precisely, we derive some results from classical theorems of Burnside and Schur on finite exponent groups and linear groups (see Lemma 3.2). These results (cf. Lemmas 3.5 and 3.6), combined with the local stability theorem of Reeb stating the finiteness of the holonomy group of the (compact) leaf passing through the above-mentioned fixed point, imply that also the global holonomy group is a finite group of diffeomorphisms of the fiber.

2. Holonomy and global holonomy

Let $\mathcal{F}$ be a codimension k holomorphic foliation transverse to a fibration $\pi\colon E \xrightarrow{F} B$ with fiber F, base B and total space E. We always assume that B, F and E are connected manifolds. Given a point $p \in E$, put $b = \pi(p) \in B$ and denote by F_b the fiber $\pi^{-1}(b) \subset E$, which is biholomorphic to F. Given a point $p \in E$ the leaf through p is denoted by L_p and its *holonomy group* is denoted by $\mathrm{Hol}(L_p)$, the holonomy group of L_p, and by $\mathrm{Hol}(L_p, F_b, p)$, the local representation of this holonomy calculated with respect to the local transverse section induced by F_b at the point $p \in F_b$. The group $\mathrm{Hol}(L_p, F_b, p)$ is therefore a subgroup of the group of germs $\mathrm{Bih}(F_b, p)$, which is identified with the group of germs $\mathrm{Bih}(\mathbb{C}^k, 0)$.

Let $\varphi\colon \pi_1(B, b) \to \mathrm{Bih}(F)$ be the global holonomy representation of the fundamental group of B in the group of biholomorphisms of the manifold F, obtained by lifting closed paths in B to the leaves of $\mathcal{F}$ via the covering maps $\pi\big|_L$, where L is a leaf of $\mathcal{F}$. The image of this representation is the *global holonomy* of $\mathcal{F}$, and its construction shows that $\mathcal{F}$ is conjugated to the suspension of its global holonomy ([2, Theorem 3, p. 103]).

Let L be any leaf of $\mathcal{F}$. Fix a point $p \in L$, let $b = \pi(p) \in B$ so that $F_b = \pi^{-1}(b) \subset E$, and take a transverse disc $D \subset F_b$ with $p \in D \cap L$. Denote by $G(\mathcal{F}, F_b) \subset \mathrm{Bih}(F_b)$ the representation of the global holonomy of $\mathcal{F}$ based at b. It is well known that:

Proposition 2.1 ([2, Chapter V]). *Let $\mathcal{F}$ be a foliation on E transverse to the fibration $\pi\colon E \to B$ with fiber F. Fix a point $p \in E$, $b = \pi(p)$ and denote by L the leaf that contains p.*

(1) *The holonomy group* $\operatorname{Hol}(L, F_b, p)$ *is the subgroup of* $G(\mathcal{F}, F_b) \subset \operatorname{Bih}(F_b)$ *of those elements that have p as a fixed point.*
(2) *Given another intersection point $q \in L \cap F_b$ there is a global holonomy map $h \in G(\mathcal{F}, F_b)$ such that $h(p) = q$.*

Proof. The first part is well known (see [2, Chapter V] or [3, Corollary 1.9, p. 145]). For the second part, given $q \in L \cap F_b$ choose a smooth path $\alpha\colon [0,1] \to L$ such that $\alpha(0) = p$ and $\alpha(1) = q$. Then project α via $\pi\big|_L\colon L \to B$ onto a smooth path $\gamma = \pi \circ \alpha[0,1] \to B$. Then γ induces an element $[\gamma]$ of the fundamental group $\pi_1(B, b)$ with base at $b = \pi(p) \in B$. By construction, the lift of γ via the covering map $\pi\big|_L\colon L \to B$ with initial point at $p \in L$ is exactly the path α. Therefore, by definition of the global holonomy, we have that the global holonomy map $h \in G(\mathcal{F}, F_b)$ corresponding to the element $[\gamma] \in \pi_1(B)$ satisfies $h(p) = \alpha(1) = q$. This proves (2). □

For the next two lemmas we do not need to assume that the total space E is compact.

Lemma 2.2. *Let $\mathcal{F}$ be a holomorphic foliation transverse to a fibration $\pi\colon E \xrightarrow{F} B$. Suppose that the global holonomy $G(\mathcal{F})$ is finite. If $\mathcal{F}$ has a compact leaf, then it is a Seifert fibration; i.e., all leaves are compact with finite holonomy group.*

Proof. By hypothesis the global holonomy $G(\mathcal{F})$ is finite. Given a compact leaf $L_0 \in \mathcal{F}$ we have $B = \pi(L_0)$ and therefore B is compact. Since the holonomy of a leaf embeds into the global holonomy (recall that the holonomy group of a leaf $L \in \mathcal{F}$ is the stabilizer subgroup of L in $G(\mathcal{F})$ and so the elements in the holonomy of a leaf have a common fixed point), we conclude that each leaf has a finite holonomy group. Given any leaf $L \in \mathcal{F}$ we have a covering map $\pi\big|_L\colon L \to B$. Because the global holonomy of $\mathcal{F}$ is finite it follows from (2) of Proposition 2.1 that, for any leaf $L \in \mathcal{F}$, the intersection $\sharp(L \cap F_b) < \infty$ for any base point $b \in B$ and therefore $\pi\big|_L\colon L \to B$ is a finite map. This implies that L is also compact. □

In the proof of Theorem 1.1 we shall use:

Lemma 2.3. *Let $\mathcal{F}$ be a holomorphic foliation transverse to a fibration $\pi\colon E \xrightarrow{F} B$ with global holonomy $G(\mathcal{F}) \subset \operatorname{Bih}(F)$. If $\mathcal{F}$ has a compact leaf $L_0 \in \mathcal{F}$, then each point $p \in F_b \cap L_0$ has periodic orbit in the global holonomy $G(\mathcal{F})$. In particular there are $\ell \in \mathbb{N}$ and $p \in F$ such that $h^\ell(p) = p$ for every $h \in G(\mathcal{F})$.*

Proof. Since the leaf L_0 is compact and points of intersection $F_b \cap L_0$ are isolated in L_0, the set $L_0 \cap F_b$ is a finite set of points, say $\sharp(L_0 \cap F_b) = k$.

Given any element $h \in G(\mathcal{F}, \pi^{-1}(b))$ we have $h(L_0 \cap \pi^{-1}(b)) \subset L_0 \cap \pi^{-1}(b)$. Therefore the orbit $\mathcal{O}_x(G(\mathcal{F}), \pi^{-1}(b))$ of a point $x \in L_0 \cap \pi^{-1}(b)$ in $G(\mathcal{F})$ is finite with at most k elements. Now let $p \in \pi^{-1}(b)$.

Claim 2.4. Given $h \in G(\mathcal{F}, F_b)$, the point $p \in F_b$ is a fixed point for $h^{k!}$.

Proof. Indeed, since the orbit $\mathcal{O}_p(G(\mathcal{F}))$ is finite of order at most k, we must have $h^m(p) = p$ for some $m \in \{1, ..., k\}$. Since $k! = m.\ell$ for some $\ell \in \mathbb{N}$ we have $h^{k!}(p) = (h^m)^\ell(p) = (h^m \circ ... \circ h^m)(p)$ with ℓ factors h^m. Because $h^m(p) = p$ we have $h^{k!}(p) = p$. □

3. Groups of finite exponent

First we recall some facts from the theory of linear groups. Let G be a group with identity $e_G \in G$. We say that G is a *periodic group* if for each element $g \in G$ there is $n_g \in \mathbb{N}$ such that $g^{n_g} = e_G$; i.e., each element of G has finite order. A group G is said to be *periodic* if each element $g \in G$ is periodic. A periodic group G is *periodic of bounded exponent* if there is a uniform upper bound for the periods of its elements. This is equivalent to the existence of $m \in \mathbb{N}$ with $g^m = 1$ for all $g \in G$: indeed, if G is periodic and there is $C \in \mathbb{N}$ such that $n_g \leq C$ for all $g \in G$, then given $g \in G$ the number n_g divides $C!$ and therefore $g^{C!} = e_G$. Because of this, a group which is periodic of bounded exponent is also called a group of *finite exponent*. The minimal such m is called the *exponent* of G.

If R is a ring with identity, we say that a group G is *R-linear* if it is isomorphic to a subgroup of the matrix group $\mathrm{GL}(n, R)$ (of $n \times n$ invertible matrices with coefficients belonging to R) for some $n \in \mathbb{N}$. If the ring R is understood, we say that G is a *linear group*. We will consider $R = \mathbb{C}$, that is, complex linear groups. The following classical result is due to Burnside.

Theorem 3.1. *Regarding periodic groups and groups of finite exponent we have:*

(1) (*Burnside, 1905* [1]) *A (not necessarily finitely generated) linear group which is finite dimensional and has finite exponent is finite; i.e., any subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *with bounded exponent is finite.*

(2) (*Schur, 1911* [8]) *Every finitely generated periodic subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *is finite.*

(3) (*Burnside, 1905* [1]) *A complex linear group* $G \subset \mathrm{GL}(k, \mathbb{C})$ *of finite exponent* ℓ *has finite order; actually we have* $|G| \leq \ell^{k^2}$.

Using these results we may prove:

Lemma 3.2. *A* (*not necessarily finitely generated*) *subgroup* $G \subset \mathrm{Bih}(\mathbb{C}^k, 0)$ *of finite exponent is necessarily finite. A finitely generated periodic subgroup* $G \subset \mathrm{Bih}(\mathbb{C}^k, 0)$ *is necessarily finite.*

Proof. Let G be a not necessarily finitely generated subgroup of $\mathrm{Bih}(\mathbb{C}^k, 0)$, with finite exponent. We consider the homomorphism $D\colon \mathrm{Bih}(\mathbb{C}^k, 0) \to \mathrm{GL}(k, \mathbb{C})$ given by the derivative $Dg := g'(0)$, $g \in G$. Then the image DG is isomorphic to the quotient $G/\operatorname{Ker}(D)$, where the kernel $\operatorname{Ker}(D)$ is the group $G_1 = \{g \in G,\ g'(0) = \mathrm{Id}\}$, i.e., the normal subgroup of elements tangent to the identity. Since G is of finite exponent the same holds for DG as a consequence of the Chain Rule. By Burnside's theorem above, DG is a finite group. Let us now prove that G_1 is trivial. Indeed, take an element $h \in G_1$. Since h has finite order it is analytically linearizable and therefore $h = \mathrm{Id}$. Now we assume that $G \subset \mathrm{Bih}(\mathbb{C}^k, 0)$ is finitely generated and periodic. Again we consider the homomorphism $D\colon \mathrm{Bih}(\mathbb{C}^k, 0) \to \mathrm{GL}(k, \mathbb{C})$ and the image $DG \simeq G/G_1$ as above. Since G is finitely generated and periodic the same holds for DG as a consequence of the Chain Rule. By Schur's theorem above, DG is a finite group. As above G_1 is trivial and therefore G is finite. □

We also need

Lemma 3.3. *About subgroups of* $\mathrm{Bih}(\mathbb{C}^k, 0)$ *we have:*

(1) *Let* $G \subset \mathrm{Bih}(\mathbb{C}^k, 0)$ *be a* (*not necessarily finitely generated*) *subgroup such that for each point* x *close enough to the origin, the pseudo-orbit of* x *is finite of order* $\leq \ell$ *for a uniform* $\ell \in \mathbb{N}$. *Then* G *is finite.*
(2) *Let* $G \subset \mathrm{Bih}(\mathbb{C}^k, 0)$ *be a finitely generated subgroup. Assume that there is an invariant connected neighborhood* W *of the origin in* $\mathbb{C}^k$ *such that each point* x *is periodic for each element* $g \in G$. *Then* G *is a finite group.*

Proof. Let us prove (1). Given any $g \in G$, since the orbit $\mathcal{O}_x(G)$ is finite of order $\leq \ell$, we must have $g^m(x) = x$ for some $m \in \{1, ..., \ell\}$. Since $\ell! = m.t$ for some $t \in \mathbb{N}$ we have $g^{\ell!}(x) = (g^m)^t(x) = (g^m \circ ... \circ g^m)(x)$ with t factors g^m. Because $g^m(x) = x$ we have $g^{\ell!}(x) = x$. This shows that $g^{\ell!} =$ Id, so G has finite exponent $\leq \ell!$ and by Lemma 3.2 above G is finite (even if G is not assumed to be finitely generated).

Now we prove (2). Fix an element $g \in G$. For each $k \in \mathbb{N}$ define $X_k := \{x \in W, g^k(x) = x\}$. We claim that X_k is a closed subset of W: indeed, if $x_\nu \in X_k$ is a sequence of points converging to a point $a \in W$, then clearly $g^k(a) = a$ and therefore $a \in X_k$. By the category theorem of Baire there is $k \in \mathbb{N}$ such that X_k has non-empty interior, and therefore by the identity theorem we have $g^k = \mathrm{Id}$ in W. This shows that each element $g \in G$ is periodic. Since G is finitely generated, this implies by Lemma 3.2 that G is finite. □

Remark 3.4. The same proof as for (2) of Lemma 3.3 also shows that a finitely generated subgroup of biholomorphisms of a connected complex manifold F, having a common fixed point and such that each point $x \in F$ is periodic for each element $g \in G$, is also a finite group.

Lemma 3.5. *Let* $G \subset \mathrm{Bih}(F)$ *be a* (*not necessarily finitely generated*) *subgroup of biholomorphisms of a connected complex manifold* F. *Assume that there is a point* $p \in F$ *which is fixed by* G *and a fundamental system of neighborhoods* $\{U_\nu\}_\nu$ *of* p *in* F *such that each* U_ν *is invariant by* G *and the orbits of* G *in* U_ν *are periodic* (*not necessarily with uniformly bounded orders*). *Then* G *is a finite group.*

Proof. Since U_ν is G-invariant, each element $g \in G$ induces by restriction to U_ν an element of a group $G_\nu \subset \mathrm{Bih}(U_\nu)$. By the same arguments in the proof of Lemma 3.3 above the finiteness of the orbits in U_ν implies that G_ν is periodic. Nevertheless, this is not yet enough to assure that G_ν is a finite group because it is not assumed to be finitely generated. On the other hand, since the open subsets U_ν form a fundamental system of neighborhoods of p in F, we can choose U_ν to be contained in an embedded disk centered at p and U_ν is biholomorphic to a bounded domain in $\mathbb{C}^k$ where $k = \dim F$. A classical theorem of Cartan [5] states that if $D \subset \mathbb{C}^k$ is a bounded domain containing the origin and $\phi\colon D \to D$ a holomorphic mapping fixing the origin and such that $D\phi(0) = Id$, then ϕ is the identity mapping. This shows that in our present case the subgroup $G_{\nu,1}$ of G_ν of elements tangent to the identity is finite: indeed by Cartan's theorem, the group $G_{\nu,1}$ embeds into the linear group, and since its elements are flat this implies that it is trivial. Hence, the homomorphism $D\colon \mathrm{Bih}(\mathbb{C}^k, 0) \to \mathrm{GL}(k, \mathbb{C})$ given by the derivative $D\colon g \mapsto g'(0)$ shows that the group G_ν is isomorphic to the linear group DG_ν, and therefore G_ν embeds in the linear group $\mathrm{GL}(k, \mathbb{C})$. Since G_ν is periodic (as we showed above), this implies by Burnside's theorem above that G_ν is a finite group. By the identity principle, the group G is also finite of the same order. $\square$

Lemma 3.6. *Let $G \subset \mathrm{Bih}(F)$ be a finitely generated subgroup of biholomorphisms of a connected manifold F, admitting a periodic orbit $\{x_1, ..., x_r\} \subset F$ such that for each $j \in \{1, ..., r\}$ there is a fundamental system of neighborhoods U_ν^j of x_j with the property that $U_\nu = \bigcup_{j=1}^r U_\nu^j$ is invariant under the action of G, $U_\nu^j \cap U_\nu^\ell = \emptyset$ if $j \neq \ell$ and each orbit in U_ν is periodic. Then G is finite.*

Proof. Given any element $f \in G$ we denote by f_ν its restriction to U_ν. We put G_ν for the group generated by the maps f_ν, i.e., $G_\nu = G\big|_{U_\nu}$. There is $\ell \in \mathbb{N}$ such that given any map $f \in G$, then $f^\ell(x_j) = x_j, \forall j = 1, ..., r$. The map f^ℓ therefore has fixed points at the x_j and admits invariant neighborhoods of these fixed points where it has periodic orbits. By Lemma 3.5 above this map has finite order. This shows that any element $f \in G$ is periodic. Since G is finitely generated and has a periodic orbit, this implies that G is finite (Burnside's theorem). $\square$

4. Stability

Proof of Theorem 1.1. Since the leaf L_0 is compact and there is a covering map $\pi\big|_{L_0} \colon L_0 \to B$, the base B is also compact. This implies that the fundamental group $\pi(B)$ is finitely generated and therefore the global holonomy $G(\mathcal{F})$ is finitely generated. Given a fiber $F_b = \pi^{-1}(b)$ for some $b \in B$, the intersection $L_0 \cap F_b$ is a finite set of points, say $\{x_1, ..., x_r\} \subset F_b$. Since the holonomy of L_0 is finite and L_0 is compact by the Local Stability of Reeb, there is a fundamental system of invariant neighborhoods W_ν of L_0 in E such that on each W_ν the leaves are compact. For the intersections $U_\nu := W_\nu \cap F_b$, write $U_\nu = \bigcup_{j=1}^r U_{\nu_j}$ and give fundamental systems of neighborhoods U_{nu_j} of the points $x_j \in F_b$ as in Lemma 3.6, and by this lemma the global holonomy $G(\mathcal{F}, F_b)$ is finite. By Lemma 2.2, $\mathcal{F}$ is a Seifert fibration proving the first part. As for the second part, the restriction $\pi_L \colon L \to B$ is a finite covering map. Therefore, the compact leaf L also satisfies $H^1(L, \mathbb{R}) = 0$ and $H^1(L, \mathrm{GL}(n, \mathbb{R})) = 0$. By the stability theorem of Thurston ([9, Theorem 2, p. 348]), the holonomy group of L is trivial, and we apply the first part to conclude that $\mathcal{F}$ is a Seifert fibration. Finally, the last part of Theorem 1.1 follows from Proposition 4.1 below. $\square$

Proposition 4.1. *Let $\mathcal{F}$ be a codimension k holomorphic foliation transverse to a fibration $\pi \colon E \xrightarrow{F} B$. Suppose that $\mathcal{F}$ has finite global holonomy. The following conditions are equivalent:*

(1) *$\mathcal{F}$ admits a meromorphic first integral $E \to \mathbf{CP}^k$,*
(2) *F admits a non-constant meromorphic function $F \to \mathbf{CP}^k$.*

Proof. From the general theory of fiber bundles and foliations [3] there exists a finite covering

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\alpha}} & E \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \tilde{B} & \xrightarrow{\alpha} & B \end{array}$$

where $\alpha \colon \tilde{B} \to B$ is a finite covering, $\tilde{\alpha} \colon \tilde{E} \to \tilde{B}$ is a holomorphic fibration, $\alpha \colon \tilde{B} \to B$ corresponds to the kernel $K \subset \pi_1(B)$ of φ, which is a normal subgroup of $\pi_1(B)$, and also $\mathcal{F}$ lifts to a foliation $\tilde{\mathcal{F}} = \tilde{\alpha}^*(\mathcal{F})$ on $\tilde{E}$ which has trivial global holonomy with respect to the fibration $\tilde{\pi}\colon \tilde{E} \to \tilde{B}$. Since the global holonomy of $\tilde{\mathcal{F}}$ is trivial it induces a holomorphic trivialization of $\tilde{\pi}\colon \tilde{E} \to \tilde{B}$ and we may construct in a natural way a holomorphic function $\tilde{\xi}\colon \tilde{E} \to F_b$ which is constant along the $\tilde{\mathcal{F}}$-leaves.

Now assume that F_b admits some non-constant holomorphic function $F_b \to \mathbf{CP}^k$. Then we may obtain (non-constant) holomorphic functions $\tilde{\xi}\colon \tilde{E} \to \mathbf{CP}^k$ which are constant along the $\tilde{\mathcal{F}}$-leaves. Working now with the finite covering $\tilde{\alpha}\colon \tilde{E} \to E$ and taking symmetric functions, we may obtain a non-constant meromorphic function $\xi\colon E \to \overline{\mathbb{C}}$ which makes the following diagram commutative:

$$\begin{tikzcd} \widetilde{E} \arrow[r, "\widetilde{\xi}"] \arrow[d, "\widetilde{\alpha}"'] & \mathbf{CP}^k \\ E \arrow[ur, "\xi"'] & \end{tikzcd}$$

Clearly ξ is a meromorphic first integral for the foliation $\mathcal{F}\big|_E$. Conversely, any (non-constant) meromorphic first integral ξ for $\mathcal{F}$ induces, by $\mathcal{F}$-local trivialization of the fibration around F_b, a non-constant meromorphic function $F_b \to \mathbf{CP}^k$. This proves the proposition. $\square$

Proof of Corollary 1.2. According to Theorem 1.1 it is enough to prove that a foliation $\mathcal{G}$ on E which is close enough to $\mathcal{F}$ in the C^1 topology is still transverse to the fibration and exhibits a compact leaf with finite holonomy group. Let us first prove the persistence of the compact leaf with finite holonomy group. If $L \in \mathcal{F}$ has finite fundamental group, then this follows from the classical Reeb stability theorem for perturbations (cf. [3, Theorem 4.7, p. 125]), and we only need to assume that the perturbation is C^0 close to $\mathcal{F}$. If $\mathcal{F}$ has a compact leaf satisfying conditions (i) and (ii), then any foliation close enough to $\mathcal{F}$ in the C^1 topology has a compact leaf with finite holonomy as a consequence of a result due to Langevin and Rosenberg ([4] and [3, Theorem 4.8, p. 126]). Let us now show that there is a neighborhood $\mathcal{N}$ of $\mathcal{F}$ in the C^1 topology such that any foliation $\mathcal{G} \in \mathcal{N}$ is still transverse to the fibration. It is clear since E and the fibers are compact that any foliation $\mathcal{G}$ close enough to $\mathcal{F}$ in the C^1 topology still has leaves transverse to the fibers of $\pi\colon E \to B$. Since $\operatorname{codim} \mathcal{G} = \operatorname{codim} \mathcal{F} = \dim F$ and the fiber F is compact, this implies, as observed in the introduction, by a theorem of Ehresmann that the restriction of the projection to the leaves of $\mathcal{G}$ also gives covering maps and therefore $\mathcal{G}$ is transverse to the fibration. This finishes the proof of the corollary. $\square$

References

1. Burnside, W.: *On criteria for the finiteness of the order of a group of linear substitutions*, Proc. London Math. Soc. (2) 3 (1905), 435-440.
2. Camacho, César; Lins Neto, Alcides. *Geometry theory of foliations.* Translated from the Portuguese by Sue E. Goodman. Birkhäuser Boston, Inc., Boston, MA, 1985. MR824240 (87a:57029)
3. Godbillon, C.: *Feuilletages* (French) [Foliations], Études géométriques [Geometric studies]. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991. MR1120547 (93i:57038)
4. Langevin, R., Rosenberg, H.: *On stability of compact leaves and fibrations.* Topology 16 (1977), no. 1, 107–111. MR0461523 (57:1508)

5. Nishino, T.: *Function theory in several complex variables* (English summary). Translated from the 1996 Japanese original by Norman Levenberg and Hiroshi Yamaguchi. Translations of Mathematical Monographis, 193. American Mathematical Society, Providence, RI, 2001. MR1818167 (2001k:32001)
6. Scárdua, B.: *On complex codimension-one foliations transverse fibrations.* J. Dyn. Control Syst. 11 (2005), no. 4, 575-603. MR2170665 (2006g:32049)
7. Scárdua, B.: *Holomorphic foliations transverse to fibrations on hyperbolic manifolds.* Complex Variables Theory Appl. 46 (2001), no. 3, 219–240. MR1869737 (2002m:32048)
8. Schur, I.: *Über Gruppen periodischer substitutionen*, Sitzungsber. Preuss. Akad. Wiss. (1911), 619–627.
9. Thurston, William P.: *A generalization of the Reeb stability theorem.* Topology 13 (1974), 347–352. MR0356087 (50:8558)

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