

LIPSCHITZ (q, p) -MIXING OPERATORS

JAVIER ALEJANDRO CHÁVEZ-DOMÍNGUEZ

(Communicated by Thomas Schlumprecht)

ABSTRACT. Several useful results in the theory of p -summing operators, such as Pietsch’s composition theorem and Grothendieck’s theorem, share a common form: for certain values q and p , there is an operator such that whenever it is followed by a q -summing operator, the composition is p -summing. This is precisely the concept of (q, p) -mixing operators, defined and studied by A. Pietsch. On the other hand, J. Farmer and W. B. Johnson recently introduced the notion of a Lipschitz p -summing operator, a nonlinear generalization of p -summing operators. In this paper, a corresponding nonlinear concept of Lipschitz (q, p) -mixing operators is introduced, and several characterizations of it are proved. An interpolation-style theorem relating different Lipschitz (q, p) -mixing constants is obtained, and it is used to show reversed inequalities between Lipschitz p -summing norms.

1. INTRODUCTION

The theory of p -summing operators plays a very important role in modern Banach space theory, not only for its intrinsic beauty but also for its far-reaching applications among a wide spectrum of subjects such as Banach space geometry, harmonic analysis, approximation theory, operator theory and others. When working with p -summing operators, it is not unusual to come across an operator T with the property that $S \circ T$ is p -summing whenever S is q -summing. One example of such a situation appears in A. Pietsch’s composition theorem, a very useful tool already present in his seminal paper [Pie67]: whenever $p, q, r \in [1, \infty]$ satisfy $1/p = 1/q + 1/r$, the composition of a q -summing operator followed by an r -summing operator is p -summing. Another example with T being the identity on an L_1 space is provided by a celebrated theorem of A. Grothendieck, stating that every continuous linear operator from L_1 into Hilbert space is 1-summing; therefore, any 2-summing operator with an L_1 space as domain is 1-summing. More generally, by a theorem of B. Maurey, any 2-summing operator defined on a cotype 2 space is 1-summing. Similarly, any continuous linear operator from a $C(K)$ space into a cotype 2 space is 2-summing.

Inspired by ideas of Maurey [Mau74], Pietsch [Pie80, Chap. 20] systematically studied the situation described in the previous paragraph and called such operators (q, p) -mixing. Another exposition of the subject, with a more “tensorial” point of view, can be found in [DF93, Sec. 32]. On the other hand, J. Farmer and W. B. Johnson [FJ09] recently introduced the concept of a Lipschitz p -summing operator between metric spaces. They proved that this is a true extension of the linear

Received by the editors November 17, 2010 and, in revised form, March 21, 2011.
2010 *Mathematics Subject Classification*. Primary 46B28, 46T99, 47H99, 47J99, 47L20.
Partially supported by NSF grants DMS-0503688 and DMS-0852434.

concept and obtained a nonlinear counterpart of one of the cornerstones of the theory of (linear) p -summing operators: Pietsch's celebrated domination/factorization theorem.

In the present paper, the corresponding concept of Lipschitz (q, p) -mixing operators is defined and studied. We start by recalling the necessary theory of Lipschitz p -summing operators and then introduce the main definition. Afterwards three different characterizations of Lipschitz (q, p) -mixing operators are presented. The first one is an integral inequality along the lines of Pietsch's domination theorem, while the second one corresponds to his (q, p) -mixed sequences. The third one relies on the recently developed [CD] duality theory for Lipschitz p -summing operators. Finally these characterizations are used to prove relationships between (q, p) -mixing constants and s -summing norms in various situations, in particular obtaining reversed inequalities for Lipschitz p -summing norms.

2. NOTATION AND PRELIMINARIES

The letters X, Y, Z will denote metric spaces, whereas E, F, G will denote Banach spaces. All metric spaces under consideration will be *pointed*; i.e. each one has a special point designated by 0. For a mapping T between metric spaces, $\text{Lip}(T)$ denotes its Lipschitz constant. Given a metric space X , the Banach space of real-valued Lipschitz functions defined on X that send 0 to 0 with the Lipschitz norm $\text{Lip}(\cdot)$ will be denoted by $X^\#$. As is customary, B_E denotes the closed unit ball of a Banach space E . The letters p, q, r, s will designate elements of $[1, \infty]$, and p' denotes the exponent conjugate to p (i.e. the one that satisfies $1/p + 1/p' = 1$).

The remainder of this section is all from [FJ09]. Recall that for $1 \leq p < \infty$ a linear operator $T : E \rightarrow F$ is called p -summing if there is a nonnegative constant C such that for any vectors v_j in E , the inequality

$$\sum_j \|Tv_j\|^p \leq C^p \sup_{v^* \in B_{E^*}} \sum_j |v^*(v_j)|^p$$

holds. In this case, the p -summing norm $\pi_p(T)$ of T is the infimum of such constants C . Inspired by this useful concept, Farmer and Johnson defined the *Lipschitz p -summing norm* π_p^L of a (not necessarily linear) mapping $T : X \rightarrow Y$ as the smallest nonnegative constant C such that for any x_j, x'_j in X and any positive reals a_j ,

$$\sum_j a_j d(Tx_j, Tx'_j)^p \leq C^p \sup_{f \in B_{X^\#}} \sum_j a_j |f(x_j) - f(x'_j)|^p.$$

This definition remains unchanged if we consider only the case $a_j = 1$, a very useful observation in [FJ09] also credited to M. Mendel and G. Schechtman. The set of all Lipschitz p -summing maps from X to Y is denoted by $\Pi_p^L(X, Y)$. Note that the condition that would naturally correspond to being Lipschitz ∞ -summing is just the Lipschitz condition, and we adopt this convention for notational convenience.

It is clear from the definition that the Lipschitz p -summing norm of a mapping is equal to the supremum of the Lipschitz p -summing norms of all the restrictions of said mapping to finite subsets of its domain. Also directly from the definition, it is clear that the Lipschitz p -summing norm has the ideal property: $\pi_p^L(A \circ T \circ B) \leq \text{Lip}(A) \cdot \pi_p^L(T) \cdot \text{Lip}(B)$ whenever the composition makes sense. We next state the domination/factorization theorem for Lipschitz p -summing operators

[FJ09, Thm. 1], a particular case of the general Pietsch-type domination theorems considered in [BPR10].

Theorem 2.1. *For a mapping $T : X \rightarrow Y$ and a constant $C \geq 0$, the following are equivalent:*

- (a) $\pi_p^L(T) \leq C$.
- (b) *There is a probability μ on $B_{X^\#}$ such that for any $x, x' \in X$,*

$$d(Tx, Tx') \leq C \left[\int_{B_{X^\#}} |f(x) - f(x')|^p d\mu(f) \right]^{1/p}.$$

- (c) *For some (or any) isometric embedding J of Y into a 1-injective space Z , there is a factorization*

$$\begin{array}{ccccc} L_\infty(\mu) & \xrightarrow{I_{\infty,p}} & L_p(\mu) & & \\ \uparrow A & & \downarrow B & & \\ X & \xrightarrow{T} & Y & \xrightarrow{J} & Z \end{array}$$

with μ a probability and $\text{Lip}(A) \cdot \text{Lip}(B) \leq C$.

The domination theorem immediately implies the monotonicity of the Lipschitz p -summing norms, that is, $\pi_p^L(T) \geq \pi_q^L(T)$ whenever $p \leq q$.

It is important to stress that the concept of a Lipschitz p -summing operator is a true generalization of that of a (linear) p -summing operator: for a bounded linear operator T between Banach spaces, T is Lipschitz p -summing if and only if it is (linearly) p -summing, and moreover $\pi_p(T) = \pi_p^L(T)$ [FJ09, Thm. 2].

3. DEFINITION AND ELEMENTARY PROPERTIES

Let $1 \leq p, q \leq \infty$. An operator $T : X \rightarrow Y$ is said to be *Lipschitz (q, p) -mixing with constant K* if for any metric space Z and any Lipschitz q -summing operator $S : Y \rightarrow Z$, the composition $S \circ T$ is a Lipschitz p -summing operator and $\pi_p^L(S \circ T) \leq K \pi_q^L(S)$. The smallest such K will be denoted by $\mathbf{m}_{q,p}^L(T)$.

A first example of such an operator already appears in [FJ09], where a nonlinear Grothendieck inequality is proved. Namely, any Lipschitz map T from a metric tree X into a Hilbert space is Lipschitz 1-summing and in fact $\pi_1^L(T) \leq K_G \text{Lip}(T)$, where K_G is Grothendieck’s constant. This result, together with the factorization theorem, Theorem 2.1, implies that the identity on X is Lipschitz $(2, 1)$ -mixing with constant at most K_G . D. Chen and B. Zheng [CZ] gave another proof of this nonlinear Grothendieck inequality, showing that $\mathbf{m}_{2,1}^L(id_X) \leq A_1^{-1}$ where A_1 is the constant in Khintchine’s inequality.

Note that in order to determine if a mapping $T : X \rightarrow Y$ is Lipschitz (q, p) -mixing, it suffices to consider its compositions with mappings from Y to ℓ_q (or any other infinite-dimensional L_q space, in fact). First, we may assume without loss of generality that X and Y are finite metric spaces. Now suppose that

$$(\star) \quad \pi_p^L(R \circ T) \leq C \pi_q^L(R) \quad \text{for any } R : Y \rightarrow \ell_q,$$

and let $S : Y \rightarrow Z$ be a Lipschitz q -summing map. Let $J : Z \rightarrow W$ be an isometric embedding of Z into a 1-injective space W . By the factorization theorem

for Lipschitz q -summing operators, we can find a factorization

$$\begin{array}{ccccc} L_\infty(\mu) & \xrightarrow{I_{\infty,q}} & L_q(\mu) & & \\ \uparrow A & & \downarrow B & & \\ Y & \xrightarrow{S} & Z & \xrightarrow{J} & W \end{array}$$

with $\text{Lip}(A) \cdot \text{Lip}(B) = \pi_q^L(S)$. Since Y is a finite set, the range of $I_{\infty,q} \circ A$ is a finite subset of $L_q(\mu)$ and therefore is almost isometric to a subset of ℓ_q . Thus, for the purposes of computing Lipschitz summing norms, we may assume that $I_{\infty,q} \circ A$ is a map from Y into ℓ_q , so condition (\star) applies and therefore $\pi_p^L(I_{\infty,q} \circ A \circ T) \leq C\pi_q^L(I_{\infty,q} \circ A)$. The ideal property for Lipschitz q -summing operators implies that $\pi_q^L(I_{\infty,q} \circ A) \leq \text{Lip}(A) \cdot \pi_q^L(I_{\infty,q}) \leq \text{Lip}(A) \cdot 1$, whereas the ideal property for Lipschitz p -summing operators gives us

$$\begin{aligned} \pi_p^L(J \circ S \circ T) &= \pi_p^L(B \circ I_{\infty,q} \circ A \circ T) \\ &\leq \text{Lip}(B) \cdot \pi_q^L(I_{\infty,q} \circ A \circ T) \leq \text{Lip}(B) \cdot C \cdot \text{Lip}(A) = C\pi_q^L(S). \end{aligned}$$

But since J is an isometric embedding $J \circ S \circ T$ and $S \circ T$ have the same Lipschitz p -summing norm, so we conclude that $\pi_p^L(S \circ T) \leq C\pi_q^L(S)$, i.e. that T is Lipschitz (q, p) -mixing with constant C .

The ideal property for Lipschitz p -summing operators implies that for any operator T , $\mathbf{m}_{q,p}^L(T) = \text{Lip}(T)$ whenever $q \leq p$ and $\mathbf{m}_{\infty,p}^L(T) = \pi_p^L(T)$, so only the case $1 \leq p < q < \infty$ gives something new. Moreover, Lipschitz (q, p) -mixing operators also satisfy the ideal property and $\mathbf{m}_{q,p}^L(A \circ T \circ B) \leq \text{Lip}(A) \cdot \mathbf{m}_{q,p}^L(T) \cdot \text{Lip}(B)$ whenever the composition makes sense.

Just from the definition, we obtain a trivial composition formula for Lipschitz (q, p) -mixing operators: regardless of the values of p, q and r in $[1, \infty]$, the composition of a Lipschitz (p, r) -mixing operator T followed by a Lipschitz (q, p) -mixing operator S is Lipschitz (q, r) -mixing and moreover $\mathbf{m}_{q,r}^L(ST) \leq \mathbf{m}_{q,p}^L(S) \cdot \mathbf{m}_{p,r}^L(T)$.

Additionally, the monotonicity of the Lipschitz p -summing norms implies a monotonicity condition for the Lipschitz (q, p) -mixing constants: whenever $p_1 \leq p_2$ and $q_2 \leq q_1$, $\mathbf{m}_{q_2,p_2}^L(T) \leq \mathbf{m}_{q_1,p_1}^L(T)$ for any T .

4. CHARACTERIZATIONS

In this section three different characterizations of Lipschitz (q, p) -mixing operators are presented, all of them somewhat inspired by analogous results in the linear theory.

4.1. Domination. The first characterization is close in spirit to the characterization of Lipschitz p -summing operators via a dominating measure [FJ09]. Compare with [DF93, Prop. 32.4].

Theorem 4.1. *Let $1 \leq p \leq q \leq \infty$, $T : X \rightarrow Y$ Lipschitz and $C \geq 0$. The following are equivalent:*

(a) *T is Lipschitz (q, p) -mixing with $\mathbf{m}_{q,p}^L(T) \leq C$.*

(b) For any probability measure μ on $B_{Y^\#}$ there exists a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\left[\int_{B_{Y^\#}} |g(Tx) - g(Tx')|^q d\mu(g) \right]^{1/q} \leq C \left[\int_{B_{X^\#}} |f(x) - f(x')|^p d\nu(f) \right]^{1/p}.$$

(c) For any $x_1, \dots, x_m, x'_1, \dots, x'_m \in X$ and $g_1, \dots, g_n \in Y^\#$,

$$\begin{aligned} & \left[\sum_{j=1}^m \left[\sum_{k=1}^n |g_k(Tx_j) - g_k(Tx'_j)|^q \right]^{p/q} \right]^{1/p} \\ & \leq C \left[\sum_{k=1}^n \text{Lip}(g_k)^q \right]^{1/q} \cdot \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m |f(x_j) - f(x'_j)|^p \right]^{1/p}. \end{aligned}$$

(d) For any $x_1, \dots, x_m, x'_1, \dots, x'_m \in X$ and any probability measure μ on $B_{Y^\#}$,

$$(4.1) \quad \left[\sum_{j=1}^m \left(\int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^q d\mu(g) \right)^{p/q} \right]^{1/p} \leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m |f(x_j) - f(x'_j)|^p \right]^{1/p}.$$

In this case, $\mathbf{m}_{q,p}^L(T)$ is equal to the infimum of such constants C in either (b), (c) or (d).

Proof. The case $q = \infty$ reduces to the Domination Theorem for Lipschitz p -summing operators (Thm. 2.1), so we will assume $1 \leq p \leq q < \infty$.

(a) \Rightarrow (b): Suppose that $T : X \rightarrow Y$ is Lipschitz (q, p) -mixing, and let μ be a probability measure on $B_{Y^\#}$. By restricting to Y the canonical inclusion $C(B_{Y^\#}) \hookrightarrow L_q(\mu)$, we get a Lipschitz q -summing operator $j_\mu : Y \rightarrow L_q(\mu)$ with Lipschitz q -summing norm at most 1. Hence, since T is Lipschitz (q, p) -mixing, the composition $j_\mu \circ T : X \rightarrow L_q(\mu)$ is Lipschitz p -summing. By the Pietsch domination theorem for Lipschitz p -summing operators (Thm. 2.1), there is a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\|j_\mu(Tx) - j_\mu(Tx')\|_{L_q(\mu)} \leq \pi_p^L(j_\mu \circ T) \left[\int_{B_{X^\#}} |f(x) - f(x')|^p d\nu(f) \right]^{1/p},$$

i.e.

$$\left[\int_{B_{Y^\#}} |g(Tx) - g(Tx')|^q d\mu(g) \right]^{1/q} \leq \pi_p^L(j_\mu \circ T) \left[\int_{B_{X^\#}} |f(x) - f(x')|^p d\nu(f) \right]^{1/p},$$

so we have condition (b) with $C = \pi_p^L(j_\mu \circ T) \leq \mathbf{m}_{q,p}^L(T) \pi_q^L(j_\mu) \leq \mathbf{m}_{q,p}^L(T)$.

(b) \Rightarrow (c): By homogeneity, we may assume without loss of generality that $\sum_{k=1}^n \text{Lip}(g_k)^q = 1$. Then $\mu := \sum_{k=1}^n \text{Lip}(g_k)^q \delta_{g_k / \text{Lip}(g_k)}$ (where δ_g is the Dirac

measure at $g \in Y^\#$) is a probability measure on $B_{Y^\#}$, so there exists a corresponding ν as in (b). Therefore,

$$\begin{aligned} \sum_{j=1}^m \left[\sum_{k=1}^n |g_k(Tx_j) - g_k(Tx'_j)|^q \right]^{p/q} &= \sum_{j=1}^m \left[\int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^q d\mu(g) \right]^{p/q} \\ &\leq C^p \sum_{j=1}^m \int_{B_{X^\#}} |f(x) - f(x')|^p d\nu(f) \leq C^p \sup_{f \in B_{X^\#}} \sum_{j=1}^m |f(x) - f(x')|^p, \end{aligned}$$

so we have (c) with the same constant C .

(c) \Rightarrow (d): Condition (c) means that all finitely supported probability measures μ on $B_{Y^\#}$ already satisfy (4.1). Since the set of all finitely supported probability measures on $B_{Y^\#}$ is $\sigma(C(B_{Y^\#})^*, C(B_{Y^\#}))$ -dense in the set of all probability measures on $B_{Y^\#}$, it follows that inequality (4.1) holds for all probability measures μ on $B_{Y^\#}$.

(d) \Rightarrow (a): Now let $S : Y \rightarrow Z$ be Lipschitz q -summing. Appealing to the domination theorem again, there is a measure μ on $B_{Y^\#}$ such that for all $y, y' \in Y$,

$$d_Z(Sy, Sy)^p \leq \pi_q^L(S)^p \left[\int_{B_{Y^\#}} |g(y) - g(y')|^q d\mu(g) \right]^{p/q}.$$

Fix $x_1, \dots, x_m, x'_1, \dots, x'_m \in X$. Then, from the previous inequality,

$$\begin{aligned} \left[\sum_{j=1}^m d_Z(S(Tx_j), S(Tx'_j))^p \right]^{1/p} \\ \leq \pi_q^L(S) \left[\sum_{j=1}^m \left[\int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^q d\mu(g) \right]^{p/q} \right]^{1/p}, \end{aligned}$$

which together with (4.1) implies that

$$\left[\sum_{j=1}^m d_Z(STx_j, STx'_j)^p \right]^{1/p} \leq C \pi_q^L(S) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m |f(x) - f(x')|^p \right]^{1/p},$$

so $S \circ T$ is Lipschitz p -summing and $\pi_p^L(S \circ T) \leq C \pi_q^L(S)$. Therefore, T is Lipschitz (q, p) -mixing and $\mathfrak{m}_{q,p}^L(T) \leq C$. \square

4.2. Lipschitz (q, p) -mixed sequences. Linear (q, p) -mixing operators were given such a name by Pietsch [Pie80] because a linear operator is linearly (q, p) -mixing if and only if it maps every weakly p -summable sequence into a (q, p) -mixed sequence, i.e. one that can be expressed as the pointwise product of a weakly q -summable sequence and an r -summable scalar sequence where $1/p = 1/q + 1/r$. The analogous result in the nonlinear case will follow from Theorem 4.1 as soon as we find an appropriate nonlinear counterpart of (q, p) -mixing sequences. We will use Ky Fan’s minimax lemma as stated in [Pie80, Lemma E.4.2]. A collection of real-valued functions \mathcal{A} defined on a set K is called *concave* if given $\Phi_1, \dots, \Phi_n \in \mathcal{A}$ and $\alpha_1, \dots, \alpha_n \geq 0$ such that $\sum_{j=1}^n \alpha_j = 1$, there is $\Phi \in \mathcal{A}$ satisfying $\Phi(x) \geq \sum_{j=1}^n \alpha_j \Phi_j(x)$ for all $x \in K$. Now we prove a result analogous to [Pie80, Thm. 16.4.3] (credited mostly to [Mau74]).

Proposition 4.2. *Let $1 \leq p < q < \infty$ and $1/p = 1/q + 1/r$. Then, for any points $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X ,*

$$(4.2) \quad \sup \left\{ \left[\sum_{j=1}^n \left[\int_{B_{X^\#}} |f(x_j) - f(x'_j)|^q d\mu(f) \right]^{p/q} \right]^{1/p} : \mu \text{ is a probability on } B_{X^\#} \right\} \\ = \inf \left\{ \left[\sum_{j=1}^n \lambda_j^r \right]^{1/r} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n \lambda_j^{-q} |f(x_j) - f(x'_j)|^q \right]^{1/q} : \lambda_j > 0 \right\}.$$

Proof. Define σ to be the supremum on the left-hand side of (4.2) (noting that it is finite). Let $u = r/p$ and $v = q/p$, so that $1/u + 1/v = 1$. We now consider the compact, convex subset

$$K = \left\{ \xi = (\xi_j)_{j=1}^n : \sum_{j=1}^n \xi_j^u \leq \sigma^p \text{ and } \xi_j \geq 0 \right\}$$

of ℓ_u^n . For $\varepsilon > 0$ and μ a probability on $B_{X^\#}$, observe that the equation

$$\Phi(\xi) = \sum_{j=1}^n (\xi_j + \varepsilon)^{-v} \int_{B_{X^\#}} |f(x_j) - f(x'_j)|^q d\mu(f)$$

defines a continuous convex function Φ on K . Take the special vector $\xi \in \mathbb{R}^n$ with

$$\xi_j = \left(\int_{B_{X^\#}} |f(x_j) - f(x'_j)|^q d\mu(f) \right)^{1/(uv)}.$$

Then $\xi \in K$ and $\Phi(\xi) \leq \sigma^p$. Since the collection \mathcal{A} of all functions Φ obtained in this way is concave, by Ky Fan's lemma we can find $\xi^0 \in K$ such that $\Phi(\xi^0) \leq \sigma^p$ for all $\Phi \in \mathcal{A}$ simultaneously. In particular, considering the Dirac measure δ_f at a function $f \in B_{X^\#}$ we obtain

$$\sum_{j=1}^n (\xi_j^0 + \varepsilon)^{-v} |f(x_j) - f(x'_j)|^q \leq \sigma^p.$$

Set $\lambda_j(\varepsilon) := (\xi_j^0 + \varepsilon)^{1/p}$. Then

$$\lim_{\varepsilon \downarrow 0} \left[\sum_{j=1}^n \lambda_j(\varepsilon)^r \right]^{1/r} = \left[\sum_{j=1}^n \xi_j^{r/p} \right]^{1/r} = \left[\sum_{j=1}^n \xi_j^u \right]^{1/r} \leq \sigma^{p/r} = \sigma^{1/u}$$

and, for $f \in B_{X^\#}$,

$$\left[\sum_{j=1}^n \lambda_j(\varepsilon)^{-q} |f(x_j) - f(x'_j)|^q \right]^{1/q} = \\ \left[\sum_{j=1}^n (\xi_j^0 + \varepsilon)^{-v} |f(x_j) - f(x'_j)|^q \right]^{1/q} \leq \sigma^{p/q} = \sigma^{1/v}.$$

Therefore, the right-hand side of (4.2) is less than or equal to the left-hand side.

Conversely, let $\lambda_j > 0$ be arbitrary. Then, by Hölder's inequality for any probability measure μ on $B_{X^\#}$ we have

$$\begin{aligned} & \left[\sum_{j=1}^n \left[\int_{B_{X^\#}} |f(x_j) - f(x'_j)|^q d\mu(f) \right]^{p/q} \right]^{1/p} \\ &= \left[\sum_{j=1}^n \left[\lambda_j \left(\int_{B_{X^\#}} \lambda_j^{-q} |f(x_j) - f(x'_j)|^q d\mu(f) \right)^{1/q} \right]^p \right]^{1/p} \\ &\leq \left[\sum_{j=1}^n \lambda_j^r \right]^{1/r} \left(\sum_{j=1}^n \int_{B_{X^\#}} \lambda_j^{-q} |f(x_j) - f(x'_j)|^q d\mu(f) \right)^{1/q} \\ &= \left[\sum_{j=1}^n \lambda_j^r \right]^{1/r} \left(\int_{B_{X^\#}} \sum_{j=1}^n \lambda_j^{-q} |f(x_j) - f(x'_j)|^q d\mu(f) \right)^{1/q} \\ &\leq \left[\sum_{j=1}^n \lambda_j^r \right]^{1/r} \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^n \lambda_j^{-q} |f(x_j) - f(x'_j)|^q \right)^{1/q}. \quad \square \end{aligned}$$

Together, Theorem 4.1 and Proposition 4.2 immediately give us another characterization of Lipschitz (q, p) -mixing operators, stated below.

Corollary 4.3. *Let $1 \leq p < q < \infty$ and $1/p = 1/q + 1/r$. A Lipschitz map $T : X \rightarrow Y$ is (q, p) -mixing if and only if there exists a constant C such that for all $x_1, \dots, x_n, x'_1, \dots, x'_n \in X$,*

$$\inf \left\{ \left[\sum_{j=1}^n \lambda_j^r \right]^{1/r} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_j^{-q} |g(Tx_j) - g(Tx'_j)|^q \right]^{1/q} : \lambda_j > 0 \right\} \leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n |f(x_j) - f(x'_j)|^p \right]^{1/p}.$$

In this case, $\mathfrak{m}_{q,p}^L(T)$ is equal to the infimum of such constants C .

4.3. Chevet-Saphar spaces. The expression on the right-hand side of (4.2) looks reminiscent of the Chevet-Saphar norms introduced in [CD]. This section is devoted to a characterization of Lipschitz (q, p) -mixing operators in terms of such norms. Let us recall the pertinent definitions first.

An E -valued molecule on X is a finitely supported function $m : X \rightarrow E$ such that $\sum_{x \in X} m(x) = 0$. The space of E -valued molecules on X , denoted $\mathcal{M}(X, E)$, is clearly a vector space under pointwise addition. Given $x, x' \in X$, define $m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$. The simplest nonzero molecules, i.e. those of the form $vm_{xx'}$ for some $x, x' \in X$ and $v \in E$, are called atoms. Note that any molecule may be expressed (in a nonunique way) as a finite sum of atoms. The p -th Chevet-Saphar norm of a

molecule m is given by

$$cs_p(m) := \inf \left\{ \left(\sum_j \lambda_j^p \|v_j\|^p \right)^{1/p} \sup_{f \in B_{X^\#}} \left(\sum_j \lambda_j^{-p'} |f(x_j) - f(x'_j)| \right)^{1/p'} \right. \\ \left. : m = \sum_j v_j m_{x_j x'_j}, \lambda_j > 0 \right\}.$$

The space of E -valued molecules on E , endowed with the norm $cs_p(\cdot)$, is denoted by $\mathcal{CS}_p(X, E)$. There is a canonical way of inducing a pairing between E -valued molecules on X and functions from X to E^* : given $m \in \mathcal{M}(X, E)$ and a function $T : X \rightarrow E^*$, define $\langle T, m \rangle := \sum_{x \in X} \langle T(x), m(x) \rangle$. If we know an expression of the molecule as a sum of atoms, say $m = \sum_j v_j m_{x_j x'_j}$, then $\langle T, m \rangle = \sum_j \langle Tx_j - Tx'_j, v_j \rangle$. The main theorem in [CD] states that with this pairing, the dual space of $\mathcal{CS}_p(X, E)$ is canonically identified with the space of Lipschitz p' -summing operators from X into E^* . Also from [CD], recall that for any Banach space E a Lipschitz map $T : X \rightarrow Y$ naturally induces a well-defined linear map $T_E : \mathcal{M}(X, E) \rightarrow \mathcal{M}(Y, E)$ given by

$$T_E \left(\sum_{j=1}^n v_j m_{x_j x'_j} \right) = \sum_{j=1}^n v_j m_{Tx_j Tx'_j}.$$

Now we come to the third characterization of Lipschitz (q, p) -mixing operators.

Theorem 4.4. *Let $T : X \rightarrow Y$ be a Lipschitz map. The following are equivalent:*

- (a) *T is Lipschitz (q, p) -mixing.*
- (b) *For every Banach space G (or only $G = \ell_{q'}$), the operator*

$$T_G : \mathcal{CS}_{p'}(X, G) \rightarrow \mathcal{CS}_{q'}(Y, G)$$

is continuous.

In this case,

$$\mathfrak{m}_{q,p}^L(T) = \left\| T_{\ell_{q'}} : \mathcal{CS}_{p'}(X, \ell_{q'}) \rightarrow \mathcal{CS}_{q'}(Y, \ell_{q'}) \right\| \geq \|T_G : \mathcal{CS}_{p'}(X, G) \rightarrow \mathcal{CS}_{q'}(Y, G)\|.$$

Proof. First, suppose that T is Lipschitz (q, p) -mixing. Let $\varphi \in (\mathcal{CS}_{q'}(Y, G))^*$ with $\|\varphi\| \leq 1$. Since $(\mathcal{CS}_{q'}(Y, G))^* \equiv \Pi_q^L(Y, G^*)$, we can identify φ with a map $L_\varphi \in \Pi_q^L(Y, G^*)$ with $\pi_q^L(L_\varphi) = \|\varphi\| \leq 1$. Let $m = \sum v_j m_{x_j x'_j} \in \mathcal{M}(X, G)$. Then $T_G(m) = \sum v_j m_{Tx_j Tx'_j}$, so

$$\langle \varphi, T_G(m) \rangle = \sum_j \langle L_\varphi(Tx_j) - L_\varphi(Tx'_j), v_j \rangle = \langle L_\varphi \circ T, m \rangle,$$

and thus

$$\left| \langle \varphi, T_G(m) \rangle \right| = \left| \langle L_\varphi \circ T, m \rangle \right| \leq \pi_p^L(L_\varphi \circ T) cs_{p'}(m) \\ \leq \pi_q^L(L_\varphi) \mathfrak{m}_{q,p}^L(T) cs_{p'}(m) \leq \mathfrak{m}_{q,p}^L(T) cs_{p'}(m).$$

Taking the supremum over all such φ we obtain, $cs_{q'}(T_G(m)) \leq \mathfrak{m}_{q,p}^L(T) cs_{p'}(m)$; i.e. $T_G : \mathcal{CS}_{p'}(X, G) \rightarrow \mathcal{CS}_{q'}(Y, G)$ is continuous and $\|T_G\| \leq \mathfrak{m}_{q,p}^L(T)$.

Now, suppose that $T_{\ell_{q'}} : \mathcal{CS}_{p'}(X, \ell_{q'}) \rightarrow \mathcal{CS}_{q'}(Y, \ell_{q'})$ is continuous and has norm C , and let $S : Y \rightarrow \ell_q$ be a q -summing operator. Let m be an $\ell_{q'}$ -valued molecule on X , say $m = \sum_j v_j m_{x_j x'_j}$ with $v_j \in \ell_{q'}$ and $x_j, x'_j \in X$. Then

$$\langle S \circ T, m \rangle = \sum_j \langle v_j, STx_j - STx'_j \rangle = \left\langle S, \sum_j v_j m_{Tx_j Tx'_j} \right\rangle = \langle S, T_{\ell_{q'}}(m) \rangle.$$

By the duality between the Lipschitz q -summing norm and the q' -Chevet-Saphar norm, together with the boundedness of $T_{\ell_{q'}}$,

$$|\langle S \circ T, m \rangle| = |\langle S, T_{\ell_{q'}}(m) \rangle| \leq \pi_q^L(S) cs_{q'}(T_{\ell_{q'}}(m)) \leq \pi_q^L(S) \cdot C \cdot cs_{p'}(m).$$

Taking the supremum over all m with $cs_{p'}(m) \leq 1$ and invoking the duality between the Lipschitz p -summing norm and the p' -Chevet-Saphar norm, we conclude that $\pi_p^L(S \circ T) \leq C \pi_q^L(S)$. By the remarks in Section 3, we conclude that T is Lipschitz (q, p) -mixing with $\mathbf{m}_{q,p}^L(T) \leq C$. \square

Of course, the space $\ell_{q'}$ in the preceding theorem may be replaced by any other infinite-dimensional $L_{q'}$ space.

5. APPLICATIONS

5.1. The Lipschitz $(2, 1)$ -mixing constant of the identity on a tree. As already mentioned in Section 3, Farmer and Johnson [FJ09] proved a nonlinear Grothendieck inequality which, in our language, means that the identity on a metric tree is Lipschitz $(2, 1)$ -mixing with constant at most Grothendieck’s constant. While both their proof and the one given in [CZ] make explicit use of the lifting property for trees, using Theorem 4.1 we can reobtain the same bound without explicitly appealing to the lifting property.

Lemma 5.1. *When T is an unweighted graph-theoretic tree on $n + 1$ points and H is a Hilbert space, $\text{Lip}(T, H)$ is isometric to $\ell_\infty^n(H)$.*

Proof. From [CD, Sec. 4.2], $\mathcal{CS}_1(T, H)$ is isometric to $\ell_1^n(H)$ in a natural way. By the duality result [CD, Thm. 4.3], $\text{Lip}(T, H)$ is then isometric to $\ell_\infty^n(H)$. \square

Proposition 5.2. *Let T be a finite unweighted graph-theoretic tree. Then the identity on T is Lipschitz $(2, 1)$ -mixing with constant at most K_G .*

Proof. Let $x_1, \dots, x_m, x'_1, \dots, x'_m \in T$ and let μ be a probability measure on $B_{T^\#}$. Note that

$$\sup_{f \in B_{T^\#}} \sum_{j=1}^m |f(x_j) - f(x'_j)|$$

is the norm of the linear operator A from $T^\#$ to ℓ_1^m given by $f \mapsto (f(x_j) - f(x'_j))_{j=1}^m$. By Lemma 5.1, $T^\#$ can be identified with ℓ_∞^N for some N , so the operator A under consideration goes from ℓ_∞^N to ℓ_1^m . The classical Grothendieck inequality gives us

$$\|A : \ell_\infty^N(L_2(\mu)) \rightarrow \ell_1^m(L_2(\mu))\| \leq K_G \|A : \ell_\infty^N \rightarrow \ell_1^m\|.$$

But another application of Lemma 5.1 reveals that $\ell_\infty^N(L_2(\mu))$ can be identified with the space of Lipschitz functions from T to $L_2(\mu)$, so in fact one has

$$\sup_{\text{Lip}(F:T \rightarrow L_2(\mu)) \leq 1} \sum_{j=1}^m \|F(x_j) - F(x'_j)\|_{L_2(\mu)} \leq K_G \sup_{f \in B_{T^\#}} \sum_{j=1}^m |f(x_j) - f(x'_j)|.$$

In particular, consider the pointwise evaluation $\delta : T \rightarrow L_2(\mu)$. For any $x, x' \in T$ we have

$$\|\delta(x) - \delta(x')\|_{L_2(\mu)} = \left[\int_{g \in B_{T^\#}} |g(x) - g(x')|^2 d\mu(g) \right]^{1/2} \leq d(x, x');$$

hence $\text{Lip}(\delta : T \rightarrow L_2(\mu)) \leq 1$ and thus

$$\sum_{j=1}^m \left[\int_{B_{T^\#}} |g(x_j) - g(x'_j)|^2 d\mu(g) \right]^{1/2} \leq K_G \sup_{f \in B_{T^\#}} \sum_{j=1}^m |f(x_j) - f(x'_j)|.$$

By Theorem 4.1, we conclude that the identity on T is Lipschitz $(2, 1)$ -summing with constant at most K_G . \square

5.2. An “interpolation style” theorem. As it so often happens with many constants associated to mappings, it is not easy to calculate the Lipschitz (q, p) -mixing constant of a specific map. The following “interpolation style” theorem is based on [Puh77, Lemma 5] and gives useful bounds that are sufficient in some cases.

Theorem 5.3. *Let $1 \leq p, q, r \leq \infty$ with $1/r + 1/q = 1/p$. Then every Lipschitz p -summing map $T : X \rightarrow Y$ is Lipschitz (q, p) -mixing and satisfies*

$$\mathbf{m}_{q,p}^L(T) \leq \pi_p^L(T)^{p/r} \text{Lip}(T)^{p/q}.$$

Proof. The fact that T is (q, p) -mixing is obvious from the ideal property of Lipschitz p -summing operators. Now, let $x_1, \dots, x_n, x'_1, \dots, x'_n \in X$. For any probability measure μ on $B_{Y^\#}$, from the pointwise inequality $|g(y) - g(y')| \leq \text{Lip}(g) \cdot d(y, y')$ for any $y, y' \in Y$ and $g \in Y^\#$ we have that

$$\begin{aligned} (5.1) \quad & \left[\sum_{j=1}^n \left(\int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^q d\mu(g) \right)^{p/q} \right]^{1/p} \\ & \leq \left[\sum_{j=1}^n \left(\int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^p d\mu(g) \right)^{p/q} d(Tx_j, Tx'_j)^{(q-p)p/q} \right]^{1/p}. \end{aligned}$$

Noting that $(q - p)r/q = p$, Hölder’s inequality lets us bound the latter expression by

$$(5.2) \quad \left[\sum_{j=1}^n \int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^p d\mu(g) \right]^{1/q} \left[\sum_{j=1}^n d(Tx_j, Tx'_j)^p \right]^{1/r}.$$

On the one hand, the fact that T is Lipschitz p -summing means that

$$(5.3) \quad \left[\sum_{j=1}^n d(Tx_j, Tx'_j)^p \right]^{1/r} \leq \pi_p^L(T)^{p/r} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n |f(x_j) - f(x'_j)|^p \right]^{1/r},$$

whereas on the other hand, a simple pointwise estimate gives

$$(5.4) \quad \left[\sum_{j=1}^n \int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^p d\mu(g) \right]^{1/q} \leq \text{Lip}(T)^{p/q} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n |f(x_j) - f(x'_j)|^p \right]^{1/q}.$$

Bringing (5.1), (5.2), (5.3) and (5.4) together we have

$$\left[\sum_{j=1}^n \left(\int_{B_{Y^\#}} |g(Tx_j) - g(Tx'_j)|^q d\mu(g) \right)^{p/q} \right]^{1/p} \leq \pi_p^L(T)^{p/r} \text{Lip}(T)^{p/q} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n |f(x_j) - f(x'_j)|^p \right]^{1/p}$$

and thus the desired conclusion follows from Theorem 4.1. □

5.2.1. *The identity on a finite discrete metric space.* Denote by D_n the discrete metric space on n points. Theorem 5.3 allows us to explicitly evaluate the (q, p) -mixing norm of the identity on D_n . In fact, if $1 \leq p \leq q \leq \infty$, then the Lipschitz (q, p) -mixing norm of the identity on D_n is equal to $(2 - 2/n)^{1/p-1/q}$. To see it, let $1 \leq r \leq \infty$ satisfy $1/r + 1/q = 1/p$. From [FJ09] we have that $\pi_s^L(id_{D_n}) = (2 - 2/n)^{1/s}$ for any $s \in [1, \infty]$, and therefore

$$\mathbf{m}_{q,p}^L(id_{D_n}) \geq \frac{\pi_p^L(id_{D_n} \circ id_{D_n})}{\pi_q^L(id_{D_n})} = \frac{(2 - 2/n)^{1/p}}{(2 - 2/n)^{1/q}} = (2 - 2/n)^{1/p-1/q}.$$

On the other hand, from Theorem 5.3,

$$\mathbf{m}_{q,p}^L(id_{D_n}) \leq \pi_p^L(id_{D_n})^{p/r} \text{Lip}(id_{D_n})^{p/q} = (2 - 2/n)^{1/r} \cdot 1 = (2 - 2/n)^{1/p-1/q}$$

and thus $\mathbf{m}_{q,p}^L(id_{D_n}) = (2 - 2/n)^{1/p-1/q}$. Let us observe what this means: for every metric space X and any $T : D_n \rightarrow X$, $\pi_p^L(T) \leq (2 - 2/n)^{1/p-1/q} \pi_q^L(T)$ and this inequality is sharp.

5.2.2. *Reversed inequalities between Lipschitz p -summing norms.* The next result goes along the same theme: using Theorem 5.3 together with known estimates for Lipschitz p -summing norms.

Theorem 5.4. (a) For any $n \in \mathbb{N}$ and $1 \leq p \leq q$,

$$\mathbf{m}_{q,p}^L(id_{\ell_2^n}) \leq c_{p,n}^{p/q-1} \quad \text{where} \quad c_{p,n} = \left[\int_{S_{n-1}} |x_1|^p d\lambda(x) \right]^{1/p},$$

λ being the normalized rotation invariant measure on S_{n-1} . Hence, $\pi_p^L(T) \leq c_{p,n}^{p/q-1} \pi_q^L(T)$ for any Lipschitz map $T : \ell_2^n \rightarrow Y$.

(b) For any finite-dimensional normed space E and $2 \leq q$,

$$\mathbf{m}_{q,2}^L(id_E) \leq [\dim(E)]^{1/2-1/q}.$$

Hence, $\pi_2^L(T) \leq [\dim(E)]^{1/2-1/q} \pi_q^L(T)$ for any Lipschitz map $T : E \rightarrow Y$.

(c) There exists a universal constant C so that for any finite metric space X on n points and $1 \leq q$,

$$\mathbf{m}_{q,1}^L(id_X) \leq C^{1/q'} [\log n]^{1/q'}.$$

Hence, $\pi_1^L(T) \leq C^{1/q'} [\log n]^{1/q'} \pi_q^L(T)$ for any Lipschitz map $T : X \rightarrow Y$.

Proof. Everything follows from Theorem 5.3, together with the fact that the Lipschitz p -summing norm and the linear p -summing norm of a linear operator between Banach spaces coincide ([FJ09, Theorem 2]), and also the following estimates on p -summing norms:

- (a) $\pi_p(id_{\ell_2^n}) = c_{p,n}^{-1}$ (see, for instance, [TJ89, Theorem 10.3]).
- (b) $\pi_2(id_E) = [\dim(E)]^{1/2}$ for any finite-dimensional space E (see, for instance [TJ89, Proposition 9.11]).
- (c) $\pi_1(id_X) \leq C \log n$, essentially proved in [Bou85] as mentioned in [FJ09]. \square

5.3. The general “interpolation style” theorem. Theorem 5.3 is in fact a particular case of the following more general theorem.

Theorem 5.5. *Let $0 < \theta < 1$ and $1 \leq p \leq q_0, q_1 \leq \infty$. Define $1/q := (1 - \theta)/q_0 + \theta/q_1$. For a Lipschitz map $T : X \rightarrow Y$,*

$$\mathbf{m}_{q,p}^L(T) \leq \mathbf{m}_{q_0,p}^L(T)^{1-\theta} \mathbf{m}_{q_1,p}^L(T)^\theta.$$

Proof. Set $1/r := 1/p - 1/q$, $1/r_0 := 1/p - 1/q_0$ and $1/r_1 := 1/p - 1/q_1$. Note that $1/r := (1 - \theta)/r_0 + \theta/r_1$. Let $x_1, \dots, x_n, x'_1, \dots, x'_n \in X$. Given $\varepsilon > 0$, from Corollary 4.3 for each $k = 0, 1$ there exist $\lambda_{j,k} > 0$, $1 \leq j \leq n$ such that

$$\begin{aligned} \left[\sum_{j=1}^n \lambda_{j,k}^{r_k} \right]^{1/r_k} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_{j,k}^{-q_k} |g(Tx_j) - g(Tx'_j)|^{q_k} \right]^{1/q_k} \\ \leq (1 + \varepsilon) \mathbf{m}_{q_k,p}^L(T) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m |f(x_j) - f(x'_j)|^p \right]^{1/p}. \end{aligned}$$

Moreover, dividing by the appropriate constant we may assume that in fact

$$\begin{aligned} \left[\sum_{j=1}^n \lambda_{j,k}^{r_k} \right]^{1/r_k} \leq (1 + \varepsilon) \mathbf{m}_{q_k,p}^L(T) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m |f(x_j) - f(x'_j)|^p \right]^{1/p} \\ \text{and} \quad \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_{j,k}^{-q_k} |g(Tx_j) - g(Tx'_j)|^{q_k} \right]^{1/q_k} \leq 1. \end{aligned}$$

For $1 \leq j \leq n$, set $\lambda_j = \lambda_{j,0}^{1-\theta} \lambda_{j,1}^\theta$. Then, by Hölder’s inequality,

$$\begin{aligned} \left[\sum_{j=1}^n \lambda_j^r \right]^{1/r} \leq \left[\sum_{j=1}^n \lambda_{j,0}^{r_0} \right]^{(1-\theta)/r_0} \cdot \left[\sum_{j=1}^n \lambda_{j,1}^{r_1} \right]^{\theta/r_1} \\ \leq (1 + \varepsilon) \mathbf{m}_{q_0,p}^L(T)^{1-\theta} \mathbf{m}_{q_1,p}^L(T)^\theta \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m |f(x_j) - f(x'_j)|^p \right]^{1/p}. \end{aligned}$$

On the other hand, it follows from

$$\lambda_j^{-1} |f(x_j) - f(x'_j)| = \lambda_{j,0}^{-(1-\theta)} |f(x_j) - f(x'_j)|^{1-\theta} \lambda_{j,1}^{-\theta} |f(x_j) - f(x'_j)|^\theta$$

that

$$\begin{aligned} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_j^{-q} |g(Tx_j) - g(Tx'_j)|^q \right]^{1/q} \\ \leq \prod_{k=0,1} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_{j,k}^{-qk} |g(Tx_j) - g(Tx'_j)|^{qk} \right]^{1/qk} \leq 1. \end{aligned}$$

Therefore, using the other direction of Corollary 4.3,

$$\mathbf{m}_{q,p}^L(T) \leq (1 + \varepsilon) \mathbf{m}_{q_0,p}^L(T)^{1-\theta} \mathbf{m}_{q_1,p}^L(T)^\theta,$$

and by letting $\varepsilon \downarrow 0$, the proof is finished. \square

For $q > p \geq 1$, we say that a metric space X is (q, p) -mixing if the identity on X is (q, p) -mixing. The following lemma shows that the class of (q, p) -mixing spaces does not depend on p . This result is basically the nonlinear extrapolation theorem of Chen and Zheng [CZ, Thm. 2.2], presented in a different language.

Corollary 5.6. *Let X be a metric space and $1 \leq p_0 < p_1 < q$. Then X is (q, p_0) -mixing if and only if it is (q, p_1) -mixing. Moreover,*

$$\mathbf{m}_{q,p_1}^L(id_X) \leq \mathbf{m}_{q,p_0}^L(id_X) \leq \mathbf{m}_{q,p_1}^L(id_X)^{1/\theta},$$

where θ is defined by $1/p_1 = (1 - \theta)/q + \theta/p_0$.

Proof. The monotonicity property for (q, p) -mixing constants from Section 3 gives $\mathbf{m}_{q,p_1}^L(id_X) \leq \mathbf{m}_{q,p_0}^L(id_X)$, whereas the composition property from the same section provides us with the inequality $\mathbf{m}_{q,p_0}^L(id_X) \leq \mathbf{m}_{q,p_1}^L(id_X) \cdot \mathbf{m}_{p_1,p_0}^L(id_X)$. Now, from Theorem 5.5,

$$\mathbf{m}_{p_1,p_0}^L(id_X) \leq \mathbf{m}_{q,p_0}^L(id_X)^{1-\theta} \cdot \mathbf{m}_{p_0,p_0}^L(id_X)^\theta = \mathbf{m}_{q,p_0}^L(id_X)^{1-\theta} \cdot 1.$$

So we obtain

$$\mathbf{m}_{q,p_0}^L(id_X) \leq \mathbf{m}_{q,p_1}^L(id_X) \cdot \mathbf{m}_{q,p_0}^L(id_X)^{1-\theta}$$

from which the result follows. \square

ACKNOWLEDGEMENT

The author thanks Professor W. B. Johnson for his advice and encouragement during the preparation of this paper.

REFERENCES

- [Bou85] J. Bourgain, *On Lipschitz embedding of finite metric spaces in Hilbert space*, Israel J. Math. **52** (1985), no. 1-2, 46–52. MR815600 (87b:46017)
- [BPR10] Geraldo Botelho, Daniel Pellegrino, and Pilar Rueda, *A unified Pietsch domination theorem*, J. Math. Anal. Appl. **365** (2010), no. 1, 269–276. MR2585098
- [CD] Javier Alejandro Chávez Domínguez, *Duality for Lipschitz p -summing operators*, J. Funct. Anal. **261** (2011), no. 2, 387–407. MR2793117
- [CZ] Dongyang Chen and Bentuo Zheng, *Remarks on Lipschitz p -summing operators*, Proc. Amer. Math. Soc. **139** (2011), 2891–2898. MR2801621

- [DF93] Andreas Defant and Klaus Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993. MR1209438 (94e:46130)
- [FJ09] Jeffrey D. Farmer and William B. Johnson, *Lipschitz p -summing operators*, Proc. Amer. Math. Soc. **137** (2009), no. 9, 2989–2995. MR2506457
- [Mau74] Bernard Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p* , Société Mathématique de France, Paris, 1974, with an English summary, Astérisque, No. 11. MR0344931 (49:9670)
- [Pie67] A. Pietsch, *Absolut p -summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1966/1967), 333–353. MR0216328 (35:7162)
- [Pie80] Albrecht Pietsch, *Operator ideals*, North-Holland Mathematical Library, vol. 20, North-Holland Publishing Co., Amsterdam, 1980, translated from German by the author. MR582655 (81j:47001)
- [Puh77] Joachim Puhl, *Quotienten von Operatoridealen*, Math. Nachr. **79** (1977), 131–144. MR0493401 (58:12415)
- [TJ89] Nicole Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38, Longman Scientific & Technical, Harlow, 1989. MR993774 (90k:46039)

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843
E-mail address: jcdom@math.tamu.edu