

ON INTEGRABILITY OF NONAUTONOMOUS NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We show, in general, how to transform the nonautonomous nonlinear Schrödinger equation with quadratic Hamiltonians into the standard autonomous form that is completely integrable by the familiar inverse scattering method in nonlinear science. Derivation of the corresponding equivalent nonisospectral Lax pair is also outlined. A few simple integrable systems are discussed.

1. INTRODUCTION

Recently several nonautonomous (with time-dependent coefficients) and inhomogeneous (with space-dependent coefficients) nonlinear Schrödinger equations have been discussed as (possible) new integrable systems [7], [8], [14], [16], [20], [42], [48], [52], [53], [59], [66], [67], [76], [81], [82], [83], [84], [85], [87], [93], [96], [97], [98], [104] (see also [2], [3], [9], [18], [19], [21], [45], [55], [92] and the references therein for earlier works). They arise in the theory of Bose–Einstein condensation [35], [75], fiber optics [6], [47], superconductivity and plasma physics [18], [19], [70], [71].

As pointed out in recent papers [7], [16], [48], [49] and [55] (see also [2], [3], [21], [34], [45], [68], [74]), these systems can be reduced by a set of transformations to the standard autonomous nonlinear Schrödinger equation, which explains their integrability properties, because this equation is a well-known complete integrable system with Lax pair [58], [101], [102], [103], conservation laws and N -soliton solutions, solvable through the inverse scattering method [2], [3], [5], [54], [71], [78], [101], [102], [103]. Integration techniques of the nonlinear Schrödinger equation also include Painlevé analysis [13], [24], [25], [26], [27], [33], [34], [49], [54], [69], [89], [94], the Hirota method [50], [51], [54], Bäcklund transformation [12], [17], [54] and the Hamiltonian approach [1], [5], [43], [44], [64], [65], [71], among others [36], [62], [72], [77].

A goal of this paper is to construct these transformations explicitly (in quadratures) for the most general variable quadratic Hamiltonian. A simple relation with Green’s function of the linear problem, which seems to be missing in the available literature, is emphasized. Basics of the classical soliton theory, including one and two soliton solutions, the inverse scattering technique and the corresponding equivalent Lax pair, are also briefly summarized in order to make our presentation as

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self-contained as possible. This summary may facilitate applications of our result to specific nonlinear integrable systems.

2. TRANSFORMATION INTO AUTONOMOUS FORM

We consider the nonautonomous nonlinear Schrödinger equation

$$(2.1) \quad i \frac{\partial \psi}{\partial t} = H\psi + h |\psi|^2 \psi$$

on \mathbb{R} , where the variable Hamiltonian $H = Q(p, x, t)$ is an arbitrary quadratic form of operators $p = -i\partial/\partial x$ and x ; namely,

$$(2.2) \quad i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x + h(t)|\psi|^2\psi$$

(a, b, c, d, f and g are suitable real-valued functions of time only) under the following integrability condition [87]:¹

$$(2.3) \quad h = h_0 a(t) \beta^2(t) \mu(t) = h_0 \beta^2(0) \mu^2(0) \frac{a(t) \lambda^2(t)}{\mu(t)}$$

(h_0 is a real constant, and functions β , λ and μ will be defined below).

We present the following result.

Lemma 1. *The substitution*

$$(2.4) \quad \psi(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \chi(\xi, \tau), \quad \xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t),$$

transforms the nonautonomous and inhomogeneous nonlinear Schrödinger equation (2.2) into the standard autonomous form with respect to new variables $\xi = \beta(t)x + \varepsilon(t)$ and $\tau = \gamma(t)$:

$$(2.5) \quad i\chi_\tau + h_0 |\chi|^2 \chi = \chi_{\xi\xi}$$

provided that

$$(2.6) \quad \frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = 0,$$

$$(2.7) \quad \frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0,$$

$$(2.8) \quad \frac{d\gamma}{dt} + a\beta^2 = 0$$

and

$$(2.9) \quad \frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2\alpha g,$$

$$(2.10) \quad \frac{d\varepsilon}{dt} = (g - 2a\delta)\beta,$$

$$(2.11) \quad \frac{d\kappa}{dt} = g\delta - a\delta^2,$$

where

$$(2.12) \quad \alpha = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}.$$

¹If the nonlinear term has the form $h|\psi|^p\psi$, popular in the mathematical literature, the condition becomes $h = h_0 a \beta^2 \mu^{p/2}$.

The autonomous equation (2.5) is completely integrable by advanced methods of the soliton theory [5], [54], [71], [79], [101], [102], [103] (see also [36] and the references cited in the introduction). Equations (2.2)–(2.3) seem to represent the maximum nonautonomous and inhomogeneous one-dimensional integrable system of this kind. (Important special cases of the transformation (2.4) are discussed in [2], [18], [19], [21], [45], [49], [55], [74], [76] and [87].)

Our transformation (2.4) involves the real-valued functions $\alpha, \beta, \gamma, \delta, \varepsilon$ and κ (of time t only) defined as solutions of the system of ordinary differential equations (2.6)–(2.11). This nonlinear ODE system has already appeared in [28] from a different perspective (we shall refer to this system as a Riccati-type system). The substitution (2.12) reduces the Riccati equation (2.6) to the second order linear equation [28]:

$$(2.13) \quad \mu'' - \tau(t) \mu' + 4\sigma(t) \mu = 0,$$

with

$$(2.14) \quad \tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left(\frac{a'}{a} - \frac{d'}{d} \right).$$

(Relations with the corresponding Ehrenfest theorem for the linear Hamiltonian are discussed in [30]. It provides a clear physical interpretation of our results in the case of the Gross-Pitaevskii model of Bose condensation.) Equation (2.8) implies the monotonicity of the new time variable $\tau = \gamma(t)$.

Proof. Differentiate $\psi = \mu^{-1/2}(t) e^{iS(x,t)} \chi(\xi, \tau)$ with $S = \alpha(t)x^2 + \delta(t)x + \kappa(t)$ and $\xi = \beta(t)x + \varepsilon(t)$, $\tau = \gamma(t)$:

$$(2.15) \quad ie^{-iS} \psi_t = \frac{1}{\sqrt{\mu}} \left[-(\alpha'x^2 + \delta'x + \kappa') \chi + i \left((\beta'x + \varepsilon') \chi_\xi + \gamma' \chi_\tau - \frac{\mu'}{2\mu} \chi \right) \right],$$

$$(2.16) \quad e^{-iS} \psi_x = \frac{1}{\sqrt{\mu}} [i(2\alpha x + \delta) \chi + \beta \chi_\xi]$$

and

$$(2.17) \quad e^{-iS} \psi_{xx} = \frac{1}{\sqrt{\mu}} \left[(2i\alpha - (2\alpha x + \delta)^2) \chi + 2i(2\alpha x + \delta) \beta \chi_\xi + \beta^2 \chi_{\xi\xi} \right].$$

Substitution into (2.2), with the help of the integrability condition (2.3) and the system (2.6)–(2.11), results in (2.5). Computational details are left to the reader. \square

This observation provides a new interpretation of the Riccati-type system (2.6)–(2.11), which was originally derived in [28] during integration of the corresponding linear equation via Green’s function.

3. INTEGRATION OF THE RICCATI-TYPE SYSTEM

In order to construct the transformation (2.4) explicitly, one has to solve the nonlinear ODE system (2.6)–(2.11). As already known, the initial value problem of the Riccati-type system, which corresponds to the linear Schrödinger equation with a variable quadratic Hamiltonian (generalized harmonic oscillators [11], [40], [46], [95], [100]), can be explicitly solved in terms of solutions of characteristic equation (2.13) as follows ([28], [30], [86], [88]):

$$(3.1) \quad \mu(t) = 2\mu(0) \mu_0(t) (\alpha(0) + \gamma_0(t)),$$

$$(3.2) \quad \alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))},$$

$$(3.3) \quad \beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\mu(0)}{\mu(t)}\lambda(t),$$

$$(3.4) \quad \gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))}$$

and

$$(3.5) \quad \delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))},$$

$$(3.6) \quad \varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))},$$

$$(3.7) \quad \kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))}.$$

Here,

$$(3.8) \quad \alpha_0(t) = \frac{1}{4a(t)} \frac{\mu'_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)},$$

$$(3.9) \quad \beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \quad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) ds\right),$$

$$(3.10) \quad \gamma_0(t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)}$$

and

$$(3.11) \quad \delta_0(t) = \frac{\lambda(t)}{\mu_0(t)} \int_0^t \left[\left(f(s) - \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu'_0(s) \right] \frac{ds}{\lambda(s)},$$

$$(3.12) \quad \varepsilon_0(t) = -\frac{2a(t)\lambda(t)}{\mu'_0(t)} \delta_0(t) + 8 \int_0^t \frac{a(s)\sigma(s)\lambda(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s)) ds$$

$$+ 2 \int_0^t \frac{a(s)\lambda(s)}{\mu'_0(s)} \left(f(s) - \frac{d(s)}{a(s)} g(s) \right) ds,$$

$$(3.13) \quad \kappa_0(t) = \frac{a(t)\mu_0(t)}{\mu'_0(t)} \delta_0^2(t) - 4 \int_0^t \frac{a(s)\sigma(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s))^2 ds$$

$$- 2 \int_0^t \frac{a(s)}{\mu'_0(s)} (\mu_0(s)\delta_0(s)) \left(f(s) - \frac{d(s)}{a(s)} g(s) \right) ds$$

($\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$ and $\kappa_0(0) = 0$) provided that μ_0 and μ_1 are the standard solutions of equation (2.13) corresponding to the initial conditions $\mu_0(0) = 0$, $\mu'_0(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu'_1(0) = 0$ (proofs are outlined in [28], [32] and [86]). (Formulas (3.8)–(3.13) correspond to Green's function of generalized harmonic oscillators; see, for example, [28], [30], [41], [56], [60], [86], [88] and the references therein for more details.)

One may refer to the solutions (3.8)–(3.13) as the fundamental solution of the Riccati-type system (2.6)–(2.11). Thus the transformation property (3.1)–(3.7) allows one to find a solution of the initial value problem in terms of the fundamental solution (for the nonlinear ODE system).

4. INTEGRATION OF THE NONAUTONOMOUS LINEAR SYSTEM

The transformation (2.4) reduces the linear Schrödinger equation of generalized harmonic oscillators, namely, equation (2.1) with $h = 0$, to the Schrödinger equation for a free particle $i\chi_\tau = \chi_{\xi\xi}$ with a familiar Green function given by

$$(4.1) \quad G(\xi, \eta, \tau - \tau_0) = \frac{1}{\sqrt{-4\pi i(\tau - \tau_0)}} \exp\left[-i\frac{(\xi - \eta)^2}{4(\tau - \tau_0)}\right],$$

where $\xi = \beta(t)x + \varepsilon(t)$, $\eta = \beta(0)x + \varepsilon(0)$ and $\tau = \gamma(t)$, $\tau_0 = \gamma(0)$. One can verify directly that Green's functions of generalized harmonic oscillators [28],

$$(4.2) \quad G(x, y, t) = \frac{1}{\sqrt{2\pi i\mu_0(t)}} \exp[i(\alpha_0(t)x^2 + \beta_0(t)xy + \gamma_0(t)y^2 + \delta_0(t)x + \varepsilon_0(t)y + \kappa_0(t))],$$

are derived from the simplest free particle propagator (4.1) with the help of our transformations (3.1)–(3.7). It is worth noting, though, that the transformation (2.4) requires a knowledge of the functions μ , α , β , γ , δ , ε and κ , which allows us to determine Green's function for the generalized harmonic oscillators directly from (2.2). Thus finding this transformation is equivalent to integration of the original linear equation from the very beginning. (Lemma 1 extends this observation to the nonlinear Schrödinger equation (2.2).)

Then the superposition principle allows us to solve the corresponding Cauchy initial value problem:

$$(4.3) \quad \psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi(y, 0) dy$$

for suitable initial data $\psi(x, 0) = \varphi(x)$ (see [28], [88] and [86] for details).

As shown in [86], the following asymptotics hold:

$$(4.4) \quad \begin{aligned} \alpha_0(t) &= \frac{1}{4a(0)t} - \frac{c(0)}{4a(0)} - \frac{a'(0)}{8a^2(0)} + \mathcal{O}(t), \\ \beta_0(t) &= -\frac{1}{2a(0)t} + \frac{a'(0)}{4a^2(0)} + \mathcal{O}(t), \\ \gamma_0(t) &= \frac{1}{4a(0)t} + \frac{c(0)}{4a(0)} - \frac{a'(0)}{8a^2(0)} + \mathcal{O}(t), \\ \delta_0(t) &= \frac{g(0)}{2a(0)} + \mathcal{O}(t), \quad \varepsilon_0(t) = -\frac{g(0)}{2a(0)} + \mathcal{O}(t), \\ \kappa_0(t) &= \mathcal{O}(t) \end{aligned}$$

as $t \rightarrow 0$ for sufficiently smooth coefficients. Then

$$(4.5) \quad \begin{aligned} G(x, y, t) &\sim \frac{1}{\sqrt{2\pi ia(0)t}} \exp\left[i\frac{(x-y)^2}{4a(0)t}\right] \\ &\times \exp\left[-i\left(\frac{a'(0)}{8a^2(0)}(x-y)^2 + \frac{c(0)}{4a(0)}(x^2 - y^2) - \frac{g(0)}{2a(0)}(x-y)\right)\right] \end{aligned}$$

as $t \rightarrow 0$, which corrects a typo in [28]. (Here, $f \sim g$ as $t \rightarrow 0$, if $\lim_{t \rightarrow 0} (f/g) = 1$.)

Another form of solution of the linear problem can be found by an eigenfunction expansion [56], [88].

Numerous examples of (super) integrable (driven) generalized harmonic oscillators with a detailed bibliography can be found, for instance, in recent publications [28], [29], [30], [31] and [32]. In addition, our Lemma 1 shows that these results provide explicit transformations (2.4) of the corresponding nonlinear systems into the standard completely integrable forms.

5. ONE SOLITON SOLUTION

In the next few sections, we summarize basics of the inverse scattering technique for the autonomous nonlinear Schrödinger equation (2.5) in order to facilitate use of the transformation (2.4) for specific nonautonomous and inhomogeneous nonlinear Schrödinger equations (2.2) from various applications. More details, when needed, can be found in classical works [2], [5], [43], [71], [79], [101], [102], [103] (see also the references cited in the introduction).

As is well-known, equation (2.5) has a traveling wave solution of the form

$$(5.1) \quad \chi(\xi, \tau) = e^{i(\xi y + \tau(y^2 - g_0) + \phi)} F(\xi + 2\tau y)$$

provided

$$(5.2) \quad \left(\frac{dF}{dz}\right)^2 = C_0 + g_0 F^2 + \frac{1}{2} h_0 F^4 \quad (C_0 \text{ is a constant of integration}).$$

Examples include bright and dark solitons, as well as Jacobi elliptic transcendental solutions for nonlinear wave profiles [2], [54], [71], [79], [87]. Setting $C_0 = y = 0$ gives the stationary breather, which is located about $\xi = 0$ and oscillates at a frequency equal to g_0 [78], [79].

By (2.4), the nonautonomous Schrödinger equation (2.2) under the integrability condition (2.3) has the following solution:

$$(5.3) \quad \psi(x, t) = \frac{e^{i\phi}}{\sqrt{\mu}} \exp(i(\alpha x^2 + \beta xy + \gamma(y^2 - g_0) + \delta x + \varepsilon y + \kappa)) \\ \times F(\beta x + 2\gamma y + \varepsilon),$$

where the elliptic function F satisfies equation (5.2) and ϕ , y , g_0 and h_0 are real parameters (see also [87] for a direct derivation of this solution).

6. INTEGRABILITY OF NONAUTONOMOUS NONLINEAR SCHRÖDINGER EQUATIONS

The substitution $\Psi(X, T) = \sqrt{h_0} \chi(\sqrt{2}X, -2T)$ transforms equation (2.5) into the standard forms

$$(6.1) \quad i\Psi_T + \Psi_{XX} \pm 2|\Psi|^2\Psi = 0$$

(focusing and defocusing), which can be obtained as the flatness condition

$$(6.2) \quad U_T - V_X + UV - VU = 0$$

for the Lax–(Zakharov–Shabat) pair

$$(6.3) \quad U = -i\lambda\sigma_3 + \Psi\sigma_+ \mp \Psi^*\sigma_- \\ = \begin{pmatrix} -i\lambda & \Psi \\ \mp\Psi^* & i\lambda \end{pmatrix}$$

and

$$(6.4) \quad V = i(-2\lambda^2 \pm |\Psi|^2)\sigma_3 + (2\lambda\Psi + i\Psi_X)\sigma_+ \pm (-2\lambda\Psi^* + i\Psi_X^*)\sigma_- \\ = \begin{pmatrix} i(-2\lambda^2 \pm |\Psi|^2) & 2\lambda\Psi + i\Psi_X \\ \mp 2\lambda\Psi^* \pm i\Psi_X^* & i(2\lambda^2 \mp |\Psi|^2) \end{pmatrix}$$

(we use the asterisk for complex conjugation). Here, λ is a constant, $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$ and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$(6.5) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the Lax pair guarantees complete integrability and can alone derive all its associated properties, our transformation (2.4) trivially explains the integrability features of the nonautonomous nonlinear Schrödinger equation (2.1), including N -soliton solutions, infinite conservative properties, etc. (see [55], [71] and [79] for more details).

7. INVERSE SCATTERING METHOD

The solution of the Cauchy initial value problem through the inverse scattering method is discussed in [3], [5], [54], [71], [79], [101], [102] and [103]. In the focusing case,

$$(7.1) \quad i\Psi_T + \Psi_{XX} + 2|\Psi|^2\Psi = 0,$$

Zakharov–Shabat’s system contains four equations for an auxiliary two-component wave function $\Phi = (\varphi, v)^T$:

$$(7.2) \quad \Phi_X = U\Phi, \quad \Phi_T = V\Phi,$$

namely,

$$(7.3) \quad \begin{aligned} \varphi_X &= -i\lambda\varphi + \Psi v, \\ v_X &= -\Psi^*\varphi + i\lambda v \end{aligned}$$

and

$$(7.4) \quad \begin{aligned} \varphi_T &= i(-2\lambda^2 + |\Psi|^2)\varphi + (2\lambda\Psi + i\Psi_X)v, \\ v_T &= (-2\lambda\Psi^* + i\Psi_X^*)\varphi + i(2\lambda^2 - |\Psi|^2)v. \end{aligned}$$

Assuming that $\Psi(X, T) \rightarrow 0$ (and $\Psi_X(X, T) \rightarrow 0$) and as $X \rightarrow \pm\infty$ implies

$$(7.5) \quad \varphi_T \rightarrow -2i\lambda^2\varphi, \quad v_T \rightarrow 2i\lambda^2v \quad (X \rightarrow \pm\infty),$$

the scattering data for the problem

$$(7.6) \quad L \begin{pmatrix} \varphi \\ v \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ v \end{pmatrix},$$

$$(7.7) \quad L = i\sigma_3 \frac{\partial}{\partial X} - i\Psi\sigma_+ - i\Psi^*\sigma_- = i \begin{pmatrix} \partial_X & -\Psi \\ -\Psi^* & -\partial_X \end{pmatrix}$$

evolved with time as

$$(7.8) \quad b(\lambda, T) = b(\lambda, T_0) e^{4i\lambda^2(T-T_0)}, \quad r_n(T) = r_n(T_0) e^{4i\lambda(T-T_0)}.$$

Then the Cauchy initial value problem for the nonlinear Schrödinger equation (6.1) can be solved as follows [54], [79], [101], [102], [103]:

$$(7.9) \quad \Psi(X, T) = -2K(X, X, T),$$

where $K(X, Y, T)$ satisfies the linear integral equation

$$(7.10) \quad K(X, Y, T) = B^*(X + Y, T) - \int_X^\infty \int_X^\infty K(X, Z, T) B(Z + W, T) B^*(Y + W, T) dZ dW$$

and $B(X, T)$ can be obtained in terms of the scattering data:

$$(7.11) \quad B(X, T) = -i \sum_{n=1}^N r_n(T_0) e^{i(\lambda_n X + 4\lambda_n^2(T - T_0))} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(\lambda, T_0) e^{i(\lambda X + 4\lambda^2(T - T_0))} d\lambda$$

(see [2], [3], [5], [54], [71], [78], [79], [91], [101], [102], [103] for more details).

As a result, the combination of gauge, scaling and coordinate transformations [55],

$$(7.12) \quad \psi(x, t) = \frac{1}{\sqrt{h_0\mu(t)}} e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))} \Psi\left(\frac{1}{\sqrt{2}}(\beta(t)x + \varepsilon(t)), -\frac{1}{2}\gamma(t)\right),$$

written here explicitly in terms of the solution of the Riccati-type system (2.6)–(2.11) from section 3, allows us to solve the Cauchy initial value problem for the nonautonomous nonlinear Schrödinger equation (2.2) with the help of the standard inverse scattering technique. The following choice of parameters, $\alpha(0) = \gamma(0) = \delta(0) = \kappa(0) = \varepsilon(0) = 0$, $\beta(0) = \sqrt{2}$, $h_0\mu(0) = 1$, preserves the initial data, $\psi(x, 0) = \Psi(\xi, 0)$, and simplifies the solution (3.1)–(3.7).

8. TWO SOLITON SOLUTION

Two well-known solutions of (7.1) are given by [78], [79]:

$$(8.1) \quad \Psi_1(X, T) = \frac{e^{iT}}{\cosh X}$$

and

$$(8.2) \quad \Psi_2(X, T) = 4e^{iT} \frac{\cosh 3X + 3e^{8iT} \cosh X}{\cosh 4X + 4 \cosh 2X + 3 \cos 8T}.$$

Use of the transformation (7.12) results in one and two soliton solutions for the nonautonomous nonlinear Schrödinger equation (2.2), respectively. (See [78], [79] and [101], for more details. N -soliton solutions are also discussed in [2], [62] and [71].)

9. TRANSFORMATION OF THE LAX PAIR AND ZAKHAROV–SHABAT SYSTEM

If needed, an equivalent (nonisospectral) Lax pair for the nonautonomous Schrödinger equation (2.2), which is discussed in [9], [18], [19], [21], [84], [85] for important special cases (see also [7] and [16]), can be derived, in general, from (6.3)–(6.4) by inverting our transformation (7.12) (see [3], [55] for more details). In this paper, the required integrability condition (2.3) (found for the soliton-like solution in [87])

has been already incorporated into this transformation. Computational details are left to the reader.

10. EXAMPLES

A few simple examples are in order. (More examples can be found in [30] and [87]; see also the references therein.)

10.1. **Example 1.** As noticed by Clarkson [21] (see also [2] and [68]), the equation

$$(10.1) \quad i\psi_t + \psi_{xx} = \frac{\omega^2}{4}x^2\psi + 2|\psi|^2\psi, \quad \omega \neq 0,$$

does not pass the Painlevé test and, therefore, is not integrable. The corresponding characteristic equation, $\mu'' + \omega^2\mu = 0$, has two standard solutions:

$$(10.2) \quad \mu_0 = \frac{2}{\omega} \sin \omega t, \quad \mu_1 = \cos \omega t.$$

By our Lemma 1, a modified equation for harmonic solitons [45], [87],

$$(10.3) \quad i\psi_t + \psi_{xx} = \frac{\omega^2}{4}x^2\psi + \frac{h_0\omega\mu(0)\beta^2(0)}{4\alpha(0)\sin\omega t + \omega\cos\omega t}|\psi|^2\psi$$

($h_0, \alpha(0), \beta(0) \neq 0$ and $\mu(0)$ are real constants), can be transformed into the standard form and hence is integrable. Here, $\mu(t) = \mu(0)[4\alpha(0)\sin\omega t + \omega\cos\omega t]/\omega$ and a general solution of the corresponding Riccati-type system is given by

$$(10.4) \quad \alpha(t) = \frac{\omega 4\alpha(0)\cos\omega t - \omega\sin\omega t}{4 4\alpha(0)\sin\omega t + \omega\cos\omega t},$$

$$(10.5) \quad \beta(t) = \frac{\omega\beta(0)}{4\alpha(0)\sin\omega t + \omega\cos\omega t},$$

$$(10.6) \quad \gamma(t) = \gamma(0) - \frac{\beta^2(0)\sin\omega t}{4\alpha(0)\sin\omega t + \omega\cos\omega t},$$

$$(10.7) \quad \delta(t) = \frac{\omega\delta(0)}{4\alpha(0)\sin\omega t + \omega\cos\omega t},$$

$$(10.8) \quad \varepsilon(t) = \varepsilon(0) - \frac{2\beta(0)\delta(0)\sin\omega t}{4\alpha(0)\sin\omega t + \omega\cos\omega t},$$

$$(10.9) \quad \kappa(t) = \kappa(0) - \frac{\delta^2(0)\sin\omega t}{4\alpha(0)\sin\omega t + \omega\cos\omega t}.$$

Letting $\mu(0) = \beta(0) = 1$ and $\alpha(0) = \gamma(0) = \delta(0) = \varepsilon(0) = \kappa(0) = 0$, we arrive at the simple substitution

$$(10.10) \quad \psi(x, t) = \frac{e^{-i(\omega/4)x^2 \tan \omega t}}{\sqrt{\cos \omega t}} \chi\left(\frac{x}{\cos \omega t}, -\frac{\tan \omega t}{\omega}\right),$$

which transforms (10.3) into (2.5). (Replace $\omega = i\kappa$ for the original equation in [45].)

In a similar fashion, the equation

$$(10.11) \quad i\psi_t + \psi_{xx} + (k^2x^2 - ik)\psi = \frac{2kh_0\mu(0)\beta^2(0)}{(k + 2\alpha(0))e^{4kt} + k - 2\alpha(0)}|\psi|^2\psi$$

($k \neq 0$, $\alpha(0)$, $\beta(0) \neq 0$ and $\mu(0)$ are real constants) is integrable. Indeed, the characteristic equation and the standard solutions are given by

$$(10.12) \quad \mu'' + 4k\mu' = 0, \quad \mu_0 = \frac{1}{2k} (1 - e^{-4kt}), \quad \mu_1 = 1$$

and

$$(10.13) \quad \mu = \frac{\mu(0)}{2k} [k + 2\alpha(0) + (k - 2\alpha(0)) e^{-4kt}], \quad \lambda = e^{-2kt}.$$

The simplest case occurs when $k + 2\alpha(0) = 0$ [21]. Further details are left to the reader.

10.2. Example 2. A soliton moving with acceleration in linearly inhomogeneous plasma was discovered in [18] and [19] (see also [9] and [92]). For a modified equation with the integrability condition (2.3),

$$(10.14) \quad i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2kx\psi = \frac{h_0 \mu_0 \beta_0^2}{1 + 4\alpha_0 t} |\psi|^2 \psi,$$

where k , h_0 , α_0 , β_0 and μ_0 are constants, we get $\mu(t) = \mu_0(1 + 4\alpha_0 t)$ and

$$(10.15) \quad \alpha(t) = \frac{\alpha_0}{1 + 4\alpha_0 t}, \quad \beta(t) = \frac{\beta_0}{1 + 4\alpha_0 t},$$

$$(10.16) \quad \gamma(t) = \gamma_0 - \frac{\beta_0^2 t}{1 + 4\alpha_0 t}, \quad \delta(t) = kt + \frac{\delta_0 + kt}{1 + 4\alpha_0 t},$$

$$(10.17) \quad \varepsilon(t) = \varepsilon_0 - \frac{2\beta_0 t (\delta_0 + kt)}{1 + 4\alpha_0 t}, \quad \kappa(t) = \kappa_0 - \frac{k^2 t^3}{3} - \frac{t(\delta_0 + kt)^2}{1 + 4\alpha_0 t}.$$

The classical case [18], [19] corresponds to $\alpha_0 = 0$ and $h_0 \mu_0 \beta_0^2 = -2$ (with $k \rightarrow -k$). The simplest transformation,

$$(10.18) \quad \psi(x, t) = e^{i(2ktx - 4k^2 t^3/3)} \chi(x - 2kt^2, -t),$$

when $\alpha = \alpha_0 = \gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$ and $\mu_0 = \beta = 1$, is due to Tappert [18].

10.3. Example 3. The nonlinear Schrödinger equation [87]

$$(10.19) \quad i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} = \frac{g_0 \beta_0^2}{(1 + 4\alpha_0 t)^2} z\psi + \frac{h_0 \mu_0 \beta_0^2}{1 + 4\alpha_0 t} |\psi|^2 \psi,$$

where

$$(10.20) \quad z = \frac{\beta_0 x + 2(\gamma_0 - (\beta_0^2 - 4\alpha_0 \gamma_0)t)y}{1 + 4\alpha_0 t},$$

with the help of the gauge transformation

$$(10.21) \quad \psi = e^{-if(t)} \chi(x, t), \quad \frac{df}{dt} = 2g_0 \beta_0^2 y \frac{\gamma_0 - (\beta_0^2 - 4\alpha_0 \gamma_0)t}{(1 + 4\alpha_0 t)^3},$$

can be transformed into a similar form:

$$(10.22) \quad i\chi_t + \chi_{xx} - \frac{g_0 \beta_0^3 x}{(1 + 4\alpha_0 t)^3} \chi = \frac{h_0 \mu_0 \beta_0^2}{1 + 4\alpha_0 t} |\chi|^2 \chi.$$

Looking for a traveling wave, we indicate the following soliton-like solution [87], [89]:

$$(10.23) \quad \chi(x, t) = \frac{e^{iS(x,t)}}{\sqrt{|\mu_0|(1 + 4\alpha_0 t)}} g_0^{1/3} \sqrt{\frac{2}{h_0}} A_{k_0} \left(g_0^{1/3} z \right),$$

where

$$(10.24) \quad S(x, t) = \frac{\alpha_0 x^2 + \beta_0 xy + (\gamma_0 - (\beta_0^2 - 4\alpha_0\gamma_0)t) y^2}{1 + 4\alpha_0 t} + g_0 \beta_0^2 t \frac{2\gamma_0 - (\beta_0^2 - 8\alpha_0\gamma_0)t}{(1 + 4\alpha_0 t)^2} y$$

($\alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0, \kappa_0, \mu_0, g_0, h_0, y$ are real constants), and the soliton profile is defined, as a solution of the second Painlevé equation [89], in terms of the nonlinear Airy function $A_{k_0}(\zeta)$ with asymptotics given by

$$(10.25) \quad A_{k_0}(\zeta) = \begin{cases} k_0 \text{Ai}(\zeta), & \zeta \rightarrow +\infty, \\ r |\zeta|^{-1/4} \sin(s(\zeta) - \theta_0) + o(|\zeta|^{-1/4}), & \zeta \rightarrow -\infty. \end{cases}$$

Here, $\text{Ai}(\zeta)$ is Airy function, $-1 < k_0 < 1$ provided $k_0 \neq 0$, $r^2 = -\pi^{-1} \ln(1 - k_0^2)$,

$$(10.26) \quad s(\zeta) = \frac{2}{3} |\zeta|^{3/2} - \frac{3}{4} r^2 \ln |\zeta|$$

and

$$(10.27) \quad \theta_0 = \frac{3}{2} r^2 \ln 2 + \arg \Gamma\left(1 - \frac{i}{2} r^2\right) + \frac{\pi}{4} (1 - 2\text{sign}(k_0)).$$

These asymptotics were found in [4] and [80] and had been proven rigorously in [38] and [39] (see [5], [10], [22], [23], [25], [27], [90] and the references therein for a study of this nonlinear Airy function).

It is worth noting that, in contrast to the previous case, this A -soliton moves with a constant velocity when $\alpha_0 = 0$ (notice that the external field has essentially changed the soliton shape; see [87] for more details).

11. CONCLUSION

We have shown how to transform the nonautonomous and inhomogeneous nonlinear Schrödinger equations (2.2)–(2.3) into the standard autonomous form (2.5) that is completely integrable by the inverse scattering approach. This transformation is explicitly written in terms of Green's function of the corresponding linear problem (generalized harmonic oscillators); see Lemma 1 and equations (3.1)–(3.7) and (3.8)–(3.13). Combination of these advances allows one to solve the Cauchy initial value problem for the generic nonautonomous integrable quantum system under consideration. Simple examples are considered, and further examples can be found elsewhere; see, for instance, [30] and [87] and the references therein.

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