PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 9, September 2012, Pages 2953–2966
S 0002-9939(2011)11230-9
Article electronically published on December 29, 2011

DEFORMATIONS OF PAIRS (X, L) WHEN X IS SINGULAR

JIE WANG

(Communicated by Lev Borisov)

ABSTRACT. We give an elementary construction of the tangent obstruction theory of the deformations of the pair (X, L) with X a reduced local complete intersection scheme and L a line bundle on X. This generalizes the classical deformation theory of pairs in the case when X is smooth. A criteria for sections of L to extend is also given.

1. INTRODUCTION

Throughout this paper, we will work over the complex numbers $\mathbb{C}$. The deformation theory of the pair (X, L) for X a smooth variety and L a line bundle on X was first used to study Petri's conjecture by Arbarello and Cornalba in [1]. It was proved there that first-order deformations of the pair (X, L) are in natural one-to-one correspondence with

$$\xi \in H^1(X, \mathcal{D}_1(L)),$$

where $\mathcal{D}_1(L)$ is the sheaf of holomorphic first-order differential operators and $H^2(X, \mathcal{D}_1(L))$ is an obstruction space. Given a first-order deformation $\phi \in H^1(X, T_X)$ of X, there is a first-order deformation of L along ϕ if and only if $\phi \cup c(L) = 0 \in H^2(X, \mathcal{O}_X)$, where $c(L) \in H^1(X, \Omega^1_X)$ is the first Chern class of L in the sense of Atiyah.

Moreover, there is a natural differentiation map

$$H^1(X, \mathcal{D}_1(L)) \xrightarrow{M} Hom(H^0(X, L), H^1(X, L)) \tag{1.1}$$

such that a section $s \in H^0(X, L)$ extends to first order along ξ if and only if the element

$$M(\xi)(s) \in H^1(X, L)$$

is zero.

The map M together with the tangent obstruction spaces have numerous deformation theoretic applications. For instance, for any first-order deformation of (X, L), at least $h^0(L) - h^1(L)$ linearly independent sections of L extend; $Ker(M) \subset H^1(X, \mathcal{D}_1(L))$ is the space of first-order deformations of (X, L) to which all sections of L extend. If X is a complete curve, a dual form of (1.1) is the higher μ-map μ_1 in [3]. In case L gives an embedding of X into some projective space $\mathbb{P}$, $Coker(M)$ is naturally isomorphic to $H^1(X, N_{X|\mathbb{P}})$ (cf. [1]), and therefore the surjectivity of M implies that $X \subset \mathbb{P}$ is unobstructed. Another direct consequence is that

Received by the editors March 18, 2011.
2010 *Mathematics Subject Classification.* Primary 14B10, 14B12.
Key words and phrases. Deformations of the pair, tangent obstruction theory, local complete intersection.

the deformations of the pair (X, L) is unobstructed for a smooth curve X, since $H^2(X, \mathcal{D}_1(L)) = 0$. If X is a smooth $K3$-surface, the map $H^1(T_X) \xrightarrow{\cup c(L)} H^2(\mathcal{O}_X) \cong \mathbb{C}$ is surjective for every nontrival line bundle L. This means that L deforms along a 19-dimensional subspace of $H^1(T_X)$, because $h^1(X, T_X) = 20$.

In this paper, we give an elementary approach to the deformation theory of the pair (X, L) for X a separated reduced local complete intersection (l.c.i) scheme of finite type over $\mathbb{C}$. We prove that even though X could be singular, the functor of Artin rings

$$Def_{(X,L)}(A) = \{\text{Flat deformations of } (X, L) \text{ over } A\}/\text{isomorphisms}$$

still behaves well in the sense that there is a tangent obstruction theory for this deformation functor, with tangent space $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ and obstruction space $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$, where $\mathcal{P}^1_X(L)$ is the sheaf of *one jets* or the sheaf of principle parts of L on X. Moreover, there is a natural map analogous to M characterizing obstructions for sections of L to extend. Therefore, all the nice consequences mentioned above generalize to reduced l.c.i. schemes. If X is smooth, $\mathcal{P}^1_X(L) = \mathcal{D}_1(L)^* \otimes L$, where $\mathcal{D}_1(L)$ is the sheaf of first-order differential operators on L, and $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) = H^i(X, \mathcal{D}_1(L))$. We go back to the classical case. The tangent and obstruction spaces for deformations of (X, L) was known to experts and was stated implicitly in [8], [9]. Our approach is more elementary and does not use the more abstract machinery of cotangent complexes. See also section 3 of [4] for a different approach to the obstruction spaces, where they have a weaker three-level obstruction space but without the l.c.i assumption. It seems to the author that the generaliztion of the map M in (1.1) to singular varieties is new.

2. The sheaf of one jets

In this section, we briefly review some basic facts and definitions about the sheaf of one jets.

Let $g : X \to Y$ be a morphism between two algebraic schemes (separated schemes of finite type over $\mathbb{C}$), let L be a line bundle on X, and let $\Delta \subset X \times_Y X$ be the diagonal defined by ideal sheaf $\mathcal{I}_\Delta$. Consider the first-order neighborhood $Spec\frac{\mathcal{O}_{X\times_Y X}}{\mathcal{I}_\Delta{}^2}$ of Δ with two projections π_1, π_2 to X. The sheaf of one jets $\mathcal{P}^1_{X/Y}(L)$ of X over Y is defined to be $\mathcal{P}^1_{X/Y}(L) := \pi_{1*}\pi_2^*(L)$. $\mathcal{P}^1_{X/Y}(L)$ has a natural left $\mathcal{O}_X$-module structure induced by π_1 and a right $\mathcal{O}_X$-module structure induced by π_2 which, in general, is not equivalent to the left one. Throughout this paper, we will only use the left $\mathcal{O}_X$-module structure of $\mathcal{P}^1_{X/Y}(L)$. Consider the short exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_\Delta}{\mathcal{I}^2_\Delta} \longrightarrow \frac{\mathcal{O}_{X\times_Y X}}{\mathcal{I}^2_\Delta} \longrightarrow \frac{\mathcal{O}_{X\times_Y X}}{\mathcal{I}_\Delta} \longrightarrow 0.$$

By tensoring the above sequence with $\pi_2^* L$ then applying the functor π_{1*}, we get a short exact sequence of left $\mathcal{O}_X$-modules on X

$$0 \longrightarrow \Omega^1_{X/Y}(L) \xrightarrow{\ i\ } \mathcal{P}^1_{X/Y}(L) \longrightarrow L \longrightarrow 0, \tag{2.1}$$

where $\Omega^1_{X/Y}$ is the sheaf of relative Kähler differentials. The sequence is exact on the right because there is no higher derived image for π_{1*} (π_1 has relative dimension 0). When $Y = Spec(\mathbb{C})$, we will write $\mathcal{P}^1_X(L)$ for $\mathcal{P}^1_{X/Y}(L)$. The "fibre"of the sheaf

$\mathcal{P}^1_{X/Y}(L)$ at a closed point $x \in X$ is the stalk of $L|_{g^{-1}(g(x))}$ at x mod the maximal ideal squared, i.e.

$$\mathcal{P}^1_{X/Y}(L)_x \otimes_{\mathcal{O}_{X,x}} \frac{\mathcal{O}_{X,x}}{m_x} \cong \frac{L_x}{({m_x}^2 + m_{g(x)})L_x}.$$

This is the reason $\mathcal{P}^1_{X/Y}(L)$ is called the sheaf of (relative) one jets. There is a $\mathcal{O}_Y$-linear splitting $p_1 : L \to \mathcal{P}^1_{X/Y}(L)$, which sends a section s of L to its one jet $\pi_{1*}\pi_2^* s$. p_1 satisfies the property that

$$p_1(fs) = i(df \otimes s) + fp_1(s) \tag{2.2}$$

for any$f \in \mathcal{O}_X(U)$ and $s \in L(U)$, where $U \subset X$ is any open subset. (In fact, p_1 is $\mathcal{O}_X$-linear if we use the right $\mathcal{O}_X$-module structure of $\mathcal{P}^1_{X/Y}(L)$.) If X is smooth, $Y = Spec(\mathbb{C})$, $\mathcal{P}^1_X(L)$ is the vector bundle $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_1(L), L)$, where $\mathcal{D}_1(L)$ is the sheaf of first-order differential operators on L.

3. Computation of the tangent space

In this section, let X be a reduced algebraic scheme. Applying the functor $\mathcal{H}om_{\mathcal{O}_X}(-, L)$ to (2.1), we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(L, L) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(L), L) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(L, L) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X(L), L) \longrightarrow \cdots .$$

Notice that $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(L), L) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is the tangent space of the deformations of X, and $\mathrm{Ext}^1_{\mathcal{O}_X}(L, L) = H^1(\mathcal{O}_X)$ is the tangent space of deformations of L with the base X fixed. This suggests that $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is the tangent space of deformations of the pair (X, L) and $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is an obstruction space. If X is smooth, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is the sheaf of first-order differential operators $\mathcal{D}_1(L)$, and $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) = H^1(X, \mathcal{D}_1(L))$ is the correct tangent space. In this section and the next, we will prove this is indeed the correct generalization of the tangent obstruction theory for deformations of the pair (X, L).

Let us first recall that for any reduced algebraic scheme over $\mathbb{C}$, we have a one-to-one correspondence between isomorphism classes of extensions of X by a coherent locally free $\mathcal{O}_X$-module $\mathcal{I}$ and $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{I})$ in the following way.

Given an isomorphism class of extension of $\mathcal{O}_X$ by $\mathcal{I}$,

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

when $X \subset \mathcal{X}$ is a closed immersion defined by ideal sheaf $\mathcal{I}$, and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{\mathcal{X}}$. We associate to it (the isomorphism class of) the conormal sequence

$$\mathcal{E} : 0 \longrightarrow \mathcal{I} \longrightarrow \Omega^1_{\mathcal{X}}|_X \longrightarrow \Omega^1_X \longrightarrow 0$$

(which is also exact on the left).

This conormal sequence corresponds to an element $c_{\mathcal{E}}$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{I})$.

Conversely, for any $\mathcal{O}_X$-module extension

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{E} \xrightarrow{h} \Omega^1_X \longrightarrow 0\ , \tag{3.1}$$

let $d : \mathcal{O}_X \to \Omega^1_X$ be the canonical derivation. Let $\mathcal{O} = \mathcal{O}_X \times_{\Omega^1_X} \mathcal{E}$ be the fibre product sheaf: over an open subset $U \subset X$ we have $\mathcal{O}(U) = \{(f, a) : h(a) = df\}$ with ring structure given by

$$(f, a)(f', a') = (ff', fa' + f'a).$$

We get a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I} & \xrightarrow{j} & \mathcal{O} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{E} & \longrightarrow & \Omega^1_X & \longrightarrow & 0
\end{array}$$

It is easy to check that $d' : \mathcal{O} \to \mathcal{E}$ is a $\mathbb{C}$-derivation, thus factors through $\Omega^1_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_X$. Therefore $\Omega^1_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_X \cong \mathcal{E}$ by 5-lemma and we recover $\mathcal{X}$ from (3.1).

In case $\mathcal{I} = \mathcal{O}_X$, we can give $\mathcal{O}_{\mathcal{X}}$ a $\mathbb{C}[\epsilon]$-module structure by sending ϵ to $j(1) \in \mathcal{O}_{\mathcal{X}}$. The fact that $\epsilon\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_X$ means that $\mathcal{X}$ is flat over $Spec(\mathbb{C}[\epsilon])$. Therefore $\mathcal{X}$ is a first-order infinitesimal deformation of X.

For the deformations of the pair (X, L), we have the following result.

Theorem 3.1. *Let X be a reduced scheme of finite type over $\mathbb{C}$ and let L be a line bundle on X.*

(1) *The tangent space of the functor of Artin rings $Def_{(X,L)}$ is canonically identified with $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$.*

(2) *There exists a natural pairing*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) \otimes H^0(X, L) \xrightarrow{p} H^1(X, L) \tag{3.2}$$

such that for any first-order deformation of the pair (X, L) corresponding to $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$, a section $s \in H^0(L)$ extends to first order along ξ if and only if ξ and s pair to zero under p.

Proof. (1) Given a first-order deformation of the pair (X, L), we get the following fibered diagram with $\mathcal{O}_{\mathcal{X}}$ flat over $Spec(\mathbb{C}[\epsilon])$ and $\mathcal{L}$ line bundle on $\mathcal{X}$:

$$\begin{array}{ccc}
L & \hookrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
X & \hookrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
Spec(\mathbb{C}) & \hookrightarrow & Spec(\mathbb{C}[\epsilon])
\end{array}$$

We have a diagram of (left) $\mathcal{O}_X$-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \Omega^1_{\mathcal{X}}(\mathcal{L})|_X & \longrightarrow & \Omega^1_X(L) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X & \xrightarrow{r} & \mathcal{P}^1_X(L) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{L}|_X & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The two right columns are exact by (2.1) and the fact that restriction to X is (left) exact since $\mathcal{T}or^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{L}, \mathcal{O}_X) = 0$. ($\mathcal{L}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module!) The first row is the conormal sequence of $X \subset \mathcal{X}$ twisted by L, which is exact. Thus by the Snake Lemma, $ker(r) = L$ and the second row is exact. Therefore, we can associate any first-order deformation of the pair (X, L) with the second row exact sequence, which corresponds to an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$.

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathcal{L} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p_1'} & & \downarrow{\scriptstyle p_1} & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X & \xrightarrow{r} & \mathcal{P}^1_X(L) & \longrightarrow & 0
\end{array}$$

where p_1' is the composition of $p_1 : \mathcal{L} \to \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})$ and the restriction map to X. Thus p_1' factors through $L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$ and therefore $\mathcal{L} \cong L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$. This fact suggests that we can recover $\mathcal{L}$ from $\mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$ and L.

Conversely, for any element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ corresponding to an $\mathcal{O}_X$-module extension

$$0 \longrightarrow L \longrightarrow \mathcal{E} \xrightarrow{r} \mathcal{P}^1_X(L) \longrightarrow 0.$$

The pull back extension $\mathcal{E}' = \mathcal{E} \times_{\mathcal{P}^1_X(L)} \Omega^1_X(L)$ by the natural inclusion

$$i : \Omega^1_X(L) \longrightarrow \mathcal{P}^1_X(L)$$

sits naturally in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathcal{E}' = \mathcal{E} \times_{\mathcal{P}^1_X(L)} \Omega^1_X(L) & \longrightarrow & \Omega^1_X(L) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & L & \longrightarrow & \mathcal{E} & \xrightarrow{r} & \mathcal{P}^1_X(L) & \longrightarrow & 0
\end{array}$$

The first row exact sequence corresponds to an element in $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X(L), L) = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, which corresponds to a first-order infinitesimal deformation $\mathcal{X}$ of X as described in the beginning of this section.

To recover the deformation of L, let $\mathcal{E}'' = \mathcal{E}' \otimes L^{-1}$ and let

$$\mathcal{L} = L \times_{\mathcal{P}^1_X(L)} \mathcal{E} = \{(s,e) \in L \oplus \mathcal{E} \,|\, p_1(s) = r(e)\}.$$

$\mathcal{L}$ has the natural $\mathcal{O}_X \times_{\Omega^1_X} \mathcal{E}''(= \mathcal{O}_{\mathcal{X}})$-module structure

$$(f,a)(s,e) = (fs, fe + i'(a \cdot s))$$

where $(f,a) \in \mathcal{O}_X \times_{\Omega^1_X} \mathcal{E}'' = \mathcal{O}_{\mathcal{X}}$, $(s,e) \in \mathcal{L}$ and $a \cdot s \in \mathcal{E}'$. This is a well-defined $\mathcal{O}_{\mathcal{X}}$-module because

$$p_1(fs) = i(df \otimes s) + fp_1(s) = r(i'(a \cdot s)) + fr(e).$$

In order to see $\mathcal{L}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of rank one, it suffices to prove the case L is the trivial bundle since the question is local. In this case, (2.1) splits (as left $\mathcal{O}_X$-module) and $\mathcal{P}^1_X(\mathcal{O}_X) \cong \mathcal{O}_X \oplus \Omega^1_X$. The statement follows immediately from this.

(2) For any $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ corresponding to the extension

$$0 \longrightarrow L \longrightarrow \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X \longrightarrow \mathcal{P}^1_X(L) \longrightarrow 0,$$

we define the natural pairing by $p(\xi \otimes s) := \delta(p_1(s)) \in H^1(L)$. Where $\delta : H^0(\mathcal{P}^1_X(L)) \to H^1(L)$ is the connecting homomorphism of the long exact cohomology sequence corresponding to ξ

$$\cdots \longrightarrow H^0(\mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X) \xrightarrow{r} H^0(\mathcal{P}^1_X(L)) \xrightarrow{\delta} H^1(L) \longrightarrow \cdots,$$

$\delta(p_1(s)) = 0$ means there exists some $e \in H^0(\mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X)$ such that $r(e) = p_1(s)$; thus (s,e) determines a global section of $\mathcal{L} = L \times_{\mathcal{P}^1_X(L)} \mathcal{P}^1_{\mathcal{X}}(\mathcal{L})|_X$. $\square$

4. Obstructions

In this section, let X be as in section 3 and assume furthermore that X is a local complete intersection scheme. We will show that $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$ is an obstruction space for deformations of the pair (X, L).

The general idea is to apply Vistoli's construction of obstruction spaces for deformations of l.c.i schemes (cf. sections 3, 4 of [11]) to the total space of $L^\vee$ and keep track of the bundle structure using a $\mathbb{C}^*$-action.

For any $z \in \mathbb{C}^*$, let $\phi_z : L^\vee \to L^\vee$ be the multiplication map by z in the fiber direction. Define a $\mathbb{C}^*$-action on $\mathcal{O}_{L^\vee}$ and $\Omega^1_{L^\vee}$ by

$$z \cdot f = z^{-1}\phi_z^* f, \tag{4.1}$$

$$z \cdot \omega = z^{-1}\phi_z^* \omega \tag{4.2}$$

for local sections $f \in \mathcal{O}_{L^\vee}$, $\omega \in \Omega^1_{L^\vee}$.

Let $\mathcal{O}^{\mathbb{C}^*}_{L^\vee}$ and $\Omega^{\mathbb{C}^*}_{L^\vee}$ be the sheaf of sections which are invariant under the $\mathbb{C}^*$-action. Under some trivialization of $L^\vee$ over $U \subset X$: $L^\vee_U \cong U \times \mathbb{A}^1_t$, $\mathcal{O}^{\mathbb{C}^*}_{L^\vee}$ consists of functions on $L^\vee$ of the form $f(x)t$, and $\Omega^{\mathbb{C}^*}_{L^\vee}$ consists of 1-forms $f(x)d_{L^\vee}t + \omega(x)t$ where f is the pull back of a function on U and $\omega \in \Omega^1_U$.

Both $\mathcal{O}^{\mathbb{C}^*}_{L^\vee}$ and $\Omega^{\mathbb{C}^*}_{L^\vee}$ have natural $\mathcal{O}_X$-module structures, and we have natural isomorphisms of $\mathcal{O}_X$-modules $\mathcal{O}^{\mathbb{C}^*}_{L^\vee} \cong L$ and $\mathcal{P}^1_X(L) \cong \Omega^{\mathbb{C}^*}_{L^\vee}$. The isomorphisms can be described as follows. For any section $s \in L$, we can naturally view it as a function on the total space of $L^\vee$ which restricts to a linear function on the fiber. Such functions are invariant under the $\mathbb{C}^*$-action and vice versa. This gives the first isomorphism. The second isomorphism is the natural one which identifies $p_1(s)$ with

$d_{L^\vee}(f_s)$, where s is any section of L, f_s is the function on $L^\vee$ corresponding to s and $d_{L^\vee}$ is the exterior derivative on $L^\vee$. Under some local trivialization of $L^\vee$ over U, it sends $(f, \omega) \in \mathcal{P}^1_X(L) \cong \mathcal{O}_X(U) \oplus \Omega_X(U)$ to $f(x) d_{L^\vee} t + \omega(x) t \in \Omega^{\mathbb{C}^*}_{L^\vee}$.

Let

$$e: \ 0 \longrightarrow J \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0$$

be a small extension of local Artinian $\mathbb{C}$-algebras with $m_{\widetilde{A}} \cdot J = 0$. Suppose we have a flat deformation $(\mathcal{X}, \mathcal{L})$ of the pair (X, L) over $Spec(A)$:

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ Spec(\mathbb{C}) & \longrightarrow & Spec(A) \end{array}$$

Let $(\widetilde{\mathcal{X}}_\alpha, \widetilde{\mathcal{L}}_\alpha)$ and $(\widetilde{\mathcal{X}}_\beta, \widetilde{\mathcal{L}}_\beta)$ be two liftings of $(\mathcal{X}, \mathcal{L})$ to $Spec(\widetilde{A})$. We would like to measure the difference of two such liftings.

Let us restrict ourselves to the local situation first. Suppose that $\mathcal{X}$ is affine, embedded in $\mathcal{S} = Spec(A[x_1, \ldots, x_n])$, and with the total space of $\widetilde{\mathcal{L}}^\vee_i$ is embedded into $Spec(\widetilde{A}[x_1, \ldots, x_n]) \times \mathbb{A}^1 = \widetilde{\mathcal{S}} \times \mathbb{A}^1$ with image $\widetilde{\mathcal{X}}_i \times \mathbb{A}^1$.

Let $\mathcal{I}_0$ be the ideal sheaf of $L^\vee$ in $S \times \mathbb{A}^1$. The conormal sequence

$$0 \longrightarrow \frac{\mathcal{I}_0}{\mathcal{I}_0^2} \xrightarrow{d} \Omega_{S \times \mathbb{A}^1}|_{L^\vee} \longrightarrow \Omega_{L^\vee} \longrightarrow 0$$

is exact because $L^\vee$ is l.c.i. Taking the invariant part under the $\mathbb{C}^*$-action we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*} \xrightarrow{d'} (\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \longrightarrow \Omega^{\mathbb{C}^*}_{L^\vee} \longrightarrow 0. \tag{4.3}$$

The difference of $\widetilde{\mathcal{L}}^\vee_\alpha$ and $\widetilde{\mathcal{L}}^\vee_\beta$ as embedded deformations corresponds to an $\mathcal{O}_{L^\vee}$-module homomorphism $v_{\alpha\beta} : \frac{\mathcal{I}_0}{\mathcal{I}_0^2} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$. The fact that $\widetilde{\mathcal{L}}^\vee_i$ is embedded as $\widetilde{\mathcal{X}}_i \times \mathbb{A}^1$ implies that $v_{\alpha\beta}$ sends the invariant part $\left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*}$ to the invariant part $J \otimes_{\mathbb{C}} \mathcal{O}^{\mathbb{C}^*}_{L^\vee} = J \otimes_{\mathbb{C}} L$. Denote the restriction $v'_{\alpha\beta}$.

Now, take the push-out of (4.3) under $v'_{\alpha\beta}$. We obtain an $\mathcal{O}_X$-module extension $\mathcal{E}_{\alpha\beta}$ of $\mathcal{P}^1_X(L)$ by $J \otimes_{\mathbb{C}} L$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(\frac{\mathcal{I}_0}{\mathcal{I}_0^2}\right)^{\mathbb{C}^*} & \xrightarrow{d'} & (\Omega_{S \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} & \longrightarrow & \Omega^{\mathbb{C}^*}_{L^\vee} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v'_{\alpha\beta}} & & \downarrow{\scriptstyle \psi_{\alpha\beta}} & & \| & & \\ 0 & \longrightarrow & J \otimes_{\mathbb{C}} L & \xrightarrow{l_{\alpha\beta}} & \mathcal{E}_{\alpha\beta} & \longrightarrow & \mathcal{P}^1_X(L) & \longrightarrow & 0 \end{array} \tag{4.4}$$

Lemma 4.1. *The extension $\mathcal{E}_{\alpha\beta}$ does not depend on the choice of $\widetilde{\mathcal{S}}$.*

Proof. Suppose there are embeddings $\widetilde{\mathcal{L}}^\vee_i \to \widetilde{\mathcal{S}}_j \times \mathbb{A}^1$, where $i = \alpha, \beta$, $j = 1, 2$, reducing to embeddings $\mathcal{L}^\vee \to \mathcal{S}_1 \times \mathbb{A}^1$ and $\mathcal{L}^\vee \to \mathcal{S}_2 \times \mathbb{A}^1$. These induce embeddings

$$\widetilde{\mathcal{L}}^\vee_i \to \widetilde{\mathcal{S}}_1 \times_{Spec(\widetilde{A})} \widetilde{\mathcal{S}}_2 \times \mathbb{A}^1$$

reducing to

$$\mathcal{L}^\vee \to \mathcal{S}_1 \times_{Spec(A)} \mathcal{S}_2 \times \mathbb{A}^1.$$

Let C_1, C_2, C_{12} be the conormal bundles of $L^\vee$ in $S_1 \times \mathbb{A}^1$, $S_2 \times \mathbb{A}^1$, $S_1 \times S_2 \times \mathbb{A}^1$ respectively. Denote by $v'_j : C_j^{\mathbb{C}^*} \to J \otimes_{\mathbb{C}} L$ the invariant part of the corresponding sections of the normal bundles, $\mathcal{E}_j = v'_{j*}(\Omega_{S_j \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$, $\mathcal{E}_{12} = v'_{12*}(\Omega_{S_1 \times S_2 \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$, and let $p_j : C_j^{\mathbb{C}^*} \to C_{12}^{\mathbb{C}^*}$ be the natural map between conormal bundles. Then

$$v'_{12} \circ p_j = v'_j : C_j^{\mathbb{C}^*} \to J \otimes_{\mathbb{C}} L.$$

We have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_j^{\mathbb{C}^*} & \longrightarrow & (\Omega_{S_j \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} & \longrightarrow & \Omega_{L^\vee}^{\mathbb{C}^*} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_j} & & \downarrow & & \| & & \\
0 & \longrightarrow & C_{12}^{\mathbb{C}^*} & \longrightarrow & (\Omega_{S_1 \times S_2 \times \mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} & \longrightarrow & \Omega_{L^\vee}^{\mathbb{C}^*} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v'_{12}} & & \downarrow & & \| & & \\
0 & \longrightarrow & J \otimes_{\mathbb{C}} L & \longrightarrow & \mathcal{E}_{12} & \longrightarrow & \Omega_{L^\vee}^{\mathbb{C}^*} & \longrightarrow & 0
\end{array}$$

By the universal property of push-out, this diagram induces isomorphism of extensions $\psi_j : \mathcal{E}_j \cong \mathcal{E}_{12}$. We define the canonical isomorphism between $\mathcal{E}_2$ and $\mathcal{E}_1$ to be $\psi_1 \circ \psi_2^{-1}$. □

Proposition 4.2. *For any two liftings of line bundles $\widetilde{\mathcal{L}}_\alpha^\vee$, $\widetilde{\mathcal{L}}_\beta^\vee$ inside $\widetilde{\mathcal{S}} \times \mathbb{A}^1$ as above, there is an $\mathcal{O}_X$-module extension $\mathcal{E}_{\alpha\beta}$ of $\mathcal{P}_X^1(L)$ by $J \otimes_{\mathbb{C}} L$, well defined up to canonical isomorphism, with the following properties:*

(a) *For any three liftings $\widetilde{\mathcal{L}}_\alpha^\vee$, $\widetilde{\mathcal{L}}_\beta^\vee$, and $\widetilde{\mathcal{L}}_\gamma^\vee$, there is a canonical isomorphism of extensions*

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \cong \mathcal{E}_{\alpha\gamma}$$ [1]

such that for any four liftings,

$$F_{\alpha\gamma\delta} \circ (F_{\alpha\beta\gamma} + id_{\mathcal{E}_{\beta\gamma}}) = F_{\alpha\beta\delta} \circ (id_{\mathcal{E}_{\alpha\beta}} + F_{\beta\gamma\delta}) \tag{4.5}$$

as homomorphism of extensions from $\mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} + \mathcal{E}_{\gamma\delta}$ to $\mathcal{E}_{\alpha\delta}$.

(b) *Given an $\mathcal{O}_X$-module extension $\mathcal{E}$ of $\mathcal{P}_X^1(L)$ by $J \otimes_{\mathbb{C}} L$, and a lifting $\widetilde{\mathcal{L}}_\alpha^\vee$ of $\mathcal{L}^\vee$, there is an abstract lifting $\widetilde{\mathcal{L}}_\beta^\vee$ such that $\mathcal{E}_{\alpha\beta}$ is isomorphic to $\mathcal{E}$.*

(c) *There is a natural bijection between bundle isomorphisms $\Phi : \widetilde{\mathcal{L}}_\alpha^\vee \cong \widetilde{\mathcal{L}}_\beta^\vee$ with splittings of $\mathcal{E}_{\alpha\beta}$.*

---

[1] The sum of two extensions of $\mathcal{O}_X$-module $0 \longrightarrow \mathcal{G} \xrightarrow{l_i} \mathcal{E}_i \xrightarrow{k_i} \mathcal{F} \longrightarrow 0$ is defined to be the quotient of the submodule $\mathcal{B} = \{(e_1, e_2) \in \mathcal{E}_1 \oplus \mathcal{E}_2 : \ k_1(e_1) = k_2(e_2)\}$ by sections of the form $(l_1(y), -l_2(y))$, $y \in \mathcal{G}$.

The opposite extension $-\mathcal{E}$ is defined to be $0 \longrightarrow \mathcal{G} \xrightarrow{-l} \mathcal{E} \xrightarrow{k} \mathcal{F} \longrightarrow 0$.

Proof. (a) As embedded deformations, we certainly have $v'_{\alpha\beta} + v'_{\beta\gamma} = v'_{\alpha\gamma}$ as homomorphisms from $(\frac{\mathcal{I}_0}{\mathcal{I}_0^2})^{\mathbb{C}^*}$ to $J \otimes_{\mathbb{C}} L$. Then $\mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma}$ fits into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\frac{\mathcal{I}_0}{\mathcal{I}_0^2})^{\mathbb{C}^*} & \xrightarrow{d'} & (\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} & \longrightarrow & \Omega_{L^\vee}^{\mathbb{C}^*} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle v'_{\alpha\beta}+v'_{\beta\gamma}} & & \downarrow{\scriptstyle (\psi_{\alpha\beta},\psi_{\beta\gamma})} & & \| & & \\ 0 & \longrightarrow & J\otimes_{\mathbb{C}} L & \longrightarrow & \mathcal{E}_{\alpha\beta}+\mathcal{E}_{\beta\gamma} & \longrightarrow & \mathcal{P}_X^1(L) & \longrightarrow & 0 \end{array}$$

By the universal property of push-out, there is a unique isomorphism $F_{\alpha\beta\gamma}^{-1} : \mathcal{E}_{\alpha\gamma} \to \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma}$ such that $(\psi_{\alpha\beta}, \psi_{\beta\gamma})$ factors through $F_{\alpha\beta\gamma}^{-1}$. The compatibility condition (4.5) follows from the universal property of push-out as well.

(b) Applying the derived functor $\mathrm{Hom}_{\mathcal{O}_X}(-, J \otimes_{\mathbb{C}} L)$ to (4.3), we obtain the exact sequence

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}((\tfrac{\mathcal{I}_0}{\mathcal{I}_0^2})^{\mathbb{C}^*}, J \otimes_{\mathbb{C}} L) &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}_X^1(L), J \otimes_{\mathbb{C}} L) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}((\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}, J \otimes_{\mathbb{C}} L), \end{aligned}$$

where the last term is zero because X is affine and $(\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$ is locally free. Thus for any

$$\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}_X^1(L), J \otimes_{\mathbb{C}} L),$$

there is

$$v' \in \mathrm{Hom}_{\mathcal{O}_X}((\frac{\mathcal{I}_0}{\mathcal{I}_0^2})^{\mathbb{C}^*}, J \otimes_{\mathbb{C}} L)$$

such that $v'_*(\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \cong \mathcal{E}$. v' can be uniquely extended to a $\mathcal{O}_{L^\vee}$-module homomorphism $v : \frac{\mathcal{I}_0}{\mathcal{I}_0^2} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$. Now choose $\widetilde{\mathcal{L}}_\beta^\vee \subset \widetilde{\mathcal{S}} \times \mathbb{A}^1$ such that the difference of $\widetilde{\mathcal{L}}_\beta^\vee$ and $\widetilde{\mathcal{L}}_\alpha^\vee$ as embedded deformations corresponds to v, then by construction $\mathcal{E}_{\alpha\beta} \cong \mathcal{E}$.

(c) First notice that by the construction of push-out, to give a splitting $s : \mathcal{E}_{\alpha\beta} \to J \otimes_{\mathbb{C}} L$ is equivalent to giving a $\mathcal{O}_X$-module homomorphism $D : (\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \to J \otimes_{\mathbb{C}} L$ such that $D \circ d' = v'_{\alpha\beta}$.

Now let $\phi : \mathcal{O}_{\widetilde{\mathcal{L}}_\alpha^\vee} \cong \mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee}$ be a bundle isomorphism inducing identity on $\mathcal{O}_{\mathcal{L}^\vee}$. Consider the two projections $\pi_i : \mathcal{O}_{\widetilde{\mathcal{S}}\times\mathbb{A}^1} \to \mathcal{O}_{\widetilde{\mathcal{L}}_i^\vee}$. The difference

$$D = \pi_\beta - \phi \circ \pi_\alpha : \mathcal{O}_{\widetilde{\mathcal{S}}\times\mathbb{A}^1} \to \mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee}$$

will have its image inside $J\mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee} = J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$. It is easy to check that

$$\begin{aligned} D &\in Der_{\widetilde{A}}(\mathcal{O}_{\widetilde{\mathcal{S}}\times\mathbb{A}^1}, J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}) \\ &= Der_{\mathbb{C}}(\mathcal{O}_{S\times\mathbb{A}^1}, J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}) \cong Hom_{\mathcal{O}_{L^\vee}}(\Omega_{S\times\mathbb{A}^1}|_{L^\vee}, J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}) \end{aligned}$$

and $D \circ d = v_{\alpha\beta}$. The fact that ϕ is a bundle isomorphism implies that D sends $(\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*}$ to $J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}^{\mathbb{C}^*} = J \otimes_{\mathbb{C}} L$. This gives a splitting of $\mathcal{E}_{\alpha\beta}$.

Conversely, any $\mathcal{O}_X$-module homomorphism

$$D : (\Omega_{S\times\mathbb{A}^1}|_{L^\vee})^{\mathbb{C}^*} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}^{\mathbb{C}^*}$$

with $D \circ d' = v'_{\alpha\beta}$ can be extended uniquely to an $\mathcal{O}_{L^\vee}$-module homomorphism $D : \Omega_{S\times\mathbb{A}^1}|_{L^\vee} \to J \otimes_{\mathbb{C}} \mathcal{O}_{L^\vee}$ with $D \circ d = v_{\alpha\beta}$. Now consider

$$\pi_\beta - D \circ d : \mathcal{O}_{\widetilde{\mathcal{S}}\times\mathbb{A}^1} \longrightarrow \mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee}.$$

If $\tilde{f}$ is a local function on $\widetilde{\mathcal{S}} \times \mathbb{A}^1$ which vanishes on $\widetilde{\mathcal{L}}_\alpha^\vee$ and f is its restriction to $S \times \mathbb{A}^1$, then

$$\pi_\beta(\tilde{f}) - (D \circ d)f = \pi_\beta(\tilde{f}) - v_{\alpha\beta}(f)$$

is zero in $\mathcal{O}_{\widetilde{\mathcal{L}}_\beta^\vee}$ by the construction of $v_{\alpha\beta}$.

Thus $\pi_\beta - (D \circ d)$ factors through π_α, and therefore we recover the bundle isomorphism ϕ from such D. □

Remark. Proposition 4.2 still holds in the global case. Since the local extension does not depend on the choice of embeddings, one can construct a global extension for any two abstract liftings $\widetilde{\mathcal{L}}_2^\vee$ and $\widetilde{\mathcal{L}}_1^\vee$ by glueing together the local extensions using the canonical isomorphisms in Lemma 4.1 on the overlap of two open affine subsets. One checks easily that the glued extension satisfies the properties in the proposition. We will not need the global case in the construction of the obstruction space. □

The rest of the proof is entirely based on the construction in [11]. The idea is to use extension cocycles to measure the obstructions to coherently patching together local liftings (which always exist since X is l.c.i.).

Here we collect some useful results about extension cocycles and refer to [11] for details.

Definition 4.3. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules and $\{U_\alpha\}$ be an open covering of X. An extension cocycle

$$(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$$

of $\mathcal{F}$ by $\mathcal{G}$ on $\{U_\alpha\}$ is a collection of extensions $\{\mathcal{E}_{\alpha\beta}\}$ of $\mathcal{F}|_{U_{\alpha\beta}}$ by $\mathcal{G}|_{U_{\alpha\beta}}$ and isomorphisms

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \cong \mathcal{E}_{\alpha\gamma}$$

on $U_{\alpha\beta\gamma}$ satisfying the compatibility condition as in (4.5).

Two extension cocycles $(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$, $(\{\mathcal{E}'_{\alpha\beta}\}, \{F'_{\alpha\beta\gamma}\})$ are isomorphic if there exists an isomorphism of extensions

$$\phi_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \cong \mathcal{E}'_{\alpha\beta}$$

such that

$$\phi_{\alpha\gamma} \circ F_{\alpha\beta\gamma} = F'_{\alpha\beta\gamma} \circ (\phi_{\alpha\beta} + \phi_{\beta\gamma}).$$

Definition 4.4. We say an extension cocycle is a boundary if it is isomorphic to

$$\partial\{\mathcal{E}_\alpha\} = (\{\mathcal{E}_\alpha - \mathcal{E}_\beta\}, F_{\alpha\beta\gamma})$$

for a collection of extensions $\{\mathcal{E}_\alpha\}$ of $\mathcal{F}|_{U_\alpha}$ by $\mathcal{G}|_{U_\alpha}$, where

$$F_{\alpha\beta\gamma} : \mathcal{E}_\alpha - \mathcal{E}_\beta + \mathcal{E}_\beta - \mathcal{E}_\gamma \longrightarrow \mathcal{E}_\alpha - \mathcal{E}_\gamma$$

is the obvious isomorphism.

The set of isomorphism classes of extension cocycles form an abelian group, and the boundaries form a subgroup. The quotient group is called the group of extension classes and is denoted by $\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G})$. We refer to section 3 in [11] for the proofs of the above facts.

The following theorem is taken from Theorem 3.13 of [11]. For the convenience of the reader, we sketch the proof here.

Theorem 4.5. *For $\{U_\alpha\}$ a good cover, there is canonical group isomorphism of $\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G})$ with the kernel of the localization map*

$$\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to H^0(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

Proof. Let $\mathcal{J}$ be an injective sheaf of $\mathcal{O}_X$-modules containing $\mathcal{G}$ and let $\mathcal{Q}$ be the quotient.

$$0 \longrightarrow \mathcal{G} \xrightarrow{\ j\ } \mathcal{J} \xrightarrow{\ \pi\ } \mathcal{Q} \longrightarrow 0.$$

Then the boundary map

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) \xrightarrow{\ \partial\ } \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is an isomorphism, and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) & \xrightarrow{\ \cong\ } & \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})) & \xrightarrow{\ \cong\ } & H^0(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \end{array}$$

where the vertical arrows are localization maps. Hence the kernel of the left column is isomorphic to the kernel of the right column. But from the local-to-global spectral sequence, we get an exact sequence

$$0 \longrightarrow H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) \longrightarrow H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})).$$

Thus theorem follows from Lemma 4.6. □

Lemma 4.6. *There is a canonical isomorphism*

$$\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G}) \cong \check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})).$$

Proof. Let $(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$ be an isomorphism class of extension cocycles of $\mathcal{F}$ by $\mathcal{G}$. Since $\mathcal{J}$ is injective, we can find a homomorphism $\sigma_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \to \mathcal{J}$ such that the diagram on $U_{\alpha\beta}$

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \mathcal{G} & \xrightarrow{\ j\ } & \mathcal{J} & \xrightarrow{\ \pi\ } & \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle l_{\alpha\beta}} & \nearrow{\scriptstyle \sigma_{\alpha\beta}} & & & & & \\ & & \mathcal{E}_{\alpha\beta} & & & & & & \\ & & \downarrow{\scriptstyle k_{\alpha\beta}} & & & & & & \\ & & \mathcal{F} & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

is commutative. We claim that we can do this coherently in the sense that

$$\sigma_{\alpha\beta} + \sigma_{\beta\gamma} = \sigma_{\alpha\gamma} \circ F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \longrightarrow \mathcal{J}, \tag{4.6}$$

where $\sigma_{\alpha\beta} + \sigma_{\beta\gamma}$ sends $(e_1, e_2) \in \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma}$ to $\sigma_{\alpha\beta}(e_1) + \sigma_{\alpha\beta}(e_2)$.

Choose arbitrary homomorphisms $\sigma_{\alpha\beta}$ such that $\sigma_{\alpha\beta} \circ l_{\alpha\beta} = j$, and consider

$$\tau_{\alpha\beta\gamma} = (\sigma_{\alpha\beta} + \sigma_{\beta\gamma}) - \sigma_{\alpha\gamma} \circ F_{\alpha\beta\gamma} : \mathcal{F} \longrightarrow \mathcal{J}.$$

It is easy to check that $\{\tau_{\alpha\beta\gamma}\}$ is a Čech 2-cocycle in $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J})$, since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J})$ is flasque and thus has no higher cohomology. Hence we can find 1-cochain $\{\tau_{\alpha\beta}\}$ such that $\tau_{\alpha\beta\gamma} = \tau_{\alpha\beta} - \tau_{\alpha\gamma} + \tau_{\beta\gamma}$. If we set

$$\widetilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} + \tau_{\alpha\beta}$$

we see easily that $\widetilde{\sigma}_{\alpha\beta}$ satisfies the coherence condition (4.6). The claim is proved.

Now we describe a homomorphism from

$$\Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{F}, \mathcal{G}) \text{ to } \check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$$

as below. Choose $\sigma_{\alpha\beta}$ satisfying the coherence condition (4.6). Then

$$\pi \circ \sigma_{\alpha\beta} : \mathcal{E}_{\alpha\beta} \to \mathcal{Q}$$

sends $\mathcal{G}$ to zero and therefore induces $\eta_{\alpha\beta} : \mathcal{F} \to \mathcal{Q}$ satisfying $\eta_{\alpha\beta} + \eta_{\beta\gamma} = \eta_{\alpha\gamma}$. So we have associated to the extension cocycle $(\{\mathcal{E}_{\alpha\beta}\}, \{F_{\alpha\beta\gamma}\})$ an element $\{\eta_{\alpha\beta}\} \in \check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$. This does not depend on the choice of $\sigma_{\alpha\beta}$. If $\{\eta_{\alpha\beta}\}$ represents zero class, let $\eta_{\alpha\beta} = \eta_\beta - \eta_\alpha$ and set $\mathcal{E}_\alpha = \eta_\alpha^* \mathcal{J}$. Here we view $\mathcal{J}$ as an extension of $\mathcal{Q}$ by $\mathcal{G}$. Since $\mathcal{E}_{\alpha\beta}$ is isomorphic to $\eta_{\alpha\beta}^* \mathcal{J}$, we conclude that $\mathcal{E}_{\alpha\beta}$ is isomorphic to the boundary extension cocyle $\{\mathcal{E}_\beta - \mathcal{E}_\alpha\}$. This gives the injectivity. For the surjectivity, let $\eta_{\alpha\beta}$ represent a class in $\check{H}^1(U_\alpha, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}))$. Set

$$\mathcal{E}_{\alpha\beta} = \eta_{\alpha\beta}^* \mathcal{J},$$

and

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} = \eta_{\alpha\beta}^* \mathcal{J} + \eta_{\beta\gamma}^* \mathcal{J} \cong (\eta_{\alpha\beta} + \eta_{\beta\gamma})^* \mathcal{J} = \eta_{\alpha\gamma}^* \mathcal{J} = \mathcal{E}_{\alpha\gamma}$$

is the natural homomorphism. □

To finish the construction of the obstruction class, we cover $\mathcal{X}$ by an open affine subscheme $\{\mathcal{U}_\alpha\}$ such that $\mathcal{L}_\alpha^\vee = \mathcal{L}^\vee|_{\mathcal{U}_\alpha}$ has a lifting $\widetilde{\mathcal{L}}_\alpha^\vee$ over $\widetilde{\mathcal{U}}_\alpha$. The difference of $\widetilde{\mathcal{L}}_\alpha^\vee$ and $\widetilde{\mathcal{L}}_\beta^\vee$ on the overlap corresponds to an extension $\mathcal{E}_{\alpha\beta}$ of $\mathcal{P}^1_{U_{\alpha\beta}}(L_{\alpha\beta})$ by $J \otimes_{\mathbb{C}} L_{\alpha\beta}$. For each triple α, β, γ, consider the isomorphism

$$F_{\alpha\beta\gamma} : \mathcal{E}_{\alpha\beta} + \mathcal{E}_{\beta\gamma} \cong \mathcal{E}_{\alpha\gamma}$$

in Proposition 4.2(a).

Then $(\mathcal{E}_{\alpha\beta}, F_{\alpha\beta\gamma})$ is an extension cocycle, which we will denote simply by $(\mathcal{E}_{\alpha\beta})$. If $\overline{\mathcal{L}}_\alpha^\vee$ is another collections of liftings, coresponding to another extension cocycle $(\mathcal{E}'_{\alpha\beta})$, we get isomorphisms

$$\mathcal{E}_{\alpha\beta} \cong \mathcal{E}'_{\alpha\beta} + \mathcal{E}(\widetilde{\mathcal{L}}_\alpha^\vee, \overline{\mathcal{L}}_\alpha^\vee) - \mathcal{E}(\widetilde{\mathcal{L}}_\beta^\vee, \overline{\mathcal{L}}_\beta^\vee)$$

by Proposition 4.2(a). One checks that this is an isomorphism of extension cocycles. Thus the class of

$$[\mathcal{E}_{\alpha\beta}] \in \Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{P}^1_X(L), J \otimes_{\mathbb{C}} L)$$

is independent of the choice of local liftings.

A global lifting exists if and only if we can choose local liftings $\widetilde{\mathcal{L}}_\alpha^\vee$ and isomorphisms of line bundles $\phi_{\alpha\beta} : \widetilde{\mathcal{L}}_\alpha^\vee \to \widetilde{\mathcal{L}}_\beta^\vee$ satisfying the cocycle condition

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}.$$

By Proposition 4.2(c), to give $\phi_{\alpha\beta}$ is equivalent to assigning splittings for $\mathcal{E}_{\alpha\beta}$. It is easy to check that $\phi_{\alpha\beta}$ satisfies the cocycle condition if and only if $(\mathcal{E}_{\alpha\beta})$ is isomorphic to the trivial extension cocycle.

Conversely, if the class

$$[\mathcal{E}_{\alpha\beta}] \in \Xi_{\mathcal{O}_X}(U_\alpha; \mathcal{P}^1_X(L), J \otimes_{\mathbb{C}} L)$$

is zero, $(\mathcal{E}_{\alpha\beta})$ is isomorphic to a boundary $(\mathcal{E}_\alpha - \mathcal{E}_\beta)$. By Proposition 4.2($b$), we can choose local lifting $\overline{\mathcal{L}}^\vee_\alpha$ such that $\mathcal{E}(\widetilde{\mathcal{L}}^\vee_\alpha, \overline{\mathcal{L}}^\vee_\alpha) \cong \mathcal{E}_\alpha$. Then $\overline{\mathcal{L}}^\vee_\alpha$ will patch together to give a global lifting.

Combine the above discussion with Theorem 4.5 and the fact that

$$\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L) = 0$$

(since (4.3) is a locally free resolution of $\mathcal{P}^1_X(L)$), and we get

Theorem 4.7. *Let X be an l.c.i scheme and L a line bundle on X. For any small extension*

$$e:\ 0 \longrightarrow J \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0$$

and any deformation $(\mathcal{X}, \mathcal{L})$ of (X, L) over A,

(a) *there is an element*

$$\circ(e) \in J \otimes_{\mathbb{C}} \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L)$$

such that $\circ(e) = 0$ if and only if a lifting $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ to $\widetilde{A}$ exists.

(b) *If a lifting exists, the set of isomorphism classes of liftings is a principal homogeneous space for the group*

$$J \otimes_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{P}^1_X(L), L).$$

Acknowledgements

The author would like to thank his advisor, Herb Clemens, for valuable suggestions and constant support and E. Sernesi and M. Manetti for pointing out a gap in the previous version of the paper.

References

[1] Arbarello, E, Cornalba, M.: Su una congettura di Petri. Comment. Math. Helvetici **56**, 1-38, 1981. MR615613 (82k:14029)

[2] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of Algebraic Curves, Volume I. Springer Grundlehren **267**, 1985. MR770932 (86h:14019)

[3] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Special Divisors on Algebraic Curves. Regional Algebraic Geometry Conference. Athens, Georgia, May 1979.

[4] Altmann, K., Christophersen, J.: Deforming Stanley-Reisner schemes. Math. Ann. **348** (2010), no. 3, 513–537. MR2677892

[5] Eisenbud, D., Harris, J.: Limit linear series: Basic theory. Invent. Math. **85**, 337-371, 1986. MR846932 (87k:14024)

[6] Harris, J.: Curves in projective space. Les Press de l'Université de Montréal, 1982. MR685427 (84g:14024)

[7] Harris, J., Morrison, I.: Moduli of Curves. Graduate Text in Mathematics **187**, Springer-Verlag, New York, 1998. MR1631825 (99g:14031)

[8] Illusie, L.: Complex Cotangent et Déformations I. Lecture Notes in Mathematics **239**, Springer-Verlag, 1971. MR0491680 (58:10886a)
[9] Illusie, L.: Complex Cotangent et Déformations II. Lecture Notes in Mathematics **283**, Springer-Verlag, 1972. MR0491681 (58:10886b)
[10] Sernesi, E.: Deformations of Algebraic Schemes. Springer Grundlehren **334**, 2006. MR2247603 (2008e:14011)
[11] Vistoli, A.: The deformation theory of local complete intersections. http://arxiv.org/abs/alg-geom/9703008

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210
E-mail address: `jwang@math.ohio-state.edu`