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CASTELNUOVO-MUMFORD REGULARITY AND REDUCTION NUMBER OF SMOOTH MONOMIAL CURVES IN $\mathbb{P}^5$

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Abstract. We will compute explicitly the Castelnuovo-Mumford regularity and the reduction number of coordinate rings of smooth projective monomial curves in $\mathbb{P}^5$. Moreover we will show that these numbers are equal.

1. Introduction

Let K be a field and $A := \{0, a_1, \ldots, a_d, \alpha\}$ a subset of $\mathbb{N}$, where $d \in \mathbb{N}^+$ and $0 < a_1 < \ldots < a_d < \alpha$ is a sequence of relatively prime integers. The ring $K[A] := K[t_0^\alpha, t_0^{\alpha-a_1}t_1^{a_1}, \ldots, t_0^{\alpha-a_d}t_1^{a_d}, t_1^\alpha] \subseteq K[t_0, t_1]$ is uniquely determined by A and isomorphic to the coordinate ring of a projective monomial curve of degree α in $\mathbb{P}^{d+1}$. By a result of Gruson-Lazarsfeld-Peskine [3] the Eisenbud-Goto conjecture [2] holds for projective monomial curves; i.e., the Castelnuovo-Mumford regularity $\mathrm{reg}K[A]$ is bounded by $\alpha - d$. In terms of gaps (see Definition 2.2) $\alpha - d = \sum(\#L) + 1$, where L runs over all gaps of A. As we see, the bound $\mathrm{reg}K[A] \leq \#L + \#L' + 1$ given by L′vovsky in [6, Proposition 5.5], where L and L' are the longest and the second longest gap of A, is better than $\alpha - d$ (see [4, Introduction]). It is well known that the projective monomial curve is smooth if and only if $a_1 = 1$ and $a_d = \alpha - 1$. For smooth projective monomial curves, Hellus-Hoa-Stückrad showed in [4, Theorem 2.7] that $\mathrm{reg}K[A]$ is bounded by $\#L+1$, where L is the longest gap of A.

In Section 3 we will compute explicitly the Castelnuovo-Mumford regularity $\mathrm{reg}K[A]$ and the reduction number $\mathrm{r}(K[A])$ (see Definition 2.9) of smooth projective monomial curves in $\mathbb{P}^5$. For $K[A] = K[t_0^\alpha, t_0^{\alpha-1}t_1, t_0^{\alpha-b}t_1^b, t_0^{\alpha-c}t_1^c, t_0t_1^{\alpha-1}, t_1^\alpha]$ defined by $A = \{0, 1, b, c, \alpha-1, \alpha\} \subseteq \mathbb{N}$ with $1 < b < c < \alpha - 1$ we will show (see Corollary 3.9):

$$\mathrm{reg}K[A] = \mathrm{r}(K[A]) = \max\left\{ \left\lfloor \tfrac{c}{b} \right\rfloor + b - 2, \left\lfloor \tfrac{\alpha-b}{\alpha-c} \right\rfloor + \alpha - c - 2 \right\}.$$

In view of [5, Theorem 1.1], the property $\mathrm{reg}K[A] = \mathrm{r}(K[A])$ is very interesting, since the Eisenbud-Goto conjecture holds in this situation by [5]. But even for smooth projective monomial curves $\mathrm{reg}K[A] = \mathrm{r}(K[A])$ does not hold in general. In $\mathbb{P}^{d+1}$ for $d = 2, 3$ we have equality (see [4, Section 3]); of course, we proved the latter property in $\mathbb{P}^5$. There are examples of smooth projective monomial curves in $\mathbb{P}^{d+1}$ with $d \geq 6$ and $\mathrm{r}(K[A]) < \mathrm{reg}K[A]$; one is mentioned in [4, Example 3.2]. Moreover our interest came from the fact that still the latest bound in [4] for smooth

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projective monomial curves is imprecise. If we take a look at an example, for the coordinate ring $K[A]$ defined by $A = \{0, 1, 100, 100100, 102101, 102102\}$ we get by [4] that $\text{reg}K[A] \leq 100000$. On the other hand we know by Corollary 3.9 that $\text{reg}K[A] = 2050$. In Section 2 we consider smooth projective monomial curves in $\mathbb{P}^{d+1}$. For unspecified notation, we refer to [1].

2. Smooth projective monomial curves in $\mathbb{P}^{d+1}$

In this section we assume that $K[A]$ is defined by $A := \{0, a_1, \dots, a_d, \alpha\}$ (see the Introduction) with $a_1 := 1$ and $a_d := \alpha - 1$; i.e., we consider smooth projective monomial curves. Letting $m \in \mathbb{N}^+$ we set $mA := A + \dots + A$ (m times). Then we have $mA \subseteq \{0, 1, 2, \dots, m\alpha - 1, m\alpha\} = [0, m\alpha] \cap \mathbb{N}$, and there exists an $m \in \mathbb{N}^+$ such that the equality $\#(mA) = m\alpha + 1$ holds; in the latter case we say that mA is full. We define $\text{reg}(A) := \min\{m \in \mathbb{N}^+ \mid mA \text{ is full}\}$. By [4, Corollary 2.5] and Lemma 2.8 we have

$$\text{reg}K[A] = \text{reg}(A). \tag{1}$$

As we see, it comes down to a pure combinatorial problem.

Definition 2.1. Let X be a nonempty finite subset of $\mathbb{N}$. We set $\hat{x} := \min X$ and $\overline{x} := \max X$. The set X is called an interval if $X = \{\hat{x}, \hat{x}+1, \dots, \overline{x}-1, \overline{x}\}$, i.e. $X = [\hat{x}, \overline{x}] \cap \mathbb{N}$. We denote such intervals in $\mathbb{N}$ by $[u, v]$ with $u, v \in \mathbb{N}, u \leq v$.

Definition 2.2. A gap L of X is a maximal interval contained in the complement $G_X := [\hat{x}, \overline{x}] \setminus X$ of X in $[\hat{x}, \overline{x}]$ and its length is $\#L$. By Λ_X we denote the set of gaps of X. The number $\lambda(X) := \max_{L \in \Lambda_X} \#L$ is defined as the length of a longest gap of X. We say that X is full if $G_X = \emptyset$, i.e., if X is an interval.

For instance $A = \{0, 1, 5, 9, 10\}$ has two gaps, i.e. $\{[2, 4] \cup [6, 8]\} = G_A$. Clearly if mA is full for $m \in \mathbb{N}^+$, then $(m+i)A$ is also full for every $i \in \mathbb{N}$.

Definition 2.3. For $t \in \mathbb{N}$, let $\kappa_t : [0, t] \to [0, t]$ be the map given by $\kappa_t(i) := t - i$, for all $i \in [0, t]$. Note that κ_t is an order-reversing bijection respecting intervals with $\kappa_t^2 = id_{[0,t]}$. For $m, i \in \mathbb{N}^+$ with $i \leq m$, we define $(mA)_i := [(i-1)\alpha, i\alpha] \cap mA$ and $\epsilon := \max\{r \in \mathbb{N} \mid \{[0, r] \cup [\alpha - r, \alpha]\} \subseteq A\}$.

As we see for smooth projective monomial curves, $\epsilon \geq 1$.

Remark 2.4. With the notation of Definition 2.3, we get that mA is full if and only if $(mA)_i$ are full for all $i \in \mathbb{N}$ with $1 \leq i \leq m$ and we have

$$[(i-1)\alpha, (i-1)\alpha + \epsilon] = [0, \epsilon] + (i-1)\alpha \subseteq (mA)_i,$$
$$[i\alpha - \epsilon, i\alpha] = [\alpha - \epsilon, \alpha] + (i-1)\alpha \subseteq (mA)_i.$$

Notation 2.5. We set $\hat{A} := \kappa_\alpha(A)$ and

$$A_P := \{(\alpha, 0), (\alpha - 1, 1), (\alpha - a_2, a_2), \dots, (\alpha - a_{d-1}, a_{d-1}), (1, \alpha - 1), (0, \alpha)\}.$$

Letting $m \in \mathbb{N}^+, i \in \mathbb{N}$ with $i \leq m\alpha$, we define $g_i^m := (m\alpha - i, i)$.

Remark 2.6. Let $m \in \mathbb{N}^+$. We set $mA_P := A_P + \dots + A_P$ (m times) with the usual addition of tuples. Let $\mu_m : mA \to mA_P$ be the map given by $\mu_m(i) := g_i^m$ for all $i \in mA$. Then μ_m is bijective and therefore $\mu_m(mA) = mA_P$.

Definition 2.7. We define $G_{mA_P} := \mu_m(G_{mA})$. A subset of G_{mA_P} is called a gap of mA_P if it is of the form $\mu_m(L)$ with a gap L of mA. We say that mA_P is full if $G_{mA_P} = \emptyset$ and we define $\operatorname{reg}(A_P) := \min\{m \in \mathbb{N}^+ \mid mA_P \text{ is full}\}$.

Lemma 2.8. *Let $m, i \in \mathbb{N}^+$ with $i \leq m$ and $u, v \in \mathbb{N}$ with $u \leq v$. Then*

(*i*) $[u, v] \subseteq mA$ *if and only if* $[m\alpha - v, m\alpha - u] \subseteq m\kappa_\alpha(A)$.

(*ii*) $(mA)_i$ *is full if and only if* $(m\hat{A})_{m+1-i}$ *is full.*

(*iii*) $\operatorname{reg}(A) = \operatorname{reg}(\hat{A}) = \operatorname{reg}(A_P)$.

Proof. (*i*) We have $\kappa_{m\alpha}(mA) = m\kappa_\alpha(A)$. Now (*i*) follows by the fact that $\kappa_{m\alpha}$ is bijective and $\kappa_{m\alpha}([u, v]) = [m\alpha - v, m\alpha - u]$.

(*ii*) Follows from (*i*) replacing $[u, v]$ by $[(i-1)\alpha, i\alpha]$.

(*iii*) By Definition 2.7 we have $\operatorname{reg}(A) = \operatorname{reg}(A_P)$, and now (*iii*) follows from (*ii*). □

Definition 2.9. By [4, Section 3] and the bijective map μ_m the reduction number $\operatorname{r}(K[A])$ of $K[A]$ can be computed by

$$\operatorname{r}(K[A]) := \min\left\{r \in \mathbb{N}^+ | (r+1)A = rA + \{0, \alpha\}\right\}.$$

Definition 2.10. We will call A saturated if either $\operatorname{reg}(A) = 1$ or $\operatorname{reg}(A) > 1$ and

$$G_{(\operatorname{reg}(A)-1)A} \cap \{[0, \alpha] \cup [(\operatorname{reg}(A) - 2)\alpha, (\operatorname{reg}(A) - 1)\alpha]\} \neq \emptyset.$$

Let us take a look at an example: for $K[A]$ defined by $A =$

$$\{0, 1, 3, 5, 11, 14, 18, 21, 25, 34, 44, 67, 90, 94, 101, 103, 108, 110, 111, 113, 115, 116\},$$

we have $\operatorname{r}(K[A]) = 3$ and $\operatorname{reg}(A) = 4$. One can check that $\{190\}$ is the last gap in $3A$, and therefore A is not saturated.

Proposition 2.11. *If A is saturated, then* $\operatorname{r}(K[A]) = \operatorname{reg}(A)$.

Proof. By definition we get that $1 \leq \operatorname{r}(K[A]) \leq \operatorname{reg}(A)$. We therefore may assume that $\operatorname{reg}(A) > 1$. Since A is saturated and by Remark 2.4, there exists an $i \notin (\operatorname{reg}(A) - 1)A$ with $i \in [2, \alpha - 2]$ or with $i \in [(\operatorname{reg}(A) - 2)\alpha + 2, (\operatorname{reg}(A) - 1)\alpha - 2]$ $(\epsilon \geq 1)$. By this we have

$$(\operatorname{reg}(A) - 1)A + \{0, \alpha\} \neq [0, \operatorname{reg}(A)\alpha] = (\operatorname{reg}(A) - 1 + 1)A.$$

Hence $(r+1)A \neq rA + \{0, \alpha\}$ for $r \in \mathbb{N}$ with $1 \leq r \leq \operatorname{reg}(A) - 1$, and therefore we get $\operatorname{r}(K[A]) > \operatorname{reg}(A) - 1$. □

Lemma 2.12. *Let $i, m \in \mathbb{N}^+$ with $i \leq m$ and $t \in \mathbb{N}$. If $\lambda((mA)_i) \leq t$, then*

$$\max\{\lambda(((m+1)A)_i), \lambda(((m+1)A)_{i+1})\} \leq \max\{t - \epsilon, 0\}.$$

Proof. Suppose on the contrary that

$$\max\{\lambda(((m+1)A)_i), \lambda(((m+1)A)_{i+1})\} > \max\{t - \epsilon, 0\};$$

i.e. there is a gap L of $((m+1)A)_i$ or there is a gap L of $((m+1)A)_{i+1}$ such that $\#L > \max\{t - \epsilon, 0\} \geq 0$, $L = [u, v]$ with $u, v \in \mathbb{N}, u \leq v$. In each of these cases we shall lead to a contradiction.

Case 1. $L \subseteq G_{((m+1)A)_{i+1}}$: In this case, since L is a gap of $((m+1)A)_{i+1}$ we get by Remark 2.4 that

$$i\alpha + \epsilon < u \leq v < (i+1)\alpha - \epsilon.$$

Now suppose that $j \in \{[u-\alpha, v-\alpha] \cap (mA)_i\} \neq \emptyset$. Then $j+\alpha \in ((m+1)A)_{i+1}$, a contradiction. Hence $[u-\alpha, v-\alpha] \subseteq G_{(mA)_i}$.

Now suppose that $\{[v-\alpha+1, v-\alpha+\epsilon] \cap (mA)_i\} \neq \emptyset$. Then one can show that $v \in ((m+1)A)_{i+1}$ (since $[\alpha-\epsilon, \alpha-1] \subseteq A$), a contradiction. Hence $[v-\alpha+1, v-\alpha+\epsilon] \subseteq G_{(mA)_i}$. This proves that $[u-\alpha, v-\alpha+\epsilon] \subseteq G_{(mA)_i}$ which contradicts $\lambda((mA)_i) \leq t$.

Case 2. $L \subseteq G_{((m+1)A)_i}$: By a similar argument, we get $[u-\epsilon, v] \subseteq G_{(mA)_i}$. ☐

Lemma 2.13. *Let $i, m \in \mathbb{N}^+, t \in \mathbb{N}$ with $i \leq m$ and $\lambda((mA)_i) \leq t$. Then the sets $((m+p)A)_i, \ldots, ((m+p)A)_{i+p}$ are full for $p = \left\lfloor \frac{t-1}{\epsilon} \right\rfloor + 1$ and for $p = t$.*

Proof. Using Lemma 2.12 by induction on $p \in \mathbb{N}$ one can show that

$$\max\{\lambda(((m+p)A)_i), \ldots, \lambda(((m+p)A)_{i+p})\} \leq \max\{t - p\epsilon, 0\}.$$

Hence we get the assertion for $p = t$. The case $p = \left\lfloor \frac{t-1}{\epsilon} \right\rfloor + 1$ follows from

$$t - \left(\left\lfloor \frac{t-1}{\epsilon} \right\rfloor + 1 \right) \epsilon < t - \left(\frac{t-1}{\epsilon} \right) \epsilon = 1. \tag{2}$$ ☐

Remark 2.14. Lemma 2.13 is a new proof of [4, Theorem 2.7]. Of course this lemma is more general and will be very useful in Section 3. With $\lambda((1A)_1) = \lambda(A)$ we get that $((1+q)A)_1, \ldots, ((1+q)A)_{1+q}$ are full, and therefore $(1+q)A$ is full for $q = \left\lfloor \frac{\lambda(A)-1}{\epsilon} \right\rfloor + 1$. Hence

$$\mathrm{reg} K[A] \overset{(1)}{=} \mathrm{reg}(A) \leq \left\lfloor \frac{\lambda(A)-1}{\epsilon} \right\rfloor + 2.$$

As a direct result, $\mathrm{reg} K[A]$ is also bounded by $\lambda(A)+1$.

It is known that the bound of [4, Theorem 2.7] is sharp for smooth projective monomial curves in $\mathbb{P}^{d+1}$ for $d = 2, 3$ (see [4, Section 3]). Now we will establish two basic combinatorial lemmas which will be useful to prove Theorem 3.1.

Notation 2.15. From now on we set $m := \left\lfloor \frac{a_3}{a_2} \right\rfloor$, $x := m + a_2 - 2$ and we assume that $d \geq 4$ (see Introduction).

Lemma 2.16. *If $(x-1) > 0$, then $(a_2 m - 1) \notin (x-1)A$.*

Proof. We have $a_2 m - 1 \leq a_2 \frac{a_3}{a_2} - 1 = a_3 - 1$. Suppose that $a_2 m - 1 \in (x-1)A$. Then there are $\beta_1, \beta_2 \in \mathbb{N}$ such that

$$a_2 m - 1 = \beta_1 + \beta_2 a_2 \tag{3}$$

and

$$\beta_1 + \beta_2 \leq x - 1 = m + a_2 - 3, \tag{4}$$

and we get $(a_2-1)(m-\beta_2-1) = a_2 m - a_2\beta_2 - a_2 - m + \beta_2 + 1 \overset{(3)}{=} \beta_1 + \beta_2 - m - a_2 + 2 \overset{(4)}{\leq} m + a_2 - 3 - m - a_2 + 2 < 0$. Therefore $\beta_2 \geq m$ and hence $a_2 m - 1 \overset{(3)}{=} \beta_1 + \beta_2 a_2 \geq a_2 m$, a contradiction. ☐

Lemma 2.17. *If $\lambda(A) = a_3 - a_2 - 1$, then $(xA)_1, \ldots, (xA)_{a_2-1}$ are full.*

Proof. We will now show that $\lambda((mA)_1) \leq a_2 - 2$ and we are done by Lemma 2.13. Let $i \in \mathbb{N}, 1 \leq i \leq d-1$. We have $\lambda(A) = a_3 - a_2 - 1$ and therefore

$$a_{i+1} - a_i \leq a_3 - a_2. \tag{5}$$

Fixing one i, we will show that there cannot be a bigger gap than $a_2 - 2$ between a_i and a_{i+1} in $(mA)_1$ and therefore $\lambda((mA)_1) \leq a_2 - 2$. Let $j \in \mathbb{N}, 1 \leq j \leq m$. Then

$$[a_i + (m-j)a_2, a_i + (m-j)a_2 + j - 1] \subseteq mA.$$

One can check that the distances between these intervals are smaller than or equal to $a_2 - 1$; i.e., the length of every gap until $a_i + (m-1)a_2$ for $j = 1$ is smaller than or equal to $a_2 - 2$. If $a_{i+1} \leq a_i + (m-1)a_2$, we are done. Let us assume $a_i + (m-1)a_2 < a_{i+1}$. Then we have to show that the distance between these numbers is smaller than or equal to $a_2 - 1$, but this is clear by (6):

$$a_2 - 1 - a_{i+1} + \underbrace{a_i}_{\geq a_{i+1} - a_3 + a_2} + (m-1)a_2 \overset{(5)}{\geq} \underbrace{(m+1)}_{> \frac{a_3}{a_2} - 1 + 1} a_2 - a_3 - 1 > -1. \tag{6}$$

□

3. Smooth projective monomial curves in $\mathbb{P}^5$

In this section we assume that $A := \{0, 1, b, c, \alpha - 1, \alpha\}$ with $A \subseteq \mathbb{N}$ and $1 < b < c < \alpha - 1$. We have $K[A] = K[t_0^\alpha, t_0^{\alpha-1}t_1, t_0^{\alpha-b}t_1^b, t_0^{\alpha-c}t_1^c, t_0 t_1^{\alpha-1}, t_1^\alpha]$, $\hat{A} = \{0, 1, \alpha - c, \alpha - b, \alpha - 1, \alpha\}$, $m = \left\lfloor \frac{c}{b} \right\rfloor$ and $x = m + b - 2$. Our aim is to show:

Theorem 3.1.

$$\operatorname{reg}(A) = \max\left\{ \left\lfloor \tfrac{c}{b} \right\rfloor + b - 2, \left\lfloor \tfrac{\alpha - b}{\alpha - c} \right\rfloor + \alpha - c - 2 \right\}. \tag{7}$$

Remark 3.2. We note that the symmetric case, i.e. $c = \alpha - b$, was proven in [4, Proposition 3.5]. Let us assume that A is full. Then $A = \{0, 1, 2, 3, 4, 5\}$ and we get that $\operatorname{reg}(A) = 1 = \max\{1, 1\}$, and hence (7) holds. Without loss of generality we therefore may assume that A has at least one gap, i.e. $\lambda(A) > 0$. By this we have $b > 2$ if $\lambda(A) = b - 2$ and $x - 1 > 0$ if $x \geq y$.

Notation 3.3. For the next lemmas we set $n := \left\lfloor \frac{\alpha - b}{\alpha - c} \right\rfloor, y := n + \alpha - c - 2$ and $i := x - y$.

Lemma 3.4. (*i*) *If* $\lambda(A) = b - 2$, *then* (7) *holds.*
(*ii*) *If* $\lambda(A) = \alpha - c - 2$, *then* (7) *holds.*

Proof. We will only prove (*i*). Then (*ii*) follows by replacing A by $\hat{A}$. It is clear that $\lambda((1A)_1) = \lambda(A) = b - 2$ and we get that $(b-1)A$ is full, by Lemma 2.13. Since $\lambda(A) = b - 2$ we have $b - 1 \geq c - b$, i.e. $2b - 1 \geq c$. Then

$$m = \left\lfloor \frac{c}{b} \right\rfloor = 1, \tag{8}$$

since $c > b$. Hence $x = m + b - 2 \overset{(8)}{=} b - 1$ and $(b-2)A$ is not full by Lemma 2.16. Therefore

$$\operatorname{reg}(A) = b - 1 = x.$$

Suppose that $x < y$. Then by Lemma 2.16,

$$(\alpha - c)n - 1 \notin (y-1)\hat{A}; \tag{9}$$

hence $\operatorname{reg}(A) \overset{2.8}{=} \operatorname{reg}(\hat{A}) \overset{(9)}{>} y - 1 \geq x$, a contradiction. This proves $x \geq y$. □

So we only have to consider the case $\lambda(A) = c-b-1$, or in other words $c-b \geq b-1$ and $c - b \geq \alpha - 1 - c$. In the following we will show that (7) holds for $x \geq y$, and this will finish the proof of Theorem 3.1, since otherwise one can replace A by $\hat{A}$.

Lemma 3.5. *If $\lambda(A) = c - b - 1$ and $x \geq y$, then $(xA)_1, \ldots, (xA)_{b-1}$ and $(xA)_n, \ldots, (xA)_x$ are full.*

Proof. By Lemma 2.17 we have the first assertion and by symmetry we get that

$$(y\hat{A})_1, \ldots, (y\hat{A})_{\alpha-c-1}$$

are full. Let $h \in \mathbb{N}, 0 \leq h \leq i = x - y$ and $j \in (y\hat{A})_{\alpha-c-1}$. Then $j + h\alpha \in x\hat{A}$; hence $(x\hat{A})_1, \ldots, (x\hat{A})_{\alpha-c-1+i}$ are full. Note that

$$x = y + i = n + \alpha - c - 2 + i \Longleftrightarrow x + 1 - n = \alpha - c - 1 + i. \tag{10}$$

By (10) this is equivalent to $(x\hat{A})_{x+1-x}, \ldots, (x\hat{A})_{x+1-n}$ being full, and therefore

$$(xA)_n, \ldots, (xA)_x$$

are full by Lemma 2.8. □

We will now show that $(xA)_\psi$ is full for $b \leq \psi \leq n-1$, since our goal is to prove that xA is full. Without loss of generality we therefore may assume that $b \leq n-1$. First of all we will give a construction of two intervals contained in xA; after this we will show that there cannot be a natural number between these intervals and therefore $(xA)_\psi$ have to be full.

Lemma 3.6. *If $\psi \in \mathbb{N}$, $b \leq \psi \leq n - 1$ and $x \geq y$, then*

$$\{[(\psi - 1)\alpha, (\psi - 1)\alpha + (x - \psi + 1)b] \cup [\psi c, \psi\alpha]\} \subseteq xA.$$

Proof. Letting $k \in \mathbb{N}$ with $0 \leq k \leq (x - \psi + 1)$, we get

$$(\psi - 1)\alpha + (x - \psi + 1 - k)b \in xA.$$

We will take an element $b^* = (x - \psi + 1 - k)b$ for one fixed k. Let $h \in \mathbb{N}$, with $0 \leq h \leq b - 1$. Then

$$\underbrace{(\psi - 1 - h)}_{\geq 0}\alpha + h(\alpha - 1) + b^* \in xA$$

since $\psi \geq b$. Therefore

$$[(\psi - 1)\alpha, (\psi - 1)\alpha + (x - \psi + 1)b] \subseteq xA.$$

Note that it is not necessary at this point to bound ψ by $\psi \leq n - 1$. The above argument also holds for $\psi \leq x$. By the same argument we have

$$[(\phi - 1)\alpha, (\phi - 1)\alpha + (x - \phi + 1)(\alpha - c)] \subseteq x\hat{A},$$

with $\phi \in \mathbb{N}, \alpha - c \leq \phi \leq x$. By Lemma 2.8 and $\kappa_{x\alpha}^2 = id_{[0,x\alpha]}$ we get

$$[(x-\phi+1)c, (x-\phi+1)\alpha] = \kappa_{x\alpha}([(\phi-1)\alpha, (\phi-1)\alpha+(x-\phi+1)(\alpha-c)]) \subseteq xA. \tag{11}$$

We know that $\alpha - c \leq \phi \leq x$ and ψ is bounded by $b \leq \psi \leq n - 1$. Let $\phi = \alpha - c$. Hence $x - \phi + 1 = n + \alpha - c - 2 + i - \alpha + c + 1 = n - 1 + i$, with $i = x - y \geq 0$. Let $\phi = x$. Hence $x - \phi + 1 = 1$. Therefore for all ψ there exists a ϕ with $x - \phi + 1 = \psi$ (since $b \geq 2$). Then by (11),

$$[\psi c, \psi\alpha] \subseteq xA.$$

□

Lemma 3.7. *If $\psi \in \mathbb{N}$, $b \leq \psi \leq n - 1$ and $x \geq y$, then $(xA)_\psi$ is full.*

Proof. We will show that

$$\psi c - ((\psi-1)\alpha + (x-\psi+1)b) =: z = -xb - \psi(\alpha - c - b) + \alpha - b < 2, \tag{12}$$

i.e. $z \leq 1$, since $z \in \mathbb{Z}$. This implies that

$$[(\psi-1)\alpha, \psi\alpha] \overset{(12)}{\subseteq} \{[(\psi-1)\alpha, (\psi-1)\alpha + (x-\psi+1)b] \cup [\psi c, \psi\alpha]\} \overset{3.6}{\subseteq} xA,$$

and hence $(xA)_\psi = [(\psi-1)\alpha, \psi\alpha] \cap xA = [(\psi-1)\alpha, \psi\alpha]$; i.e. $(xA)_\psi$ is full. To prove (12) we collect the following inequalities:

$$b \leq \psi \leq n-1, \tag{13}$$

$$n > \frac{\alpha - b}{\alpha - c} - 1, \quad m > \frac{c}{b} - 1, \tag{14}$$

$$m + b - 2 = x \geq y = n + \alpha - c - 2, \tag{15}$$

$$b \geq 2, \quad \alpha - c \geq 2. \tag{16}$$

Case 1. $(\alpha - c - b) > 0$:

$$\begin{aligned}
z &\overset{(13)}{\leq} -xb - b(\alpha - c - b) + \alpha - b \\
&= -b(m + b - 2 + \alpha - c - b + 1) + \alpha \\
&\overset{(14)}{<} -b(\tfrac{c}{b} + \alpha - c - 2) + \alpha \\
&= -b(\alpha - c) + \alpha - c + 2b \\
&= (1-b)(\alpha - c) + 2b \\
&\overset{(16)}{\leq} 2 - 2b + 2b = 2.
\end{aligned}$$

Case 2. $(\alpha - c - b) \leq 0$:

$$\begin{aligned}
z &\overset{(15)}{\leq} -b(n + \alpha - c - 2) + \psi(b + c - \alpha) + \alpha - b \\
&\overset{(13)}{\leq} -b(n-1) - b(\alpha - c - 1) + (n-1)(b + c - \alpha) + \alpha - b \\
&= (n-1)(c - \alpha) - b(\alpha - c - 1) + \alpha - b \\
&\overset{(14)}{<} (\tfrac{\alpha - b}{\alpha - c} - 2)(c - \alpha) + b(c - \alpha) + b + \alpha - b \\
&= (c - \alpha)(b - 2) + b \\
&\overset{(16)}{\leq} -2b + 4 + b \\
&\overset{(16)}{\leq} 2.
\end{aligned}$$

□

Proof of Theorem 3.1.

If $\lambda(A) = b - 2$ or $\lambda(A) = \alpha - c - 2$, the assertion follows from Lemma 3.4.

If $\lambda(A) = c - b - 1$ and $x \geq y$, then by Lemmas 3.5 and 3.7, $(xA)_1, \ldots, (xA)_x$ are full. Therefore we get that xA is full. By Lemma 2.16, $(x-1)A$ is not full and hence

$$\operatorname{reg}(A) = x.$$

Similarly for $y \geq x$ and $\lambda(A) = \lambda(\hat{A}) = c - b - 1$, replace A by $\hat{A}$. □

Corollary 3.8. *A is saturated.*

Proof. Without loss of generality assume that $\operatorname{reg}(A) > 1$. If $\operatorname{reg}(A) = x$, then A is saturated by Lemma 2.16. If $\operatorname{reg}(A) = y$, then $\hat{A}$ is saturated by Lemma 2.16. Hence A is saturated by Lemma 2.8 and we are done. □

Corollary 3.9.

$$\operatorname{reg} K[A] = \operatorname{r}(K[A]) = \max\left\{\left\lfloor \tfrac{c}{b} \right\rfloor + b - 2, \left\lfloor \tfrac{\alpha-b}{\alpha-c} \right\rfloor + \alpha - c - 2\right\}.$$

Proof.
$$\max\left\{\left\lfloor \tfrac{c}{b} \right\rfloor + b - 2, \left\lfloor \tfrac{\alpha-b}{\alpha-c} \right\rfloor + \alpha - c - 2\right\} \overset{3.1}{=} \operatorname{reg}(A) \underset{2.11}{\overset{3.8}{=}} \operatorname{r}(K[A]) = \operatorname{reg}(A) \overset{(1)}{=} \operatorname{reg} K[A].$$

□

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