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THE ZEROS OF CERTAIN LOMMEL FUNCTIONS

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ABSTRACT. Lommel's function $s_{\mu,\nu}(z)$ is a particular solution of the differential equation $z^2y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}$. Here we present estimates and monotonicity properties of the positive zeros of $s_{\mu-1/2,1/2}(z)$ when $\mu \in (0,1)$. The positivity of a closely related integral is also considered.

1. INTRODUCTION

Lommel's function [10, Sec. 10.7]

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2}z)^{2n}\Gamma((\mu-\nu+1)/2)\Gamma((\mu+\nu+1)/2)}{\Gamma((\mu-\nu+2n+3)/2)\Gamma((\mu+\nu+2n+3)/2)}$$

is a particular solution of the inhomogeneous Bessel equation

$$z^2y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}$$

and occurs in several places in physics and engineering (see [3] for a list of references). In [9] Steinig examined the sign of $s_{\mu,\nu}(z)$ for positive z and $\mu,\ \nu \in \mathbb{R}$. Among other things, he showed that for $\mu < \frac{1}{2}$ the function $s_{\mu,\nu}(z)$ has infinitely many changes of sign on $(0,\infty)$. Here we present the following refinement of this result. In this paper the zeros of analytic functions will always be counted according to their multiplicities and $\mathbb{N}$ will denote the set of positive integers.

Theorem 1.1. *For $\mu \in (0,1)$ and $n \in \mathbb{N}$ set*

$$I_{2n-1}(\mu) := \left(\left(2n-1+\frac{\mu}{2}\right)\pi, (2n-1+\mu)\,\pi\right]$$

and

$$I_{2n}(\mu) := \left(2n\pi, \left(2n+\frac{\mu}{2}\right)\pi\right).$$

Then, for $\mu \in (0,1)$ and positive z, $s_{\mu-1/2,1/2}(z) = 0$ implies $z \in I_n(\mu)$ for an $n \in \mathbb{N}$ and $s_{\mu-1/2,1/2}(z)$ has exactly one zero in every interval $I_n(\mu)$.

Hence, for each $n \in \mathbb{N}$ and $\mu \in (0,1)$ there is a $z_n := z_n(\mu) \in I_n$ such that $s_{\mu-1/2,1/2}(z_n) = 0$, and this z_n satisfies $s'_{\mu-1/2,1/2}(z_n) \neq 0$. The implicit function theorem therefore implies that each z_n is an analytic function of $\mu \in (0,1)$.

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Theorem 1.2. *Set* $w_n := w_n(\mu) := z_n - n\pi$*. For* $\mu \in (0,1)$ *we have* $\lim_{n\to\infty} w_n = \mu\pi/2$ *and*

$$w_{2n} < w_{2n+2} < \frac{\mu\pi}{2} < w_{2n+1} < w_{2n-1}, \quad n \in \mathbb{N}. \tag{1.1}$$

Furthermore, for $n \in \mathbb{N}$ *the functions* z_{2n-1} *are strictly increasing in* $(0,1)$*.*

Proposition 1.4 below shows that z_{2n} is not monotonic in μ for any $n \in \mathbb{N}$.

Our proofs of Theorems 1.1 and 1.2 will be based on the fact that for μ, $\nu \in \mathbb{C}$ with $\Re(\mu \pm \nu + 1) > 0$ and $z \in \mathbb{C}^- := \mathbb{C} \setminus \{z : z \le 0\}$, one has

$$s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_\nu(z)\int_0^z t^\mu J_\nu(t)\,dt - J_\nu(z)\int_0^z t^\mu Y_\nu(t)\,dt\right], \tag{1.2}$$

where $J_\nu(z)$ and $Y_n(z)$ are the usual Bessel functions [10, Sec. 10.7]. It is well-known that

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z \quad \text{and} \quad Y_{1/2}(z) = -\sqrt{\frac{2}{\pi z}}\cos z,$$

and therefore we obtain from (1.2) that for $\mu > 0$,

$$s_{\mu-1/2,1/2}(z) = \frac{1}{z^{1/2}}\int_0^z t^{\mu-1}\sin(z-t)\,dt.$$

Theorem 1.1 thus takes the following equivalent form.

Theorem 1.3. *For all* $\mu \in (0,1)$ *the function*

$$F_\mu(z) := \int_0^z t^{\mu-1}\sin(z-t)\,dt$$

has exactly one zero in each interval $I_n(\mu)$*,* $n \in \mathbb{N}$*, and vanishes nowhere else on the positive real axis.*

Obviously, $F_\mu(z)$ is analytic for $\Re\mu > 0$ and $z \in \mathbb{C}^-$. Partial integration shows that

$$F_\mu(z) = \frac{1}{\mu}\int_0^z t^\mu \cos(z-t)\,dt =: \frac{1}{\mu}G_\mu(z). \tag{1.3}$$

The following proposition is thus clear.

Proposition 1.4. *For* $n \in \mathbb{N}$ *we have* $z_n \to n\pi$ *as* $\mu \to 0$ *and* $z_{2n-1}(1) = z_{2n}(1) = 2n\pi$*.*

Our interest in the Lommel functions $s_{\mu-1/2,1/2}(z)$ stems from the fact that the function $F_\mu(z)$ plays an important role in the theory of positive trigonometric sums.

In [11, V.2.29] it is shown that there is exactly one $\alpha^* \in (0,1)$ such that $F_{1-\alpha^*}(\frac{3\pi}{2})$ vanishes. In [2] it is proven that α^* is such that for all $\alpha \ge \alpha^*$, $n \in \mathbb{N}$, and $x \in (0,\pi)$,

$$\tau_n(\alpha,x) := 1 + \sum_{k=1}^n \frac{\cos(kx)}{k^\alpha} > 0,$$

while for $\alpha < \alpha^*$ the cosine polynomials $\tau_n(\alpha,x)$ do not have a uniform lower bound on $(0,\pi)$. Nowadays, $\alpha^* = 0.30844\ldots$ is called the Littlewood-Salem-Izumi constant [1]. See also [4] and [6] for related results and considerations.

In [7] it is claimed that for each $\rho \in (0,1)$ there is exactly one $\mu^*(\rho) \in (0,1)$ such that

$$F_{\mu^*(\rho)}((\rho+1)\pi) = 0. \tag{1.4}$$

It is then shown there that the thus defined function $\mu^*(\rho)$ plays an important role in a conjecture concerning the mapping properties of partial sums of certain univalent functions. In [5] it is proven that this conjecture is equivalent to the fact that for $n \in \mathbb{N}$, $\rho \in (0,1)$, and $\theta \in (0,\pi)$ one has

$$\sum_{k=0}^{n} \frac{(\mu)_k}{k!} \sin[(2k+\rho)\theta] > 0$$

for all $\mu \in (0, \mu^*(\rho)]$ but for no $\mu \in (\mu^*(\rho), 1)$. Here, $(\mu)_k := \mu(\mu+1)\cdots(\mu+k-1)$ is the Pochhammer symbol.

It seems that in [7] the proof of the well-definition of $\mu^*(\rho)$ is not completely correct. In order to prove that for each $\rho \in (0,1)$ there is exactly one $\mu^*(\rho) \in (0,1)$ such that (1.4) holds, it is claimed in the proof of [7, Lem. 1] that for all ρ, $\mu \in (0,1)$,

$$I(\rho,\mu) := \frac{d}{d\mu} F_\mu((\rho+1)\pi) = \int_0^{(\rho+1)\pi} t^{\mu-1} \sin((\rho+1)\pi - t) \log(t)\, dt > 0.$$

But for $\rho = 0$ it is easy to see that

$$I(0,\mu) = \int_0^{\pi} t^{\mu-1} \sin(t) \log(t)\, dt$$

takes a negative value when $\mu = 0$. Hence, because of continuity, there is an $\epsilon > 0$ such that for all $\mu \in (0,\epsilon)$ there is a $\rho > 0$ for which $I(\rho,\mu) < 0$.

Despite this, the function $\mu^*(\rho)$ is well-defined, as we can show here. It follows readily from Theorem 1.2 and Proposition 1.4 that z_1 is a strictly increasing analytic function from $(0,1)$ onto $(\pi, 2\pi)$. Hence, $\mu^*(\rho)$ is well-defined, and we have $\mu^*(\rho) = z_1^{(-1)}((\rho+1)\pi)$. The next proposition is thus clear.

Proposition 1.5. *For each $\rho \in (0,1]$ there is exactly one $\mu^*(\rho) \in (0,1]$ such that $F_{\mu^*(\rho)}((\rho+1)\pi) = 0$. $\mu^*(\rho)$ is strictly increasing and analytic in $(0,1)$.*

In order to prove our results we will first show a weak form of Theorem 1.3 in the next section. In Section 3 we will then present the proofs of Theorems 1.2 and 1.3. For the proof of Theorem 1.2 we will need some information concerning the set of $(\rho,\mu) \in (0,1)^2$ for which $I(\rho,\mu)$ is positive. The relevant results will be presented in Section 4.

2. A weak form of Theorem 1.3

In this section we will prove the following lemma.

Lemma 2.1. *For $\mu \in (0,1)$ the function $F_\mu(z)$ has exactly one zero in every interval $(n\pi, (n+1)\pi)$, $n \in \mathbb{N}$, and vanishes nowhere else on the positive axis.*

First, note that, since $t^{\mu-1}$ is decreasing on $(0,\infty)$ when $\mu \in [0,1)$, obviously

$$\int_0^a t^{\mu-1} \sin(t) dt > 0 \quad \text{when} \quad a > 0 \text{ and } \mu \in [0,1). \tag{2.1}$$

The next lemma is the key result for the proof of Lemma 2.1.

Lemma 2.2. *Suppose $\mu \in (0,1)$ and $z^* > 0$ are such that $F_\mu(z^*) = 0$. Then*

$$F'_\mu(z^*) = \int_0^{z^*} t^{\mu-1} \cos(z^* - t)\, dt \neq 0.$$

Proof. If both $F_\mu(z^*)$ and $F'_\mu(z^*)$ would vanish, then so would

$$F_\mu(z^*) + iF'_\mu(z^*) = e^{iz^*} \int_0^{z^*} t^{\mu-1} e^{-it}\, dt.$$

In particular,

$$\int_0^{z^*} t^{\mu-1} \sin(t)\, dt = 0$$

would hold. Since this contradicts (2.1), the proof is complete. □

It follows from (2.1) that $F_\mu(n\pi) \neq 0$ and

$$\operatorname{sgn} F_\mu(n\pi) = (-1)^{n+1} \quad \text{for} \quad \mu \in (0,1),\ n \in \mathbb{N}. \tag{2.2}$$

Furthermore, it is easy to see that $F_\mu(z)$ is positive for $z \in (0,\pi)$ and $\mu \in (0,1)$.

Now, in order to verify the remaining statements of Lemma 2.1, let $n \in \mathbb{N}$ and $\mu^* \in (0,1)$ and suppose that F_{μ^*} has exactly $m \in \{1,2,3,\ldots\}$ zeros in the interval $I := (n\pi, (n+1)\pi)$. Then, because of Lemma 2.2, F_{μ^*} has m simple zeros $x_1,\ldots,x_m$ in I and therefore there are m open subintervals $J_1,\ldots,J_m$ of $(0,1)$, all containing μ^*, and m differentiable functions $x_k(\mu)$, $k \in \{1,\ldots,m\}$, that satisfy $x_k(\mu^*) = x_k$ and $F_\mu(x_k(\mu)) = 0$ for $\mu \in J_k$. Since $F_\mu(n\pi) \neq 0$ for all $\mu \in (0,1)$ and $n \in \mathbb{N}$, $x_k(\mu)$ lies in I for all $\mu \in J_k$ and $k \in \{1,\ldots,m\}$.

If $k \in \{1,\ldots,m\}$ and $\sigma \in (0,1)$ is a boundary point of J_k, then $\lim_{\mu\to\sigma} x_k(\mu)$ exists. Otherwise, because $x_k(\mu)$ is continuous and satisfies $x_k(\mu) \in I$ for $\mu \in J_k$, there is an open non-empty subinterval I^* of I such that for every $z \in I^*$ the set $x_k^{(-1)}(z)$ contains an infinite number of points that accumulate at σ. Consequently, $F_\sigma(z) = 0$ for all $z \in I^*$, a contradiction.

We can therefore assume that $J_k = (0,1)$ for all $k \in \{1,\ldots,m\}$. Otherwise, because of what we have just shown, one interval J_k would have a boundary point $\sigma \in (0,1)$ for which

$$F_\sigma\left(\lim_{\mu\to\sigma} x_k(\mu)\right) = 0 = F'_\sigma\left(\lim_{\mu\to\sigma} x_k(\mu)\right).$$

This would contradict Lemma 2.2.

We have thus proven that if F_μ has exactly m zeros in $(0,1)$ for one $\mu \in (0,1)$, then F_μ has at least m zeros in $(0,1)$ for all $\mu \in (0,1)$. This readily implies that if F_μ has exactly m zeros in $(0,1)$ for one $\mu \in (0,1)$, then F_μ has exactly m zeros in $(0,1)$ for all $\mu \in (0,1)$.

In order to complete the proof of Lemma 2.1 we will now determine a $\mu \in (0,1)$ for which F_μ has exactly one zero in I.

To that end, observe that, because of (1.3), F_μ vanishes if, and only if, G_μ vanishes. For positive z the function $G_0(z) = \sin(z)$ vanishes exactly at the points $n\pi$, $n \in \mathbb{N}$, and satisfies $G'_0(n\pi) = (-1)^n$. Therefore for every $n \in \mathbb{N}$ there is a differentiable function $z_n(\mu)$, defined in an open real neighborhood U_n of the origin, such that $z_n(0) = n\pi$, $G_\mu(z_n(\mu)) = 0$ for all $\mu \in U_n$, and

$$z'_n(0) = -\frac{\frac{d}{d\mu} G_\mu(z_n(0))|_{\mu=0}}{G'_0(z_n(0))} = (-1)^{n+1} \int_0^{n\pi} \cos(n\pi - t) \log(t)\, dt. \tag{2.3}$$

Partial integration gives

$$\int_0^{n\pi} \cos(n\pi - t)\log(t)\,dt = (-1)^{n+1}\int_0^{n\pi} \frac{\sin(t)}{t}\,dt.$$

The integral on the right-hand side of this equation is positive because of (2.1), and thus it follows from (2.3) and Hurwitz's theorem that for every $n \in \mathbb{N}$ there is a $\mu_n > 0$ such that for $0 < \mu < \mu_n$ the functions G_μ and F_μ have exactly one zero in $(n\pi, (n+1)\pi)$. The proof of Lemma 2.1 is complete.

3. Proof of Theorems 1.2 and 1.3

The following result is an easy consequence of Lemma 2.1 and (2.2).

Proposition 3.1. *For a $z \in (n\pi, (n+1)\pi)$ the relations*

$$F_\mu(z) > 0 \quad \textit{and} \quad F_\mu(z) < 0$$

hold if, and only if, $z < z_n$ and $z > z_n$, respectively, in the case where n is odd, and if, and only if, $z > z_n$ and $z < z_n$, respectively, in the case where n is even.

Now, let $n \in \mathbb{N}$, $n \geq 3$. Then

$$\begin{aligned} 0 &= \int_0^{z_n} t^{\mu-1}\sin(z_n - t)\,dt \\ &= \int_0^{z_n-2\pi} t^{\mu-1}\sin(z_n - t)\,dt + \int_{z_n-2\pi}^{z_n} t^{\mu-1}\sin(z_n - t)\,dt. \end{aligned}$$

Since $t^{\mu-1}$ is decreasing on $(0,\infty)$, it is clear that the integral on the right-hand side of the sum is negative, and therefore

$$\int_0^{z_n-2\pi} t^{\mu-1}\sin(z_n - 2\pi - t)\,dt > 0.$$

Hence, it follows from Proposition 3.1 that

$$z_n - 2\pi < z_{n-2} \quad \text{or} \quad z_n - 2\pi > z_{n-2},$$

depending on whether n is odd or even.

We have thus shown that, for fixed $\mu \in (0,1)$, the sequence w_{2n-1} is strictly decreasing, while w_{2n} is strictly increasing. Consequently, since $w_n \in (0,\pi)$ for all $n \in \mathbb{N}$, $w_e = \lim_{n\to\infty} w_{2n}$ and $w_o = \lim_{n\to\infty} w_{2n-1}$ exist.

In order to prove that $w_e = w_o = \mu\pi/2$, recall that the generalized sine and cosine integrals are defined by

$$\mathrm{Si}(z,\mu) := \int_0^z t^{\mu-1}\sin(t)\,dt \quad \text{and} \quad \mathrm{Ci}(z,\mu) := \int_0^z t^{\mu-1}\cos(t)\,dt,$$

respectively, and satisfy

$$\lim_{z\to\infty} \mathrm{Si}(z,\mu) = \sin\left(\frac{\mu\pi}{2}\right)\Gamma(\mu) \quad \text{and} \quad \lim_{z\to\infty} \mathrm{Ci}(z,\mu) = \cos\left(\frac{\mu\pi}{2}\right)\Gamma(\mu) \tag{3.1}$$

for z tending to ∞ on the real axis and $\mu \in (0,1)$ [8]. For all $n \in \mathbb{N}$ we have

$$0 = (-1)^n F_\mu(z_n) = \sin(w_n)\,\mathrm{Ci}(z_n,\mu) - \cos(w_n)\,\mathrm{Si}(z_n,\mu).$$

Letting $n \to \infty$, we obtain from (3.1) that both for $w = w_e$ and $w = w_o$,

$$0 = \Gamma(\mu)\sin\left(w - \frac{\mu\pi}{2}\right),$$

and thus that $w_e = w_o = \mu\pi/2$.

Next, observe that, according to the implicit function theorem,

$$z_n' = -\frac{\frac{d}{d\sigma}F_\sigma(z_n)|_{\sigma=\mu}}{F_\mu'(z_n)}.$$

Since $F_\mu'(z_n)$ does not change sign in $(0,1)$, it follows from Proposition 3.1 that in order to prove that z_{2n-1} is increasing in $(0,1)$, it will suffice to show

$$\frac{d}{d\sigma}F_\sigma(z_{2n-1})\Big|_{\sigma=\mu} = \int_0^{z_{2n-1}} t^{\mu-1}\log(t)\sin(t-w_{2n-1})\,dt > 0 \tag{3.2}$$

for $\mu \in (0,1)$.

To that end, note first that

$$\int_0^{z_1} t^{\mu-1}\log(t)\sin(t-w_1)\,dt = \int_0^{w_1} t^{\mu-1}\log(t)\sin(t-w_1)\,dt + \int_{w_1}^{z_1} t^{\mu-1}\log(t)\sin(t-w_1)\,dt.$$

Since $t^{\mu-1}\sin(t-w_1) < 0$ and $t^{\mu-1}\sin(t-w_1) > 0$ for $t \in (0,w_1)$ and $t \in (w_1, z_1)$, respectively, we therefore obtain

$$\int_0^{z_1} t^{\mu-1}\log(t)\sin(t-w_1)\,dt > \log(w_1)F_\mu(z_1) = 0.$$

We have thus shown that z_1 is a strictly increasing and analytic function in $(0,1)$. As explained in the paragraph preceding Proposition 1.5, this implies that the function $\mu^*(\rho)$ is well-defined. We can therefore make use of the inequality

$$\rho \le \mu^*(\rho), \quad \rho \in (0,1), \tag{3.3}$$

which was established in [7].

Now, before we show (3.2) also for $n = 2, 3, \ldots$, we will first complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Because of Proposition 3.1 and what we have shown so far, only the inequality $F_\mu((2n-1+\mu)\pi) \le 0$, $n \in \mathbb{N}$, remains to be verified. In fact, since

$$\begin{aligned} F_\mu\left((2n-1+\mu)\pi\right) &= F_\mu((1+\mu)\pi) + \int_{(1+\mu)\pi}^{(2n-1+\mu)\pi} t^{\mu-1}\sin(t-\mu\pi)\,dt \\ &\le F_\mu((1+\mu)\pi), \end{aligned}$$

it suffices to prove that $F_\mu((1+\mu)\pi) \le 0$ or, equivalently, that $z_1 \le (1+\mu)\pi$. However, since $\mu^*(\rho) = z_1^{(-1)}((\rho+1)\pi)$, the latter is equivalent to (3.3). □

Proof of Theorem 1.2. It remains to verify (3.2) for $n \in \{2, 3, \ldots\}$. Because of (3.1) we have

$$\int_0^\infty t^{\mu-1}\sin(t-x)\,dt = \Gamma(\mu)\sin\left(\frac{\mu\pi}{2} - x\right), \quad x \in (0,\pi),$$

and thus

$$\int_0^\infty t^{\mu-1}\log(t)\sin(t-x)\,dt = \Gamma'(\mu)\sin\left(\frac{\mu\pi}{2} - x\right) + \frac{\pi}{2}\Gamma(\mu)\cos\left(\frac{\mu\pi}{2} - x\right).$$

Since by Theorem 1.3 $\mu\pi/2 < w_{2n-1} \le \mu\pi$, we find that

$$J_n := \int_0^\infty t^{\mu-1} \log(t) \sin(t - w_{2n-1})\, dt > 0, \quad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ define the sequence

$$D_{n,k} := \int_{\max\{0, w_{2n-1}+(2k-3)\pi\}}^{w_{2n-1}+(2k-1)\pi} t^{\mu-1} \log(t) \sin(t - w_{2n-1})\, dt, \quad k \in \mathbb{N}.$$

Then

$$\sum_{k=1}^{\infty} D_{n,k} = J_n > 0$$

and

$$\sum_{k=1}^{n} D_{n,k} = \int_0^{z_{2n-1}} t^{\mu-1} \log(t) \sin(t - w_{2n-1})\, dt.$$

The proof of the theorem will therefore be complete if we can show that for each $n \in \{2, 3, \dots\}$ there is a $k(n) \in \mathbb{N}$ such that $D_{n,k} \ge 0$ for $k \in \{1, \dots, k(n)\}$ and $D_{n,k} \le 0$ for $k > k(n)$.

It is easy to check that for every $\mu \in (0,1)$ there is a $t_\mu > e$ such that $t^{\mu-1}\log(t)$ is positive and increasing in $(1, t_\mu)$ and positive and decreasing in (t_μ, ∞). Let $k^* \in \{1, 2, 3, \dots\}$ be such that $t_\mu \in [w_{2n-1} + (2k^*-3)\pi, w_{2n-1} + (2k^*-1)\pi)$. Then it is clear that $D_{n,k} > 0$ for $k \in \{2, \dots, k^*-1\}$ and $D_{n,k} < 0$ for $k > k^*$. We therefore set $k(n) = k^*$ if $D_{n,k^*} \ge 0$ and $k(n) = k^* - 1$ otherwise.

$D_{n,1} > 0$ remains to be verified. But since $\mu\pi/2 < w_{2n-1} \le \mu\pi$ for $\mu \in (0,1)$, this follows readily from Theorem 4.1 below. □

We have thus shown that for $n \in \mathbb{N}$ the functions z_{2n-1} are strictly increasing in $(0,1)$. While the functions z_{2n} are not monotonic in $(0,1)$ for any $n \in \mathbb{N}$ (cf. Proposition 1.4), numerical computation strongly supports the conjecture that the functions z_{2n-1} and z_{2n}, $n \in \mathbb{N}$, are convex and concave, respectively, in $(0,1)$. This, however, seems quite hard to prove.

4. On the sign of $I(\rho, \mu)$

Recall that

$$I(\rho,\mu) := \frac{d}{d\mu} F_\mu((\rho+1)\pi) = \int_0^{(\rho+1)\pi} t^{\mu-1} \sin(t - \rho\pi) \log(t)\, dt.$$

In this section we will determine a large subset of $(\rho,\mu) \in (0,1)^2$ for which $I(\rho,\mu)$ is positive.

To that end, observe first that the function

$$p(\mu) := \int_0^\pi t^{\mu-1}\, \log(t)\, \sin(t)\, dt$$

is strictly increasing on $(0,1)$ and satisfies $p(0) = -0.53\dots$ and $p(1) = 0.64\dots$. Consequently, the equation $p(\mu) = 0$ has a unique solution μ_0 in $(0,1)$. The numerical value of μ_0 is $\mu_0 = 0.32\dots$. Further, let M be the union of the three sets

$$\begin{aligned} M_1 &:= \{(\rho,\mu) : (10\pi)^{-1} \le \rho \le 1, 0 \le \mu \le 1\}, \\ M_2 &:= \{(\rho,\mu) : 0 < \rho < (10\pi)^{-1}, \mu_0 \le \mu \le 1\}, \\ M_3 &:= \{(\rho,\mu) : 0 < \mu \le 2\rho \le (5\pi)^{-1}\}. \end{aligned}$$

We will show the following.

Theorem 4.1. *For all* $(\rho,\mu)\in M$ *we have* $I(\rho,\mu)>0$. *To each* $\mu\in(0,\mu_0)$ *there is a* $\delta_\mu>0$ *such that* $I(\rho,\mu)<0$ *for* $\rho\in(0,\delta_\mu)$.

The second statement of this theorem has essentially been shown in the introduction. Hence, it only remains to verify that $I(\rho,\mu)>0$ for $(\rho,\mu)\in M$. To that end, we need some auxiliary lemmas. For $x\in[0,\pi]$ define

$$f(x):=\int_0^x \log(t)\sin(t-x)\,dt \quad \text{and} \quad g(x):=\int_0^\pi \frac{\log(t+x)}{t+x}\,\sin(t)\,dt.$$

Lemma 4.2. *For* $x\in(0,1)$ *we have* $f(x)>0$.

Proof. If $x\in(0,1)$, then clearly

$$f(x)\ge \log(x)\int_0^x \sin(t-x)\,dt = -\log(x)\,(1-\cos(x))>0.$$

□

Lemma 4.3. *The function* $f(x)$ *is concave on* $(1,\,\pi)$.

Proof. We recall that

$$\mathrm{Si}(x)=\int_0^x \frac{\sin(t)}{t}\,dt=\frac{\pi}{2}-\int_x^\infty \frac{\sin(t)}{t}\,dt,$$
$$\mathrm{Ci}(x)=-\int_x^\infty \frac{\cos(t)}{t}\,dt=\gamma+\log(x)-\int_0^x \frac{1-\cos(t)}{t}\,dt,$$

where γ is Euler's constant.

It is then easy to see that

$$\int_0^x \log(t)\,\cos(t)\,dt=\log(x)\,\sin(x)-\mathrm{Si}(x) \tag{4.1}$$

and

$$\int_0^x \log(t)\,\sin(t)\,dt=-\log(x)\,\cos(x)+\mathrm{Ci}(x)-\gamma. \tag{4.2}$$

From these, the relation

$$f(x)=-\log(x)+\cos(x)\,\big(\mathrm{Ci}(x)-\gamma\big)+\sin(x)\,\mathrm{Si}(x)$$

follows. We observe also that

$$f''(x)=-f(x)-\log(x).$$

Therefore

$$f''(x)=\cos(x)\,\big(-\mathrm{Ci}(x)+\gamma\big)-\sin(x)\,\mathrm{Si}(x). \tag{4.3}$$

It is clear that the function $w(x):=-\mathrm{Ci}(x)+\gamma$ is strictly decreasing on $(0,\,\pi/2)$ and strictly increasing on $(\pi/2,\,\pi)$; hence $w(x)\ge w(\pi/2)=0.1...>0$. Since $\mathrm{Si}(x)>0$ for all $x>0$, it readily follows from (4.3) that $f''(x)<0$ for $\pi/2\le x\le\pi$. On the other hand, for $1\le x<\pi/2$ we have

$$f''(x)\le\big(-\mathrm{Ci}(1)+\gamma\big)\cos(x)-\mathrm{Si}(1)\,\sin(x)<0.$$

□

Lemma 4.4. *The function* $g(x)$ *is concave on* $(0,\pi)$.

Proof. For $x \in (0,\pi)$ we have

$$g^{(4)}(x) = \int_0^\pi \frac{24\,\log(t+x) - 50}{(t+x)^5}\,\sin(t)\,dt \le \big(24\,\log(2\pi) - 50\big)\int_0^\pi \frac{\sin(t)}{(t+x)^5}\,dt < 0.$$

This implies $g'''(x) > g'''(\pi) = 0.011\ldots > 0$, which, in turn, gives $g''(x) < g''(\pi) = -0.0012\ldots < 0$ for $x \in (0,\pi)$. □

We will now prove Theorem 4.1 separately for the three sets M_1, M_2, and M_3. Note that

$$I(\rho,\mu) = \int_0^{\rho\pi} t^{\mu-1}\,\log(t)\sin\big(t-\rho\pi\big)\,dt + \int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1}\,\log(t)\sin\big(t-\rho\pi\big)\,dt. \tag{4.4}$$

Proof of Theorem 4.1 *for* $(\rho,\mu) \in M_1$. It follows readily from (4.4) that for $(\rho,\mu) \in M_1$,

$$I(\rho,\mu) \ge f(\rho\pi) + g(\rho\pi).$$

It will therefore be enough to prove

$$f(x) + g(x) > 0 \quad \text{for} \quad x \in [1/10,\pi]. \tag{4.5}$$

For $x \in [\frac{1}{10},1]$ we use Lemmas 4.2 and 4.4 to obtain

$$f(x) + g(x) > g(x) > \min\{g(1/10),\, g(1)\} = g(1/10) = 0.0819\ldots > 0.$$

For $x \in (1,\pi]$ Lemmas 4.3 and 4.4 give

$$f(x) + g(x) > \min\{f(1) + g(1), f(\pi) + g(\pi)\} = f(\pi) + g(\pi) = 0.0169\ldots > 0.$$

□

Proof of Theorem 4.1 *for* $(\rho,\mu) \in M_2$. Observe that for $\rho \in (0,\frac{1}{10\pi})$,

$$\int_0^{\rho\pi} t^{\mu-1}\,\log(t)\sin\big(t-\rho\pi\big)\,dt > 0.$$

Equation (4.4) therefore implies

$$\begin{aligned} I(\rho,\mu) &> \int_{\rho\pi}^{(\rho+1)\pi} t^{\mu-1}\,\log(t)\sin\big(t-\rho\pi\big)\,dt = \int_0^\pi (t+\rho\pi)^{\mu-1}\,\log(t+\rho\pi)\sin(t)\,dt \\ &\ge \int_0^\pi (t+\rho\pi)^{\mu_0-1}\,\log(t+\rho\pi)\sin(t)\,dt \ge \int_0^\pi t^{\mu_0-1}\,\log(t)\sin(t)\,dt = 0 \end{aligned}$$

for $\rho \in [0,\frac{1}{10\pi})$. Here we have made use of the fact that, for all $t \in (0,\pi)$, $(t+\rho\pi)^{\mu-1}\log(t+\rho\pi)$ is an increasing function of μ when $\rho \in [0,\frac{1}{10\pi})$ and an increasing function of ρ when $\mu = \mu_0$. □

Proof of Theorem 4.1 *for* $(\rho,\mu) \in M_3$. Since $\rho \in [\frac{\mu}{2},\frac{1}{10\pi}]$, we have

$$\int_0^{(\rho+1)\pi} t^{\mu-1}\log(t)\sin(t-\rho\pi)\,dt > \int_0^1 t^{\mu-1}\log(t)\sin(t-\rho\pi)\,dt.$$

The integral on the right-hand side of this inequality can be written as

$$\cos(\rho\pi)\int_0^1 t^{\mu-1}\log(t)\sin(t)\,dt-\sin(\rho\pi)\int_0^1 t^{\mu-1}\log(t)\cos(t)\,dt$$

and is thus larger than

$$\cos\left(\frac{\mu\pi}{2}\right)\int_0^1 t^{\mu-1}\log(t)\sin(t)\,dt-\sin\left(\frac{\mu\pi}{2}\right)\int_0^1 t^{\mu-1}\log(t)\cos(t)\,dt. \tag{4.6}$$

Applying the estimates $\sin(t)<t$ and $\cos(t)>1-2t/\pi$, $t\in(0,\pi/2)$, and calculating the resulting integrals, we find that the term in (4.6) is larger than

$$\frac{2\mu\pi\sin\frac{\mu\pi}{2}-\pi\mu^2\cos\frac{\mu\pi}{2}+\sin\frac{\mu\pi}{2}(\mu^2(\pi-2)+\pi)}{\pi\mu^2(\mu+1)^2}.$$

It is easy to check that $2\sin\frac{\mu\pi}{2}>\mu\cos\frac{\mu\pi}{2}$ for $\mu\in(0,\frac{1}{5\pi}]$, and therefore the proof is complete. ☐

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