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STRICTLY CONVEX NORMS, G_δ-DIAGONALS AND NON-GRUENHAGE SPACES

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Abstract. We present an example in ZFC of a locally compact, scattered Hausdorff non-Gruenhage space D having a G_δ-diagonal. This means that Gruenhage spaces are not necessary for the construction of strictly convex dual norms on dual Banach spaces, answering a question posed by Orihuela, Troyanski and the author. In addition, we show that the Banach space of continuous functions $C_0(D)$ admits a C^∞-smooth bump function.

1. Introduction

All topological spaces considered in this paper will be Hausdorff. Recall that a norm on a Banach space is *strictly convex* if every element of its unit sphere is an extreme point of the unit ball. The authors of [7] introduced the following topological property to help understand the nature of strictly convex norms.

Definition 1.1 ([7])**.** We say that a topological space X has $(*)$ if there exists a sequence $(\mathscr{U}_n)_{n=1}^\infty$ of families of open subsets of X, with the property that given any $x, y \in X$, there exists $n \in \mathbb{N}$ such that

(1) $\{x, y\} \cap \bigcup \mathscr{U}_n$ is non-empty, and
(2) $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{U}_n$.

If $(\mathscr{U}_n)_{n=1}^\infty$ satisfies properties (1) and (2) of Definition 1.1, then we will call it a *$(*)$-sequence*. This notion can be regarded as a "point-separation" property, in the sense that it specifies in advance a family of open sets which can separate pairs of distinct points in a controlled way. It generalizes the extensively studied *G_δ-diagonal* property.

Definition 1.2. A space X has a G_δ-diagonal if its diagonal is a G_δ set in X^2 or, equivalently, if there is a sequence $(\mathscr{G}_n)_{n=1}^\infty$ of open covers of X, such that given $x, y \in X$, there exists n with the property that $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{G}_n$.

See [1, Section 2] for a comprehensive introduction to spaces with G_δ-diagonals. All spaces having a G_δ-diagonal have $(*)$, and if L is a locally compact space having $(*)$, then so does its 1-point compactification $L \cup \{\infty\}$: simply adjoin to any $(*)$-sequence for L the singleton family $\{L\}$, which separates all points in L from ∞.

While compact spaces having G_δ-diagonals are metrizable (cf. [1, Theorem 2.13]), compact spaces having $(*)$ can be highly non-metrizable. The next definition

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presents another way in which points can be separated by a family of open sets, over which we have some control.

Definition 1.3 (cf. [2, p. 372])**.** A topological space X is called *Gruenhage* if there exists a sequence $(\mathscr{U}_n)_{n=1}^\infty$ of families of open subsets of X, and sets R_n, $n \geq 1$, with the property that

(1) if $x, y \in X$, then there exists $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$ such that $\{x, y\} \cap U$ is a singleton, and
(2) $U \cap V = R_n$ whenever $U, V \in \mathscr{U}_n$ are distinct.

Any Gruenhage space has $(*)$, and there are plenty of compact non-metrizable Gruenhage spaces [7, Section 4].

The relevance of spaces having $(*)$ to the geometry of Banach spaces is partly explained by the next result. Recall that a topological space is *scattered* if every non-empty subset admits a relatively isolated point.

Theorem 1.4 ([7])**.** *Let K be a scattered compact space. Then $C(K)$ admits an equivalent norm with a strictly convex dual norm if and only if K has $(*)$. Moreover, the norm can be a lattice norm.*

Here, $C(K)$ denotes the Banach space of continuous real-valued functions on K. Gruenhage spaces were introduced in [2] for reasons other than Banach space geometry, but have found a place in this field nonetheless.

Theorem 1.5 ([8, Theorem 7])**.** *If K is Gruenhage compact, then $C(K)$ admits an equivalent lattice norm with a strictly convex dual norm.*

Theorem 1.4 is not a consequence of Theorem 1.5. Under additional axioms, there are scattered compact, non-Gruenhage spaces having $(*)$ [7, Section 4]. The purpose of this paper is to show that such an example exists in ZFC. For more information about how these and related classes of topological spaces fit into Banach space theory, we refer the reader to [9, 7].

It turns out that if X has cardinality at most the continuum $\mathfrak{c}$, then we have a much more straightforward description of Gruenhage's property available, which we will put to use in the next section.

Proposition 1.6 ([9, Proposition 2])**.** *Let X be a topological space with* $\operatorname{card} X \leq \mathfrak{c}$. *Then X is Gruenhage if and only if there is a sequence $(U_n)_{n=1}^\infty$ of open subsets of X with the property that if $x, y \in X$, then $\{x, y\} \cap U_n$ is a singleton for some n.*

The basic idea behind the example remains the same as that in [7, Section 4]. We take a topological space X of cardinality $\mathfrak{c}$ and endow a "duplicate" $D = X \times \{1, -1\}$ with a new topology, the basic open sets of which use the existing structure of X to "oscillate" rapidly between the levels $+1$ and -1. This oscillation induces a non-trivial interaction between the levels and will make it difficult to separate all the "problem pairs" of the form $(x, 1), (x, -1)$, $x \in X$, in the manner of Proposition 1.6. This will render the space non-Gruenhage. However, at the same time, we will construct D delicately enough to ensure that we do not lose the other properties that we want it to have.

We shall use a particular tree in its standard interval topology as our starting point. Before proceeding with the construction, we should point out that trees by themselves cannot furnish us with a desired example. If Υ is any tree which is Hausdorff in its standard interval topology, then Υ is Gruenhage if and only if it has $(*)$. In particular, Υ has a G_δ-diagonal in its standard interval topology if and only if it is $\mathbb{R}$-*embeddable* [4], which in turn means that it is certainly Gruenhage. See [7, Section 4] for more details.

2. The Λ-duplicate

Recall that a *tree* is a partially ordered set $(\Upsilon, \preccurlyeq)$ with the property that, given any $t \in \Upsilon$, the set of predecessors $(0,t] = \{s \in \Upsilon \,:\, s \preccurlyeq t\}$ is well ordered. For convenience, we shall regard 0 as an extra element, not in Υ, such that $0 \prec t$ for all $t \in \Upsilon$. Trees are natural generalizations of ordinal numbers. We will use standard interval notation throughout this paper. For instance $(r,t]$, where $t \in \Upsilon$ and $r \in \Upsilon \cup \{0\}$, is the set of all $s \in \Upsilon$ satisfying $r \prec s \preccurlyeq t$. Other intervals such as $[r,t)$ are defined accordingly. For further notation and details about trees, we refer the reader to e.g. [6].

The tree in question was first considered by Kurepa. Denoted by Λ in [6, Section 10], Kurepa's tree is the set of injective functions $t : \alpha \longrightarrow \omega$ with (countable) ordinal domain and coinfinite range, and where $s \preccurlyeq t$ if and only if t extends s. We shall regard functions in the usual set-theoretic sense, that is, as sets of ordered pairs, and with $\operatorname{dom} f$ and $\operatorname{ran} f$ the domain and range of a function f, respectively. In this paper, we treat 0 as distinct from the empty function $\varnothing$, the latter being the least element of Λ.

A subset $A \subseteq \Lambda$ is an *antichain* if no two distinct elements of A are comparable in the tree order. We shall define Λ^+ to be the set of elements of Λ with successor ordinal domain. It is widely known and easy to show that $\{\varnothing\} \cup \Lambda^+$ can be written as a countable union of antichains: if

$$A_0 = \{\varnothing\} \quad \text{and} \quad A_{n+1} = \left\{t \in \Lambda^+ \,:\, t(\operatorname{dom} t - 1) = n\right\},$$

then each A_n is an antichain and $\{\varnothing\} \cup \Lambda^+ = \bigcup_{n=0}^\infty A_n$.

The underlying set of our example is $D = \Lambda \times \{1, -1\}$. We set up a function τ on pairs (s,t), $s \in \Lambda \cup \{0\}$, $t \in \Lambda$, $s \preccurlyeq t$, which will be central to the definition of our topology on D. Given $s \prec t$, we define a finite sequence of ordinals $\operatorname{dom} t = \beta_0 > \beta_1 > \beta_2 > \cdots > \beta_k = \operatorname{dom} s$. Given $\beta_i > \operatorname{dom} s$, let $\beta_{i+1} \in [\operatorname{dom} s, \beta_i)$ be the unique ordinal with the property that

$$t(\beta_{i+1}) \le t(\xi) \qquad \text{for all } \xi \in [\operatorname{dom} s, \beta_i).$$

As the β_i are strictly decreasing, this process eventually stops at some finite stage $k > 0$, with $\beta_k = \operatorname{dom} s$. Let

$$\tau(s,t) = (\beta_k, \ldots, \beta_1).$$

For convenience, we also set $\tau(t,t)$ to be the empty sequence for each $t \in \Lambda$.

The next lemma will help when we use the τ sequences to define a basis for our topology. We let $\frown$ denote concatenation of sequences.

Lemma 2.1. *Given $t, u \in \Lambda$, $t \prec u$, there exists $r \in \Lambda \cup \{0\}$, $r \prec t$, such that $\tau(s,u) = \tau(s,t)^\frown\tau(t,u)$ for every $s \in (r,t]$.*

Proof. If t is the empty function $\varnothing$, then let $r = 0$. If $t \in \Lambda^+$, then we let $r = t\restriction_{\operatorname{dom} t - 1}$ be the immediate predecessor of t in the tree order. Now suppose that $\operatorname{dom} t$ is a limit ordinal, and let $\tau(t,u) = (\beta_k, \ldots, \beta_1)$. By construction, we have

$$u(\beta_1) < u(\beta_2) < \cdots < u(\beta_k) = u(\operatorname{dom} t).$$

Since $\operatorname{dom} t$ is a limit, there exists $\alpha < \operatorname{dom} t$ such that $t(\eta) = u(\eta) > u(\operatorname{dom} t)$ whenever $\eta \in [\alpha, \operatorname{dom} t)$. Set $r = t\restriction_\alpha$, so that $\operatorname{dom} r = \alpha$. Let $s \in (r, t]$ and

$$\tau(s, u) = (\gamma_m, \ldots, \gamma_1).$$

By the choice of α, we have ensured that $m \geq k$ and $\gamma_i = \beta_i$ whenever $i \leq k$. $\square$

It is time to define the basic open sets. Let $\ell(s,t)$ denote the length of $\tau(s,t)$. Given $(t,i) \in D$ and $r \in \Lambda \cup \{0\}$, $r \prec t$, let

$$W(r,t,i) = \left\{ (s,j) \in D \ : \ s \in (r,t] \text{ and } j = (-1)^{\ell(s,t)} i \right\}.$$

Observe that if $\pi : D \longrightarrow \Lambda$ is the natural projection, then the restriction of π to any $W(r,t,i)$ is injective. Moreover, the images $\pi(W(r,t,i)) = (r,t]$ form the usual basis of the standard interval topology on Λ.

Proposition 2.2. *The $W(r,t,i)$ form a basis for a locally compact scattered topology on D.*

Proof. First, we show that these sets form a basis. If $(t,k) \in W(r_1,u_1,i_1) \cap W(r_2,u_2,i_2)$, then, by Lemma 2.1 and the fact that $r_1, r_2 \prec t$ are comparable, we can find $r \in [\max\{r_1, r_2\}, t)$ such that $\tau(s,u_j) = \tau(s,t)^\frown\tau(t,u_j)$ whenever $s \in (r,t]$ and $j = 1$, 2. It follows that

$$W(r,t,k) \subseteq W(r_1,u_1,i_1) \cap W(r_2,u_2,i_2).$$

Indeed, if $(s,l) \in W(r,t,k)$, then $s \in (r,t] \subseteq (r_j, u_j]$ and

$$l = (-1)^{\ell(s,t)} k = (-1)^{\ell(s,t)} (-1)^{\ell(t,u_j)} i = (-1)^{\ell(s,u_j)} i$$

since $\ell(s,u_j) = \ell(s,t) + \ell(t,u_j)$. Therefore $(s,l) \in W(r_1,u_1,i_1) \cap W(r_2,u_2,i_2)$ as required. We conclude that the $W(r,t,i)$ form a basis for a topology on D.

Now we show that this topology is Hausdorff and scattered. Let $(t_1,i_1), (t_2,i_2) \in D$ be distinct. If $t_1 \neq t_2$, then we let r be the largest common predecessor of these elements. It is clear that $W(r,t_1,i_1) \cap W(r,t_2,i_2)$ is empty. Instead, if $t_1 = t_2$, then $i_1 = -i_2$, so $W(0,t_1,i_1) \cap W(0,t_2,i_2)$ is empty. To see that the topology is scattered, let $E \subseteq D$ be non-empty and find minimal $t \in \Lambda$, subject to there being some i for which $(t,i) \in E$. Then $W(0,t,i) \cap E = \{(t,i)\}$.

Finally, we show that each $W(r,t,i)$ is compact. Suppose that $(u,k) \in W(r,v,i) \cap U$, where U is some open set. Again from Lemma 2.1, we know that we can find $s \in [r,u)$ such that $\tau(t,v) = \tau(t,u)^\frown\tau(u,v)$ whenever $t \in (s,u]$. Moreover, we can choose s so that $W(s,u,k) \subseteq U$. If $t \in (s,u]$ and $(t,l) \in W(r,v,i)$, then we have

$$\begin{aligned} l &= (-1)^{\ell(t,v)} i \\ &= (-1)^{\ell(t,v)} (-1)^{-\ell(u,v)} k \qquad \text{since } k = (-1)^{\ell(u,v)} i \\ &= (-1)^{\ell(t,u)} k \qquad \text{since } \ell(t,v) = \ell(t,u) + \ell(u,v) \end{aligned}$$

and so $(t,l) \in W(s,u,k) \subseteq U$.

This will allow us to show that $W(r,v,i)$ is compact. The method follows that used to show that each $(r,t]$ is compact in the usual interval topology of Λ. If $W(r,v,i)$ is covered by a family of open sets $\mathscr{U}$, we can find $U_1 \in \mathscr{U}$ covering (v_1,i_1), where $v_1 = v$ and $i_1 = i$. From the above, there is some $v_2 \prec v_1$ such that $(t,l) \in U_1$ whenever $(t,l) \in W(r,v,i)$ and $t \in (v_2,v_1]$. Then we pick $U_2 \in \mathscr{U}$ covering (v_2,i_2), where i_2 is the unique number satisfying $(v_2,i_2) \in W(r,v,i)$, and continue. The process stops at some finite $k > 1$, with $v_k = r$ and $W(r,v,i)$ covered by $U_1,\ldots,U_{k-1}$. □

Definition 2.3. We shall call D above, together with this topology, the Λ-*duplicate*.

Theorem 2.4. *The Λ-duplicate has a G_δ-diagonal but is not Gruenhage.*

Proof. First, we show that D has a G_δ-diagonal. Given $s \prec t$ and $\tau(s,t) = (\beta_k,\ldots,\beta_1)$, we define $p(s,t) = t(\beta_1)$. Note that $p(s,t) \leq t(\beta_k) = t(\operatorname{dom} s)$. We'll set $p(t,t) = \infty$ for every $t \in \Lambda$, again for convenience. For $(u,i) \in D$ and finite p, define

$$V(u,i,p) = \left\{(t,j) \ : \ t \preccurlyeq u,\, p(t,u) \geq p \text{ and } j = (-1)^{\ell(t,u)} i\right\}.$$

If $(t,j) \in V(u,i,p)$, then, by Lemma 2.1, there exists $r \prec t$ such that whenever $s \in (r,t]$, we have $\tau(s,u) = \tau(s,t)^\frown\tau(t,u)$. Certainly, for such s, we get $p(s,u) = p(t,u) \geq p$ and

$$W(r,t,j) \subseteq V(u,i,p).$$

Therefore, each $V(u,i,p)$ is open. We claim that if

$$\mathscr{G}_p = \{V(u,i,p) \ : \ (u,i) \in D\},$$

then $(\mathscr{G}_p)_{p=1}^\infty$ forms a G_δ-diagonal sequence for D. Let $(u_1,i_1),(u_2,i_2) \in D$ be distinct, and suppose that for some $(u,i) \in D$ and p we have $(u_1,i_1),(u_2,i_2) \in V(u,i,p)$. Since u_1, $u_2 \preccurlyeq u$, they are comparable. Necessarily, $u_1 \neq u_2$; otherwise we would have

$$i_1 = (-1)^{\ell(u_1,u)} i = (-1)^{\ell(u_2,u)} i = i_2,$$

giving $(u_1,i_1) = (u_2,i_2)$. Without loss of generality, assume that $u_1 \prec u_2$. Then we get

$$p \leq p(u_1,u) \leq u(\operatorname{dom} u_1) = u_2(\operatorname{dom} u_1).$$

Consequently, if we are given distinct $(u_1,i_1),(u_2,i_2) \in D$, then by choosing p large enough, we can ensure that there is no $V \in \mathscr{G}_p$ for which $(u_1,i_1),(u_2,i_2) \in V$. This establishes that the Λ-duplicate has a G_δ-diagonal.

We shall suppose for a contradiction that D is Gruenhage. As $\operatorname{card} D = \mathfrak{c}$, we can use Proposition 1.6 to find a sequence $(U_n)_{n=1}^\infty$ of open subsets of D so that given any $t \in \Lambda$, there exists n for which

$$\{(t,1),(t,-1)\} \cap U_n$$

is a singleton. Set

$$E_{n,i} = \{t \in \Lambda \ : \ (t,i) \in U_n \text{ and } (t,-i) \notin U_n\}.$$

We know that $\Lambda = \bigcup_{n,i} E_{n,i}$. Now we are going to decompose each $E_{n,i}$ into countably many subsets. If $t \in E_{n,i}$, then $(t,i) \in W(\theta(t),t,i) \subseteq U_n$ for some $\theta(t) \prec t$. Define $r_t = \theta(t) \cup \{(\alpha,t(\alpha))\}$, where $\alpha = \operatorname{dom}\theta(t) < \operatorname{dom} t$ (or $r_t = \varnothing$ if $\theta(t) = 0$). Then set

$$E_{n,m,i} = \{t \in E_{n,i} \ : \ r_t \in A_m\},$$

where the A_m are the antichains defined at the beginning of the section. Suppose that $t, u \in E_{n,m,i}$ and $t \prec u$. Since $r_t, r_u \in A_m \cap (0, u]$, it follows that $r_t = r_u$, which implies $\theta(u) = \theta(t) \prec t \prec u$. Now, we have

$$(t, j) \in W(\theta(u), u, i) \subseteq U_n,$$

where $j = (-1)^{\ell(t,u)} i$. Since $t \in E_{n,i}$, we gather that $j = i$, whence $\ell(t, u)$ is an even number.

To simplify the notation, we shall alter the indices and denote the $E_{n,m,i}$ by E_n, $n < \infty$. In summary, we have shown that if D is Gruenhage, then we can write $\Lambda = \bigcup_{n=1}^{\infty} E_n$, where each E_n has the property that $\ell(t, u)$ is an even number whenever $t, u \in E_n$ and $t \prec u$. In the final part of the proof, we use a Baire category type argument (cf. [6, Lemma 10.1]) to show that no decomposition of Λ into such sets E_n is possible.

Set $\Lambda_1 = \Lambda$ and let m_1 be minimal, subject to the condition that there exists some $t_1 \in \Lambda_1 \cap E_{m_1}$. Let

$$k_1 = \min \omega \setminus \operatorname{ran} t_1, \quad l_1 = \min \omega \setminus (\operatorname{ran} t_1 \cup \{k_1\}), \quad u_1 = t_1 \cup \{(\operatorname{dom} t_1, l_1)\}$$

and define

$$\Lambda_2 = \{v \in [u_1, \infty) \cap \Lambda_1 \ : \ k_1 \notin \operatorname{ran} v\}.$$

We observe that $\Lambda_2 \cap E_{m_1}$ is empty. If $v \in \Lambda_2$, then $v(\operatorname{dom} t_1) = u_1(\operatorname{dom} t_1) = l_1 \leq v(\eta)$ for any $\eta \in [\operatorname{dom} t_1, \operatorname{dom} v)$, by minimality of l_1 and the fact that $k_1 \notin \operatorname{ran} v$. Therefore, $\tau(t_1, v) = (\operatorname{dom} t_1)$ and $\ell(t_1, v) = 1$. Since $t_1 \in E_{m_1}$ and $\ell(t_1, v)$ is not an even number, we have $v \notin E_{m_1}$.

Continue by letting m_2 be minimal, subject to the condition that we can find some $t_2 \in \Lambda_2 \cap E_{m_2}$. Necessarily $m_2 > m_1$. Let

$$k_2 = \min \omega \setminus (\operatorname{ran} t_2 \cup \{k_1\}) > l_1, \quad l_2 = \min \omega \setminus (\operatorname{ran} t_2 \cup \{k_1, k_2\}),$$

$u_2 = t_2 \cup \{(\operatorname{dom} t_2, l_2)\}$ and define

$$\Lambda_3 = \{v \in [u_2, \infty) \cap \Lambda_2 \ : \ k_2 \notin \operatorname{ran} v\}.$$

As above, we find that $\Lambda_3 \cap E_{m_2}$ is empty because if $v \in \Lambda_3$, then $\tau(t_2, v) = (\operatorname{dom} t_2)$ and $\ell(t_2, v) = 1$; however $t_2 \in E_{m_2}$ and $\ell(t_2, v)$ must be even if v is to be an element of E_{m_2} as well.

Let $m_3 > m_2$ be minimal, subject to there being some $t_3 \in \Lambda_3 \cap E_{m_3}$, and define

$$k_3 = \min \omega \setminus (\operatorname{ran} t_3 \cup \{k_1, k_2\}) > l_2, \quad l_3 = \min \omega \setminus (\operatorname{ran} t_3 \cup \{k_1, k_2, k_3\}).$$

It should be clear how to proceed. We obtain a decreasing sequence of sets $(\Lambda_j)_{j=1}^{\infty}$ and corresponding least elements u_j, with the property that $\Lambda_{j+1} \cap E_m$ is empty whenever $m \leq m_j$. Moreover, if $v \in \Lambda_{j+1}$, then $k_1, \ldots, k_j \notin \operatorname{ran} v$.

Let $u = \bigcup_{j=1}^{\infty} u_j$. Being the union of an increasing sequence of injective functions, u is also injective. By construction, we have ensured that $k_j \notin \operatorname{ran} u$ for all j, whence $\omega \setminus \operatorname{ran} u$ is infinite and $u \in \Lambda$. Moreover, $u \in \Lambda_j$ for all j. However, this means that $u \notin E_m$ for any m, because the m_j form a strictly increasing sequence. This contradiction establishes the fact that D is not Gruenhage. $\square$

Corollary 2.5. *The* 1*-point compactification K of D is a scattered compact non-Gruenhage space with $(*)$. By Theorem* 1.4 *(but not Theorem* 1.5*), $C(K)$ admits an equivalent lattice norm with a strictly convex dual norm.*

3. The space $C_0(D)$ has a C^∞-smooth bump

If L is locally compact and scattered, then the Banach space $C_0(L)$ of continuous real-valued functions vanishing at infinity is an *Asplund* space. A non-zero real-valued continuously Fréchet differentiable function which vanishes outside some norm-bounded set is called a C^1*-smooth bump function.* To date, the long-standing question of whether every Asplund space admits such a function remains unresolved. Thus, it makes sense to test $C_0(L)$ whenever a new locally compact scattered space L comes along. The purpose of this final section is to confirm that (unfortunately!) $C_0(D)$ does admit such a function.

Definition 3.1. Given a non-empty set Γ, we say that $T : C_0(L) \longrightarrow c_0(L \times \Gamma)$ is a (generally non-linear) *Talagrand operator of class* C^∞ if

(1) whenever $f \in C_0(L)$ is non-zero, then there exists $(t, \gamma) \in L \times \Gamma$ such that $|f(t)| = \|f\|_\infty$ and $(Tf)(t, \gamma) \neq 0$, and
(2) for every pair (t, γ), the map $f \mapsto (Tf)(t, \gamma)$ is C^∞-smooth, i.e., has Fréchet derivatives of all orders, on the set on which it is non-zero.

It follows from [5, Corollary 3] that if $C_0(L)$ admits such an operator, then it admits a C^∞-smooth bump function. We shall prove that $C_0(D)$ admits such an operator. Our method follows that of [6, Theorem 9.3], which shows that $C_0(\Upsilon)$ admits a C^∞-smooth bump function for every tree Υ. However, since the topology of D is slightly more complicated than that of ordinary trees, we present some of the details.

Lemma 3.2. *Suppose that U and V are open subsets of D such that the restrictions $\pi\restriction_U$ and $\pi\restriction_V$ of the natural projection π are injective, and $\pi(U) = \pi(V) = (r, t]$ for some $r \in \Lambda \cup \{0\}$, $t \in \Lambda$. Then there exist basic open sets $W_1, \ldots, W_k$ and $W'_1, \ldots, W'_k$ such that*

$$U = W_1 \cup \cdots \cup W_k \quad \textit{and} \quad V = W'_1 \cup \cdots \cup W'_k,$$

and given any $i \leq k$, either $W_i = W'_i$ or $W_i \cap W'_i$ is empty. Moreover, if $i \neq j$, then both $W_i \cap W_j$ and $W'_i \cap W'_j$ are empty.

Proof. The argument is similar to the one used to show that the basis elements are compact. Set $t_1 = t$ and take $p_1, q_1 \in \{1, -1\}$ such that $(t_1, p_1) \in U$ and $(t_1, q_1) \in V$. From Proposition 2.2, we can find $t_2 \in [r, t_1)$ such that $W(t_2, t_1, p_1) \subseteq U$ and $W(t_2, t_1, q_1) \subseteq V$. Set $W_1 = W(t_2, t_1, p_1)$ and $W'_1 = W(t_2, t_1, q_1)$. If $p_1 = q_1$, then $W_1 = W'_1$, and if not, then $W_1 \cap W'_1$ is empty. If $t_2 = r$, then stop. Otherwise, continue by finding $p_2, q_2 \in \{1, -1\}$ and $t_3 \in [r, t_2)$ such that $W(t_3, t_2, p_2) \subseteq U$ and $W(t_3, t_2, q_2) \subseteq V$. This process stops at a finite stage k. Since π is injective on U, we have $U = W_1 \cup \cdots \cup W_k$, and similarly for V. $\square$

If $s \in \Lambda$, then s^+ is the set of all *immediate successors* of s, i.e., the set of all elements of Λ extending s and having domain $\operatorname{dom} s + 1$. Given $(s, i) \in D$, we define the set of "immediate successors" $(s, i)^+ = s^+ \times \{1, -1\}$. In the next lemma, we gather together some properties of elements in $C_0(D)$ that we need in order to define our Talagrand operator.

Lemma 3.3. *Let $f \in C_0(D)$ and $\delta > 0$.*

(1) *Given $(s,i) \in D$, there are only finitely many $(t,j) \in (s,i)^+$ satisfying $|f(t,j)| \geq \delta$.*
(2) *If f is non-zero, then there exists a maximal $s \in \Lambda$, subject to there being $i \in \{1,-1\}$ satisfying $|f(s,i)| = \|f\|_\infty$.*
(3) *For all but finitely many $(s,i) \in D$, there exists $(t,i) \in (s,j)^+$ such that $|f(s,i) - f(t,j)| < \delta$.*

Proof.

(1) Observe that
$$K = \{(t,j) \in D \ : \ |f(t,j)| \geq \delta\} \cap (s,i)^+$$
is compact and discrete, hence finite.

(2) If f is non-zero, then
$$M = \{(s,i) \in D \ : \ |f(s,i)| = \|f\|_\infty\}$$
is compact, and thus there exist finitely many pairs $(s_k, i_k) \in M$, $k \leq n$, such that $M \subseteq \bigcup_{k=1}^n W(0, s_k, i_k)$. Take the maximal s amongst the s_k.

(3) The intersection of any two basis elements of D is a finite union of pairwise disjoint basis elements. Hence, by a standard Stone-Weierstrass argument, $C_0(D)$ is equal to the closed linear span of the family of indicator functions $\mathbf{1}_W$, as W ranges over the basis elements.

Thus we are done by uniform approximation if we can show that (3) applies to finite linear combinations of the $\mathbf{1}_W$. Let $W_1, \ldots, W_n$ be basis elements, $a_1, \ldots, a_n \in \mathbb{R}$ and set $f = \sum_{k=1}^n a_k \mathbf{1}_{W_k}$. By splitting the W_k into smaller basis elements if necessary, and by using Lemma 3.2, we can assume that whenever $k \neq l$, either $W_k = W_l$ or $W_k \cap W_l$ is empty.

If $\pi(W_k) = (r_k, t_k]$, $k \leq n$, then set $F = \{r_1, \ldots, r_n\} \cup \{t_1, \ldots, t_n\}$. Take any $(s,i) \in D$ satisfying $s \notin F$. Define E to be the set of $k \leq n$ such that $(s,i) \in W_k$, so that $f(s,i) = \sum_{k \in E} a_i$. From the above, we know that $W_k = W_l$ and $t_k = t_l$ whenever $k, l \in E$. If E is non-empty, then let us denote this common set and endpoint by W and u, respectively. Because $(s,i) \in W$ and $s \notin F$, we have $s \prec u$. Take $t \in s^+$ such that $t \preccurlyeq u$, and then $j \in \{1,-1\}$ such that $(t,j) \in W$. It is clear that $(t,j) \notin W_k$ whenever $k \notin E$, else $W_k = W \ni (s,i)$. In conclusion, $f(t,j) = \sum_{k \in E} a_k = f(s,i)$. If E is empty, then pick any $t \in s^+$. If $(t,1) \notin W_k$ for all k, then we are done because $f(t,1) = 0 = f(s,i)$. If $(t,1) \in W_k$ for some k, then we claim that $(t,-1) \notin W_l$ for any l. Certainly, $(t,-1) \notin W_k$. We have $r_k \prec t \preccurlyeq t_k$, meaning $r_k \preccurlyeq s$, and as $s \notin F$ we know that $r_k \prec s$. Because $(s,i) \notin W_k$, we must have $(s,-i) \in W_k$ instead. Suppose that $(t,-1) \in W_l$ for some l. Then by the same argument we have $r_l \prec s \prec t_l$ and $(s,-i) \in W_l$. However, this implies that $(t,-1) \in W_l = W_k$, which is not so. Therefore $(t,-1) \notin W_l$ for all l and $f(t,-1) = 0 = f(s,i)$.

□

Proposition 3.4. *The space $C_0(D)$ admits a Talagrand operator of type C^∞.*

Sketch of proof. We define $T : C_0(D) \longrightarrow c_0(D \times \mathbb{N})$ in almost exactly the same way as in [6, Theorem 9.3]. Let $\phi : \mathbb{R} \longrightarrow [0,1]$ be an even C^∞-smooth function satisfying $\phi(x) = 0$ for $|x| \leq \frac{1}{2}$ and $\phi(x) = 1$ for $|x| \geq 1$. Set $\psi = 1 - \phi$. Given $f \in C_0(D)$, $(s,i) \in D$ and $n \in \mathbb{N}$, define

$$(Tf)(s,i,n) = \begin{cases} 0 \text{ if } f(s,i) = 0 \text{ or if there is } (t,j) \in (s,i)^+ \text{ with } f(t,j) = f(s,i), \\ 2^{-n}\phi(2^n f(s,i)) \displaystyle\prod_{(t,j)\in(s,i)^+} \psi\left(\frac{2^{-n}f(t,j)}{f(t,j) - f(s,i)}\right) \text{ otherwise.} \end{cases}$$

To verify that T is indeed a Talagrand operator of class C^∞, we simply use Lemma 3.3 and follow the proof of [6, Theorem 9.3], replacing s by (s,i) and t by (t,j) throughout. □

By [3], it follows that $C_0(D)$ also admits C^∞-smooth partitions of unity.

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