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PRESCRIBED COMPRESSIONS OF DUAL HYPERCYCLIC OPERATORS

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ABSTRACT. If M is a closed subspace of a separable, infinite dimensional Hilbert space H with $\dim(H/M) = \infty$, then we show that every bounded linear operator $A : M \to M$ is the compression of a dual hypercyclic operator $T : H \to H$.

On a separable, infinite dimensional Banach space X, a bounded linear operator $T : X \to X$ is said to be *hypercyclic* on X if there is a vector $x \in X$ whose orbit $\operatorname{orb}(T, x) = \{x, Tx, T^2x, T^3x, \ldots\}$ is dense in X. Such a vector x is called a *hypercyclic vector* for T. When the operator $T : X \to X$ is hypercyclic on X and its adjoint operator $T^* : X^* \to X^*$ is hypercyclic on the dual space X^* of X, then we say that T is *dual hypercyclic.* Since an orbit is a countable set, dual hypercyclicity can only take place when both X and X^* are separable. However, X is separable whenever its dual X^* is, but the converse is not always true.

Separability does not present an issue when the Banach space X is indeed a Hilbert space H, because the adjoint T^* is a bounded linear operator on H itself. In fact, it was the Hilbert space setting that the concept of dual hypercyclicity started to develop. A fundamental question is whether dual hypercyclic operators on H can ever exist. This question was originally raised by Herrero [4]. The first example of such an operator was found by Salas [6]. Later he [7] provided another example using a general result for hypercyclic bilateral weighted shift operators.

Recently generalizations of dual hypercyclic operators to a Banach space X were studied. For instance, Petersson [5] showed that any infinite dimensional Banach space X with a shrinking symmetric basis, such as c_0 and ℓ^p with $1 < p < \infty$, admits a dual hypercyclic operator $T : X \to X$. Then Salas [8] showed that any Banach space X with a separable dual space X^* admits a dual hypercyclic operator. More recently, Shkarin [10] studied dual hypercyclic tuples of operators on Banach spaces, and Salas [9] studied dual disjoint hypercyclic operators.

In the present paper we return to the setting of a separable, infinite dimensional Hilbert space H and study compressions of dual hypercyclic operators $T : H \to H$, making use of unique Hilbert space properties. Our main result is Theorem 2 below, which states that the compression of a dual hypercyclic operator T onto a closed subspace M of infinite codimension in H can coincide with any prescribed operator A on M. In other words, if $P : H \to H$ is the orthogonal projection onto a closed subspace M with $\dim(H/M) = \infty$, then for any bounded linear

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operator $A : M \to M$, there is a dual hypercyclic operator $T : H \to H$ such that $PTP|_M = A$. Compressions of a hypercyclic operator were studied by Salas [7], who showed that if $T : H \to H$ is a hypercyclic operator and N is an invariant subspace of its adjoint T^*, then the compression of T to N is hypercyclic on N.

To prove Theorem 2, we additively decompose the Hilbert space H as an orthogonal sum of infinitely many copies of a nonzero closed subspace M with $\dim(H/M) = \infty$; that is, $H = \bigoplus_{j=-\infty}^{\infty} M_j$, where each M_j is a closed subspace of H that is isomorphic to M and $M_i \perp M_j$ whenever $i \neq j$. We identify the original closed subspace M with M_0 in H, and the prescribed operator $A : M \to M$ becomes an operator $A : M_0 \to M_0$. By extending A so that $A|_{M_0^\perp} = 0$, we can view A as a bounded linear operator $A : H \to H$. Assume $\dim M = \infty$, and let $\{e_{j,1}, e_{j,2}, e_{j,3}, \ldots\}$ be an orthonormal basis of M_j. Hence $e_{j,i} \perp e_{m,n}$ whenever $(j, i) \neq (m, n)$. Let $S : H \to H$ be the unitary operator given by $Se_{j,i} = e_{j+1,i}$ whenever $i, j \in \mathbb{Z}$. In fact, $S|_{M_j}$ is an isomorphism between M_j and M_{j+1}. Lastly, we need a number $a > \max\{1, \|A\|\}$ to help set up the following technical lemma.

Lemma 1. *Suppose k is a positive integer, and for each integer j with $|j| \leq k$ we are given a positive weight w_j satisfying $0 < w_j \leq a$ and also vectors x_j and y_j in M_j. Let $\epsilon > 0$. Then there exists an integer $n \geq 2k + 2$, and there exist weights w_j with $0 < w_j \leq a$ and also vectors $x_j \in M_j$ for all integers j with $1 + k \leq |j| \leq n + k$ such that if D is defined by*

$$(1) \qquad De_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n + k, \\ 0 & \text{otherwise,} \end{cases}$$

then

(i) $\left\| (D + A)^n \left(\sum_{i=-k}^{n-k-1} x_i \right) \right\| < \epsilon,$

(ii) $(D + A)^n \left(\sum_{i=n-k}^{n+k} x_i \right) = \sum_{i=-k}^{k} y_i,$

(iii) $\sum_{i=1+k}^{n+k} \|x_i\|^2 < \epsilon,$ *and*

(iv) $x_i = 0$ *whenever* $-n - k \leq i \leq -1 - k$.

Proof. Before we determine a desired integer n for the definition of D in (1), we first observe that if $j \neq 0$, then

$$(D + A)|_{M_j} = D|_{M_j}.$$

We also observe that if m is an integer with $1 + k \leq m \leq n + k$, then for any vector $x_m \in M_m$, (1) gives

$$(2) \qquad (D + A)^m \left(x_m + \sum_{i=-k}^{k} x_i \right) \perp M_j \quad \text{whenever } j \geq 1.$$

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Note that w_j are given positive numbers whenever $|j| \leq k$. We now define $w_j = a$ for all indices j with $1+k \leq j \leq m$. If we let $P : H \to H$ be the orthogonal projection onto M_0, then we trivially have

$$P(D+A)^m\bigg(\sum_{i=-k}^{k} x_i\bigg) \in M_0.$$

Furthermore, if we specifically take the vector x_m in M_m to be

$$x_m = \frac{-1}{w_1 \cdots w_k w_{k+1} \cdots w_m} S^m P(D+A)^m\bigg(\sum_{i=-k}^{k} x_i\bigg),$$

then we can take $P(D+A)^m$ on both sides of the above equation to obtain

$$P(D+A)^m\bigg(\sum_{i=-k}^{k} x_i\bigg) \;=\; -P(D+A)^m x_m.$$

By combining with (2), we have

$$(D+A)^m\bigg(x_m + \sum_{i=-k}^{k} x_i\bigg) \perp M_j \qquad \text{whenever } j \geq 0. \tag{3}$$

Recall that $w_{k+1} = \cdots = w_m = a > \|A\|$ and $\|S\| = 1$, and note that x_m can be rewritten as

$$x_m \;=\; \frac{-1}{w_1 \cdots w_k w_{k+1} \cdots w_m} S^m P A^{m-k} (D+A)^k\bigg(\sum_{i=-k}^{k} x_i\bigg).$$

Thus we can first choose a large enough integer m, even before we determine the desired integer n for the definition of D, so that the above formulation of x_m gives

$$\|x_m\|^2 < \frac{\epsilon}{2}. \tag{4}$$

Then we determine a large enough integer n with $n \geq m+k+1 \geq 2k+2$, so that if we define $w_{-1-k} = \cdots = w_{-n-k} = a^{-1} < 1$, then it follows from (1) and (3) that

$$\bigg\|(D+A)^n\bigg(x_m + \sum_{i=-k}^{k} x_i\bigg)\bigg\| < \epsilon.$$

Thus if we define $x_{1+k} = \cdots = x_{m-1} = 0$ and $x_{m+1} = \cdots = x_{n-k-1} = 0$, then (i) is clearly satisfied.

We now check how (ii) can be satisfied whenever $n \geq 2k+2$ by appropriately choosing $x_{n-k}, \ldots, x_{n+k}$. For that, let $w_{m+1} = \cdots = w_{n+k} = a$ to complete the definitions for w_j in the whole range $1+k \leq |j| \leq n+k$ as required by the lemma. For the given vectors y_j in M_j with $|j| \leq k$ as in the statement of our lemma, we first let

$$x_{n-k} = \frac{1}{w_{1-k} \cdots w_{n-k}} S^n y_{-k}. \tag{5}$$

Next, for every index i with $1-k \leq i \leq 0$, let

$$x_{n+i} = \frac{1}{w_{1+i} \cdots w_{n+i}}\bigg(S^n y_i - \frac{1}{w_i} S^{n+i} A S^{1-i} y_{i-1}\bigg). \tag{6}$$

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Lastly, for every index i with $1 \leq i \leq k$, let

$$x_{n+i} = \frac{1}{w_{1+i} \cdots w_{n+i}} S^n y_i. \tag{7}$$

One can easily verify that the above definitions give $x_i \in M_i$ whenever $n-k \leq i \leq n+k$.

To establish (ii), we first make an observation that if $1-k \leq i \leq 0$, then

$$\begin{aligned}
&(D+A)^n \left(\frac{S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i-1} w_{n+i}} \right) \\
&= (D+A)^{-i+1} \left(\frac{(D+A)^{n+i-1} S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{(D+A)^{n+i-1} S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i-1} w_{n+i}} \right).
\end{aligned}$$

To continue our computations, we note that $S^n y_{i-1} \in M_{n+i-1}$ and $S^{n+i} A S^{1-i} y_{i-1} \in M_{n+i}$. Thus we can simplify the above expression as

$$(D+A)^{-i+1} \left(\frac{S^{1-i} y_{i-1}}{w_i \cdots w_0} - \frac{S A S^{1-i} y_{i-1}}{w_i \cdots w_0 w_1} \right).$$

Since $S^{1-i} y_{i-1} \in M_0$ and $SAS^{1-i} y_{i-1} \in M_1$, we can further reduce the above expression to

$$\begin{aligned}
&(D+A)^{-i} \left(\left(\frac{S^{-i} y_{i-1}}{w_i \cdots w_{-1}} + \frac{A S^{1-i} y_{i-1}}{w_i \cdots w_0} \right) - \frac{A S^{1-i} y_{i-1}}{w_i \cdots w_0} \right) \\
&= (D+A)^{-i} \left(\frac{S^{-i} y_{i-1}}{w_i \cdots w_{-1}} \right) \\
&= y_{i-1}.
\end{aligned}$$

Hence we conclude from our above observation that if $1-k \leq i \leq 0$, then

$$(D+A)^n \left(\frac{S^n y_{i-1}}{w_i \cdots w_{n+i-1}} - \frac{S^{n+i} A S^{1-i} y_{i-1}}{w_i \cdots w_{n+i-1} w_{n+i}} \right) = y_{i-1}. \tag{8}$$

We are now ready to establish (ii) by first using the definitions of x_i in (5), (6), and (7):

$$\begin{aligned}
&(D+A)^n \left(\sum_{i=n-k}^{n+k} x_i \right) \\
&= (D+A)^n \Bigg(\frac{S^n y_{-k}}{w_{1-k} \cdots w_{n-k}} + \\
&\quad + \sum_{i=1-k}^{0} \frac{1}{w_{1+i} \cdots w_{n+i}} (S^n y_i - \frac{1}{w_i} S^{n+i} A S^{1-i} y_{i-1}) + \sum_{i=1}^{k} x_{n+i} \Bigg),
\end{aligned}$$

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which can be rearranged as

$$
\begin{aligned}
&(D+A)^n\left(\sum_{i=1-k}^{0}\left(\frac{S^n y_{i-1}}{w_i\cdots w_{n+i-1}} - \frac{S^{n+i}AS^{1-i}y_{i-1}}{w_i\cdots w_{n+i}}\right)\right)\\
&+\ (D+A)^n\left(\frac{S^n y_0}{w_1\cdots w_n}\right)+(D+A)^n\left(\sum_{i=1}^{k}\frac{S^n y_i}{w_{1+i}\cdots w_{n+i}}\right)\\
=&\sum_{i=1-k}^{0} y_{i-1}+y_0+\sum_{i=1}^{k} y_i,\quad \text{by (8) and the fact that}\quad s^n y_i\in M_{n+i}.
\end{aligned}
$$

Hence we have established (ii).

Since we have set $w_{m+1}=\cdots=w_{n+k}=a>1$, and S is an isometry, we can assume that n is chosen large enough so that the definitions for $x_{n-k},\ldots,x_{n+k}$ in (5), (6), (7) give

$$\sum_{i=n-k}^{n+k}\|x_i\|^2<\frac{\epsilon}{2}.$$

This inequality, along with (4) and the fact that $x_{1+k}=\cdots=x_{m-1}=x_{m+1}=\cdots=x_{n-k-1}=0$, yields (iii). The proof of the lemma is completed by setting $x_{-1-k}=\cdots=x_{-n-k}=0$ to establish (iv). $\square$

For an arbitrary operator $A:M_0\to M_0$, the above lemma provides an operator D in (1) that shifts some orthonormal basis members in the backward direction, that is, taking some $e_{j,i}$ to $w_j e_{j-1,i}$. For the adjoint operator $A^*:M_0\to M_0$, one can easily modify the techniques in the lemma to provide an operator D' which shifts in the forward direction, taking some $e_{j,i}$ to $w_{j+1}e_{j+1,i}$. We omit the statement of the forward shifting version of the lemma.

To prove our main result in Theorem 2 below, we apply the lemma to A and then apply the forward shifting version of the lemma to A^*. The two applications to A and A^* are then repeated inductively.

Theorem 2. *Let H be a separable, infinite dimensional Hilbert space and let M be a closed subspace of H with $\dim(H/M)=\infty$. Let $P:H\to H$ be the orthogonal projection onto M. If $A:M\to M$ is a bounded linear operator, then there is an operator $T:H\to H$ such that*

(1) *T is dual hypercyclic,*
(2) *$PTP|_M=A$, and*
(3) *$PT^*P|_M=A^*$.*

Proof. Since M has infinite codimension in H, we can use an orthonormal basis argument to additively decompose H as an orthogonal sum $H=\bigoplus_{j=-\infty}^{\infty}M_j$, where M_0 is the closed subspace M given in the statement of our theorem and each M_j is isomorphic to M.

In the rest of the proof, we assume that $\dim M=\infty$, and the same argument works if $\dim M$ is finite. For that, let $\{e_{j,1},e_{j,2},e_{j,3},\ldots\}$ be an orthonormal basis of M_j. Since A takes M_0 to M_0, we can view A as an operator from H to H with $A|_{M_0^\perp}=0$. It follows that $\overline{\operatorname{ran}A^*}=(\ker A)^\perp\subset M_0$ and also that $M_0^\perp\subset(\operatorname{ran}A)^\perp=\ker A^*$, and so we can also view the original adjoint operator $A^*:M_0\to M_0$ as

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an operator from H to H with $A^*|_{M_0^\perp} = 0$. That allows us to define an operator $T : H \to H$ by $T = A + B$ and so $T^* = A^* + B^*$, where $B : H \to H$ is a linear map defined by $Be_{j,i} = w_j e_{j-1,i}$ for all integers i, j and $\{w_j : j \in \mathbb{Z}\}$ is a bounded two-sided sequence of positive numbers. Thus B takes each M_j to M_{j-1} and $B|_{M_j}$ is in the form of "w_j times a Hilbert space isomorphism". In fact, if $S : H \to H$ is the unitary operator given by $Se_{j,i} = e_{j+1,i}$ whenever $i, j \in \mathbb{Z}$, then $S|_{M_j}$ is an isomorphism from M_j to M_{j+1}, and $B|_{M_j} = w_j S^{-1}|_{M_j}$. It is easy to verify that $B^*|_{M_j} = w_{j+1} S|_{M_j}$; that is, $B^* e_{j,i} = w_{j+1} e_{j+1,i}$. In addition, the definition of T gives $PTP|_{M_0} = A$ and $PT^*P|_{M_0} = A^*$. In fact, B is a bilateral backward shift, and $\|B\| = \sup |w_j| < \infty$. In the rest of the proof we need to choose w_j so that T and T^* are hypercyclic.

Let a be a number with $a > \max\{1, \|A\|\}$, and let $v_1, v_2, v_3, \ldots$ be an enumeration of all vectors, each of which has only a finite number of nonzero coefficients $\langle v, e_{j,i} \rangle$ and all coefficients are rationals. The set of all v_i is dense in H. Let $k_1 \geq 1$ be such that $v_1 \in \bigoplus_{|j| \leq k_1} M_j$. After choosing k_1, let $w_j = a$ and $x_j = 0$ whenever $|j| \leq k_1$, and write $v_1 = \sum_{|j| \leq k_1} y_j$ with each $y_j \in M_j$. Let $\epsilon_1 = (2a^{2k_1})^{-1} > 0$. The lemma provides an integer n_1 and weights w_j with $0 < w_j \leq a$ and vectors $x_j \in M_j$ whenever $1 + k_1 \leq |j| \leq n_1 + k_1$ such that if D_1 is given by

$$D_1 e_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \leq n_1 + k_1, \\ 0 & \text{otherwise,} \end{cases}$$

then statements (i), (ii), (iii), and (iv) are satisfied.

With the above setting, we apply the lemma to the operator A^* in a direction opposite to the statement of the lemma, that is, to obtain a forward shift operator D'. Hence, for the same vector $v_1 = \sum_{|j| \leq k_1} y_j$ we can continue to define w_j and x_j by setting $k_1' = n_1 + k_1$ and $\epsilon_1' = (2a^{2k_1'})^{-1} > 0$. The lemma provides an integer n_1' and weights w_j with $0 < w_j \leq a$, and vectors $x_j \in M_j$, for all indices j in the range $1 + k_1' \leq |j| \leq n_1' + k_1'$. Since we are applying the statement of the lemma in the opposite direction, statement (iv) gives $x_i = 0$ for those indices i in the range $1 + k_1' \leq i \leq n_1' + k_1'$, and statement (iii) gives

$$\sum_{i=-n_1'-k_1'}^{-1-k_1'} \|x_i\|^2 < \frac{1}{2a^{2k_1'}}.$$

In addition, the lemma provides a forward shift operator D_1' by

$$D_1' e_{j,i} = \begin{cases} w_{j+1} e_{j+1,i} & \text{if } i \in \mathbb{Z} \text{ and } -n_1' - k_1' \leq j \leq n_1' + k_1' - 1, \\ 0 & \text{otherwise,} \end{cases}$$

such that statements (i) and (ii) become

$$\left\| (D_1' + A^*)^{n_1'} \left(\sum_{i=-n_1'+k_1'+1}^{k_1'} x_i \right) \right\| \; < \; \epsilon_1'$$

and

$$(D_1' + A^*)^{n_1'} \left(\sum_{i=-n_1'-k_1'}^{-n_1'+k_1'} x_i \right) \; = \; v_1.$$

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Note that the lemma provides the term $w_{-n_1'-k_1'}$ that we do not actually use in the above definition of D_1', and this term causes no consequence in our subsequent argument. In the second step, we let $k_2 = n_1' + k_1'$, and $\epsilon_2 = (2^2 a^{2k_2})^{-1} > 0$ and assume that, without loss of generality, $v_2 \in \bigoplus_{|j|\le k_2} M_j$. In the case that v_2 is not in the subspace $\bigoplus_{|j|\le k_2} M_j$, we can choose v_i with the least integer i such that v_i is in that subspace. We can use the lemma to define operator D_2 and then D_2' for the same vector v_2, as we define D_1 and D_1' in the previous case for v_1.

Inductively, in the m-th step we take $k_m = n_{m-1}' + k_{m-1}'$ and $\epsilon_m = (2^m a^{2k_m})^{-1} > 0$ and we can assume, without loss of generality, that $v_m \in \bigoplus_{|j|\le k_m} M_j$. Together with those w_j and x_j already defined for indices j with $|j| \le n_{m-1}' + k_{m-1}' = k_m$ in the $(m-1)$-th step, we obtain from the lemma an integer n_m and weights w_j with $0 < w_j \le a$, and vectors x_j in M_j, where $|j| \le n_m + k_m$ and in fact,

$$x_i = 0 \text{ for those indices } i \text{ in the range } -n_m - k_m \le i \le -1 - k_m, \tag{9}$$

such that if D_m is given by

$$D_m e_{j,i} = \begin{cases} w_j e_{j-1,i} & \text{whenever } i \in \mathbb{Z} \text{ and } |j| \le n_m + k_m, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\sum_{i=1+k_m}^{n_m+k_m} \|x_i\|^2 \;<\; \frac{1}{2^m a^{2k_m}} \tag{10}$$

and

$$\left\| (D_m + A)^{n_m} \left(\sum_{i=-k_m}^{n_m-k_m-1} x_i \right) \right\| \;<\; \frac{1}{2^m a^{2k_m}} \tag{11}$$

and also

$$(D_m + A)^{n_m} \left(\sum_{i=n_m-k_m}^{n_m+k_m} x_i \right) \;=\; v_m. \tag{12}$$

Next we apply the lemma to obtain a forward shift D_m' for the same vector v_m, but for the adjoint A^* instead of A itself. For that purpose, let $k_m' = n_m + k_m$ and $\epsilon_m' = (2^m a^{2k_m'})^{-1} > 0$. Together with those w_j and x_j that have already been defined for indices j in the range $-k_m' \le j \le k_m'$, we obtain from the lemma an integer n_m' and weights w_j with $0 < w_j \le a$ and vectors x_j in M_j whenever $|j| \le n_m' + k_m'$, and in fact

$$x_i = 0 \;\text{ for those indices } i \text{ in the range } 1 + k_m' \le i \le n_m' + k_m', \tag{13}$$

such that

$$\sum_{i=-n_m'-k_m'}^{-1-k_m'} \|x_i\|^2 < \frac{1}{2^m a^{2k_m'}}. \tag{14}$$

In addition, the lemma provides an operator D_m' given by

$$D_m' e_{j,i} = \begin{cases} w_{j+1} e_{j+1,i} & \text{if } i \in \mathbb{Z} \text{ and } -n_m' - k_m' \le j \le n_m' + k_m' - 1, \\ 0 & \text{otherwise,} \end{cases}$$

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such that

$$\left\|(D'_m + A^*)^{n'_m}\left(\sum_{i=-n'_m+k'_m+1}^{k'_m} x_i\right)\right\| < \epsilon'_m \tag{15}$$

and also

$$(D'_m + A^*)^{n'_m}\left(\sum_{i=-n'_m-k'_m}^{-n'_m+k'_m} x_i\right) = v_m. \tag{16}$$

With all weights w_j given by the inductive process above, we define an operator $B : H \to H$ by $B|_{M_j} = w_j S^{-1}|_{M_j}$ as we have outlined in the beginning of this proof. The operator B is bounded because $0 < w_j \leq a$. Also, with all vectors x_j given by the inductive process, we let $x = \sum_{j\in\mathbb{Z}} x_j$, which represents a vector in H because the terms x_j are mutually orthogonal and square summable. To verify that, we note that $k'_m = n_m + k_m$, and $k_{m+1} = n'_m + k'_m$ by their definitions, and hence

$$\begin{aligned} &\sum_{i=1+k_1}^{\infty} \|x_i\|^2 \\ &= \sum_{m=1}^{\infty}\left(\sum_{i=1+k_m}^{n_m+k_m} \|x_i\|^2 + \sum_{i=1+k'_m}^{n'_m+k'_m} \|x_i\|^2\right) \\ &= \sum_{m=1}^{\infty}\left(\sum_{i=1+k_m}^{n_m+k_m} \|x_i\|^2\right) \quad \text{by (13)}, \end{aligned}$$

and hence by (10) we have

$$\sum_{i=1+k_1}^{\infty} \|x_i\|^2 < \sum_{m=1}^{\infty} \frac{1}{2^m a^{2k_m}} < \infty.$$

Similarly, using (9) and (14), we obtain

$$\sum_{i=-\infty}^{-k'_1-1} \|x_i\|^2 < \infty,$$

and so the sum $\sum_{j\in\mathbb{Z}} x_j$ defines a vector in H.

We now proceed to prove that if we write $x = \sum_{j\in\mathbb{Z}} x_j$, then x is a hypercyclic vector for both $A + B$ and $A^* + B^*$. We begin by rewriting (12) as

$$(A+B)^{n_m}\left(\sum_{i=n_m-k_m}^{n_m+k_m} x_i\right) = v_m,$$

and hence by orthogonality we have

$$\begin{aligned} &\|(A+B)^{n_m}x - v_m\|^2 \\ &= \left\|(A+B)^{n_m}\left(\sum_{i=-\infty}^{n_m-k_m-1} x_i\right)\right\|^2 + \left\|(A+B)^{n_m}\left(\sum_{i=n_m+k_m+1}^{\infty} x_i\right)\right\|^2. \end{aligned} \tag{17}$$

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We continue our computations with the two summands in (17) separately. For the first summand we note that by orthogonality, we can write it as the sum of two terms:

$$\left\|(A+B)^{n_m}\Bigl(\sum_{i=-n_m-k_m}^{n_m-k_m-1} x_i\Bigr)\right\|^2 + \left\|B^{n_m}\Bigl(\sum_{i=-\infty}^{-n_m-k_m-1} x_i\Bigr)\right\|^2. \tag{18}$$

To continue our estimations, note that (9) and (11) give

$$\begin{aligned} &\left\|(A+B)^{n_m}\Bigl(\sum_{i=-n_m-k_m}^{n_m-k_m-1} x_i\Bigr)\right\|^2 \\ = &\left\|(A+B)^{n_m}\Bigl(\sum_{i=-k_m}^{n_m-k_m-1} x_i\Bigr)\right\|^2 \\ < &\left(\frac{1}{2^m a^{2k_m}}\right)^2. \end{aligned}$$

Thus we obtain an estimate for the first summand in (17) using (9) and (18) as follows:

$$\left\|(A+B)^{n_m}\Bigl(\sum_{i=-\infty}^{n_m-k_m-1} x_i\Bigr)\right\|^2 \;<\; \left(\frac{1}{2^m a^{2k_m}}\right)^2 \;+\; a^{2n_m}\sum_{j=m}^{\infty}\sum_{i=-k_{j+1}}^{-n_j-k_j-1}\|x_i\|^2.$$

Note that $k'_j = n_j + k_j$ and $k_{j+1} = n'_j + k'_j$, and so

$$\begin{aligned} a^{2n_m}\sum_{j=m}^{\infty}\sum_{i=-k_{j+1}}^{-n_j-k_j-1}\|x_i\|^2 &= a^{2n_m}\sum_{j=m}^{\infty}\left(\sum_{i=-n'_j-k'_j}^{-1-k'_j}\|x_i\|^2\right) \\ &< a^{2n_m}\sum_{j=m}^{\infty}\frac{1}{2^j a^{2k'_j}},\ \text{by (14)} \\ &< \sum_{j=m}^{\infty}\frac{1}{2^j},\ \text{(because } k'_j \ge k'_m = n_m + k_m\text{)} \\ &= 2^{1-m}. \end{aligned}$$

Hence the first summand in (17) is bounded above by $(2^m a^{2k_m})^{-2} + 2^{1-m}$, which goes to 0 as m goes to ∞.

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To estimate the second summand in (17) we first use (13) to obtain that $x_i = 0$ for those indices i in the range $1 + k'_j = n_j + k_j + 1 \le i \le n'_j + k'_j = k_{j+1}$, and hence

$$\begin{aligned}
&\left\| (A+B)^{n_m} \Big(\sum_{i=n_m+k_m+1}^{\infty} x_i \Big) \right\|^2 \\
&= \left\| B^{n_m} \Big(\sum_{j=m}^{\infty} \sum_{i=1+k_{j+1}}^{n_{j+1}+k_{j+1}} x_i \Big) \right\|^2 \\
&\le \|B\|^{2n_m} \Big(\sum_{j=m}^{\infty} \sum_{i=1+k_{j+1}}^{n_{j+1}+k_{j+1}} \|x_i\|^2 \Big) \\
&< a^{2n_m} \sum_{j=m}^{\infty} \Big(\frac{1}{2^{j+1} a^{2k_{j+1}}} \Big) \quad \text{by (10)} \\
&\le \sum_{j=m}^{\infty} \frac{1}{2^{j+1}} \quad (\text{because } \; k_{j+1} \ge k_{m+1} > k'_m > n_m) \\
&= 2^{-m},
\end{aligned}$$

which goes to 0 as m goes to ∞. Since $T = A + B$, it follows from (17) that $\|T^{n_m}x - v_m\|^2 \to 0$ as $m \to \infty$. Recall that the set $\{v_1, v_2, v_3, \ldots\}$ is dense in H, and so is the set $\{x, Tx, T^2x, \ldots\}$. In other words, T is hypercyclic.

A similar argument using (9), (10), (13), (14), (15), and (16) shows that $T^* = A^* + B^*$ is also hypercyclic. Thus T is dual hypercyclic. □

In the above proof, we construct a vector x that is hypercyclic for both $A + B$ and $A^* + B^*$. As a matter of fact, for any two hypercyclic operators T_1 and T_2 on H, there must be a vector that is hypercyclic for both operators because for any hypercyclic operator T, the set of all hypercyclic vectors $HC(T)$ is a dense G_δ set. By the Baire Category Theorem, $HC(T_1) \cap HC(T_2)$ is also a dense G_δ set.

In the above theorem, if we take M to be a one dimensional subspace and $A : M \to M$ to be the zero operator, then the proof provides a bilateral weighted shift operator that is dual hypercyclic. This result was obtained by Salas [7].

It may be a surprising fact that dual hypercyclic operators can exist on a separable, infinite dimensional Hilbert space H, but Theorem 2 shows that they are indeed very common in terms of prescribed behaviors of their compressions on a closed subspace M with infinite codimension in H. For such subspaces M, Chan and Turcu [2] and Grivaux [3] showed that any bounded linear operator $A : M \to M$ can be extended to a chaotic operator $T : H \to H$. Here, a *chaotic operator* is a hypercyclic operator with a dense set of periodic points x; that is, $T^n x = x$ for some positive integer n depending on x. However, a chaotic operator cannot be dual hypercyclic, because if T has a periodic point, say $T^n x = x$, then for any vector y in H the set $\{\langle T^{*kn}y, x\rangle \mid k \ge 0\}$ cannot be dense in the scalar field, and hence T^{*n} cannot have a hypercyclic vector. That contradicts Ansari's result [1] that the n-th power of a hypercyclic operator is also hypercyclic.

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