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COMPLETELY POSITIVE MATRIX NUMERICAL INDEX ON MATRIX REGULAR OPERATOR SPACES

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Abstract. In the article, we compute the completely positive matrix numerical index of matrix regular operator spaces and show that they take values in the interval $[\frac{1}{2}, 1]$. Moreover, we show that the dual of a unital operator system has the completely positive matrix numerical index $\frac{1}{2}$ if its dimension is greater than 1. Furthermore, both $S_p(\mathbf{H})$ and $L_p(\mathbf{M})$ have the completely positive matrix numerical index $2^{-\frac{1}{p}}$ if their dimensions are greater than 1, where $p \in [1, +\infty)$, $\mathbf{H}$ is a Hilbert space and $\mathbf{M}$ is a finite von Neumann algebra.

1. Introduction

The completely positive matrix numerical index $n_{cb}^{+}(\mathbf{V})$ of a matrix pre-ordered operator space $\mathbf{V}$ is a constant based on the matrix norm and the matrix order of the space. It was used in [10] to characterize (non-unital) operator systems (in fact non-selfadjoint, non-unital operator systems are considered in [10]). By a (non-unital) operator system, we mean a selfadjoint linear subspace of some $\mathcal{L}(\mathbf{H})$ equipped with the induced matrix order operator space structure. Let us recall the relevant definitions. Let $\mathbf{V}$ be a matrix pre-ordered operator space. We define the *completely positive n-matrix state space*

$$\mathcal{Q}_n^{\mathbf{V}} := \{\varphi \in CB(\mathbf{V}, M_n) : \varphi \in CP(\mathbf{V}, M_n), \|\varphi\|_{cb} \leq 1\},$$

and the *completely positive matrix numerical radius* of an element x in $M_n(\mathbf{V})(n \in \mathbb{N})$ is defined as

$$\gamma_n^{+}(x) \ := \ \sup\left\{\|\varphi_n(x)\| : \varphi \in \mathcal{Q}_k^{\mathbf{V}}, k \in \mathbb{N}\right\},$$

as well as the *completely positive matrix numerical index* of $\mathbf{V}$ is given by

$$n_{cb}^{+}(\mathbf{V}) \ := \ \inf\left\{\gamma_n^{+}(v) : v \in M_n(\mathbf{V}), \|v\|_n = 1, n \in \mathbb{N}\right\}.$$

Note that $n_{cb}^{+}(\mathbf{V})$ is the greatest constant $t \geq 0$ such that $t\|x\| \leq \gamma_k^{+}(x)$ for every $x \in M_k(\mathbf{V})$.

Ng shows in [10, Theorem 2.6] that $n_{cb}^{+}(\mathbf{V}) = 1$ (respectively, $n_{cb}^{+}(\mathbf{V}) > 0$) if and only if it is an operator system (respectively, there exist a Hilbert space $\mathbf{H}$ and a map $\Phi : \mathbf{V} \to B(\mathbf{H})$ which is $n_{cb}^{+}(\mathbf{V})^{-1}$-completely isomorphically and complete order isomorphically). Thus, all unital operator systems have completely positive

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matrix numerical index 1. A matrix pre-ordered operator space $\mathbf{V}$ is called a *quasi-operator system* if $n_{cb}^{+}(\mathbf{V}) > 0$. It may be worth reminding the readers that in some papers (see [3, 4, 14]) the term "operator system" will be used for the matrix pre-order operator space with strictly positive completely positive matrix numerical index and its abstract characterization given by Werner [14, Theorem 4.15].

The purpose of this paper is to compute the completely positive numerical index of classical matrix regular operator spaces. First, we show that $n_{cb}^{+}(\mathbf{V}) \geq \frac{1}{2}$ for any matrix regular operator space $\mathbf{V}$. This implies Karn's Theorem [7] that every matrix regular operator space can be embedded into $\mathcal{L}(\mathbf{H})$ 2-completely isomorphically and complete order isomorphically. We prove that a lot of matrix regular operator spaces of dimension greater than one are quasi-operator systems but not operator systems. In fact, we will prove that the predual of a von Neumann algebra and the dual of a unital operator system, with dimension greater than one, have the completely positive numerical index $\frac{1}{2}$. Moreover, we exhibit that the spaces $S_p(\mathbf{H})$ and $L_p(\mathbf{M})$ for $1 \leq p < \infty$ have the same completely positive numerical index $2^{-\frac{1}{p}}$, where $\mathbf{H}$ is a Hilbert space of dimension greater than one and $\mathbf{M}$ is a finite von Neumann algebra such that $\dim(\mathbf{M}) > 1$. This implies that for a matrix regular operator space $\mathbf{V}$, the completely positive numerical index $n_{cb}^{+}(\mathbf{V})$ can be any number in the interval $[\frac{1}{2}, 1]$. Finally, we show that $n_{cb}^{+}(\mathrm{CB}(\mathbf{V}, \mathbf{W})) \geq \frac{1}{2} n_{cb}^{+}(\mathbf{W})$, where $\mathbf{V}$ is a matrix regular operator space and $\mathbf{W}$ is a matrix pre-order operator space. In this case, $\mathrm{CB}(\mathbf{V}, \mathbf{W})$ is a quasi-operator system when $\mathbf{W}$ is a quasi-operator system.

2. Preliminaries

A complex involutive vector space $\mathbf{V}$ is called a *matrix pre-ordered vector space* if for each $n \in \mathbb{N}$ there is a set $M_n(\mathbf{V})_+ \subseteq M_n(\mathbf{V})_{sa}$ so that

(a) $M_n(\mathbf{V})_+ \oplus M_m(\mathbf{V})_+ \subseteq M_{n+m}(\mathbf{V})_+$ for all $m, n \in \mathbb{N}$,

(b) $\gamma^* M_m(\mathbf{V})_+ \gamma \subseteq M_n(\mathbf{V})_+$ for each $m, n \in \mathbb{N}$ and all $\gamma \in M_{m,n}$.

A matrix pre-ordered vector space $\mathbf{V}$ is called a *matrix pre-ordered operator space* if $\mathbf{V}$ is an operator space, its involution is an isometry on $M_n(\mathbf{V})$ and the cones $M_n(\mathbf{V})_+$ are closed for all $n \in \mathbb{N}$. Following W. Werner [14] we say that $\mathbf{V}$ is a *matrix ordered operator space* if $M_n(\mathbf{V})_+ \cap -M_n(\mathbf{V})_+ = \{0\}$ for all $n \in \mathbb{N}$. In fact, if $\mathbf{V}_+ \cap -\mathbf{V}_+ = \{0\}$, then $M_n(\mathbf{V})_+ \cap -M_n(\mathbf{V})_+ = \{0\}$ for all $n \in \mathbb{N}$ (see [14, Remark 2.2 (ii)]).

Let $\Phi : \mathbf{V} \to \mathbf{W}$ be a linear map between two matrix pre-ordered vector spaces $\mathbf{V}$ and $\mathbf{W}$, and define Φ^* by $\Phi^*(v) := \Phi(v^*)^*$. We say that $\Phi : \mathbf{V} \to \mathbf{W}$ is *positive* if $\Phi^* = \Phi$ and $\Phi(\mathbf{V}_+) \subseteq \mathbf{W}_+$. We let $\Phi_n : M_n(\mathbf{V}) \to M_n(\mathbf{W})$ be defined by $\Phi_n((x_{ij})) := (\Phi(x_{ij}))$ and we call Φ *completely positive* if Φ_n is positive for all $n \in \mathbb{N}$. We denote the set of completely positive mappings from $\mathbf{V}$ to $\mathbf{W}$ by $\mathrm{CP}(\mathbf{V}, \mathbf{W})$. An injective completely positive mapping Φ is called a *complete order monomorphism* if for all $n \in \mathbb{N}$, $\Phi_n(M_n(\mathbf{V})_+) = \Phi_n(M_n(\mathbf{V})) \cap M_n(\mathbf{W})_+$.

Let $\mathbf{V}$ and $\mathbf{W}$ be matrix ordered operator spaces. We set

$$M_n(\mathrm{CB}(\mathbf{V}, \mathbf{W}))_+ := \mathrm{CB}(\mathbf{V}, M_n(\mathbf{W})) \cap \mathrm{CP}(\mathbf{V}, M_n(\mathbf{W}))$$

as well as

$$M_n(\mathrm{CB}(\mathbf{V}, \mathbf{W}))_{sa} := \{\varphi \in \mathrm{CB}(\mathbf{V}, M_n(\mathbf{W})) : \varphi = \varphi^*\}.$$

Then $\mathrm{CB}(\mathbf{V}, \mathbf{W})$ is a matrix pre-ordered operator space (see [12, Proposition 3.1]). In particular, if $\mathbf{W} = \mathbb{C}$ we will use the symbol $\mathbf{V}'$ to indicate $\mathrm{CB}(\mathbf{V}, \mathbb{C})$. In this case, $M_n(\mathbf{V}')_+ = \mathrm{CB}(\mathbf{V}, M_n)_+$.

Definition 2.1. *A matrix ordered operator space is called a* matrix regular (or matrical Riesz) operator space *if for each $n \in \mathbb{N}$ and for all $v \in M_n(\mathbf{V})_{sa}$,*

(a) $u \in M_n(\mathbf{V})_{sa}$ and $-u \leq v \leq u$ imply that $\|v\|_n \leq \|u\|_n$;

(b) $\|v\|_n < 1$ implies that there exists $u \in M_n(\mathbf{V})_{sa}$ such that $\|u\|_n < 1$ and $-u \leq v \leq u$.

Note that examples of matrix ordered regular operator spaces include all unital operator systems and their duals, C^*-algebras, the predual spaces of von Neumann algebras, the Schatten class spaces and the commutative L_p spaces [13].

The following result is due to the work of W. J. Schreiner [12].

Proposition 2.2. *Let $\mathbf{V}$ be a matrix ordered operator space. Then the following are equivalent:*

(a) *$\mathbf{V}$ is a matrix regular operator space.*

(b) *$\mathbf{V}'$ is a matrix regular operator space.*

(c) *For each $n \in \mathbb{N}$ and for all $v \in M_n(\mathbf{V})$, $\|v\|_n < 1$ if and only if there exist $u_1, u_2 \in M_n(\mathbf{V})_+$, $\|u_1\|_n < 1$ and $\|u_2\|_n < 1$ such that $\begin{pmatrix} u_1 & v \\ v^* & u_2 \end{pmatrix} \in M_{2n}(\mathbf{V})_+$.*

3. Computing the completely positive matrix numerical index

We will give another proof of Karn's Theorem [7]. Our proof here is quite distinct and much simpler.

Theorem 3.1. *If $\mathbf{V}$ is a matrix regular operator space, then $n_{cb}^+(\mathbf{V}) \geq \frac{1}{2}$.*

Proof. It follows from Proposition 2.2 that $\mathbf{V}'$ is a matrix regular operator space, and hence given any $f \in M_n(\mathbf{V}')$ with $\|f\| < 1$, there exist $f_1, f_2 \in \mathcal{Q}_n^{\mathbf{V}}$ such that $\frac{1}{2}\begin{pmatrix} f_1 & f \\ f^* & f_2 \end{pmatrix} \in \mathcal{Q}_n^{\mathbf{V}}$. Thus for any $x \in M_n(\mathbf{V})$ we have that $\|x\| \leq 2\gamma_k^+(x)$. This means that $n_{cb}^+(\mathbf{V}) \geq \frac{1}{2}$. □

Corollary 3.2. *Let $\mathbf{V}$ be the predual of a von Neumann algebra, the dual of a unital operator system, the Schatten class space S_p or the commutative l_p space for $1 < p < \infty$. If $\dim(\mathbf{V}) > 1$, then $\mathbf{V}$ is a quasi-operator system but not an operator system.*

Proof. We claim that a matrix pre-ordered operator space $\mathbf{V}$ satisfying

$$\exists \;\; x_1, x_2 \in \mathbf{V}_+ \text{ such that } \|x_1 - x_2\| > \max\{\|x_1\|, \|x_2\|\} \tag{1}$$

is not an operator system. Indeed, suppose that $\mathbf{V}$ is an operator system. Then there exist a Hilbert space $\mathbf{H}$ and a completely isometric complete order monomorphism $\Phi : \mathbf{V} \to \mathcal{L}(\mathbf{H})$. Set $T_1 = \Phi(x_1)$ and $T_2 = \Phi(x_2)$. We see that

$$\|T_1 - T_2\| \leq \max\{\|T_1\|, \|T_2\|\} = \max\{\|x_1\|, \|x_2\|\}.$$

This contradicts the fact that

$$\|T_1 - T_2\| = \|x_1 - x_2\| > \max\{\|x_1\|, \|x_2\|\}.$$

Notice that the predual of a von Neumann algebra with dimension greater than one satisfies condition (1) by Jordan Decomposition, and so the dual of a unital operator system by [1, Theorem 4], the Schatten class S_p space (with $x_1 = E_{1,1}$ and

$x_2 = E_{2,2}$) and the commutative l_p space (with $x_1 = e_1$ and $x_2 = e_2$) are all not operator systems. On the other hand, since these vector spaces are matrix regular operator spaces, they are quasi-operator systems by Theorem 3.1. □

In view of the examples given in the last corollary, we will compute the completely positive numerical index on these classical matrix regular operator spaces.

Theorem 3.3. *Let* $\mathbf{M}$ *be a von Neumann algebra with predual* $\mathbf{M}_*$*, and let* $\mathbf{W}$ *be a unital operator system. If* $\mathbf{M}$ *and* $\mathbf{W}$ *have dimension greater than one, then*

$$n_{cb}^+(\mathbf{M}_*) = n_{cb}^+(\mathbf{W}') = \frac{1}{2}.$$

Proof. Let $\mathbf{V} = \mathbf{M}_*$ or $\mathbf{W}'$. Then there exist $x_1, x_2 \in \mathbf{V}_+$ such that

$$\|x_1\| = \|x_2\| = \frac{1}{2}\|x_1 - x_2\|.$$

In fact, if $\mathbf{V} = \mathbf{M}_*$, we can find $f, g \in \mathbf{M}_*^+$ such that $f(p) = 1$ and $g(I-p) = 1$, where $p \in M$ is a non-trivial projection (i.e., $p \neq 0, I$). Now $f_1, f_2 \in \mathbf{M}_*^+$ are defined by setting:

$$f_1(x) := f(pxp),\ \ f_2(x) := g((I-p)x(I-p)) \quad \text{for any } x \in \mathbf{M}.$$

By [8, Theorem 3.3.3],

$$\|f_1\| = \|f_2\| = f_1(I) = f_2(I) = 1,$$

and so

$$\|f_1 - f_2\| \leq \|f_1\| + \|f_2\| = 2.$$

Moreover, $u := 2p - I$ is a selfadjoint unitary in $\mathbf{M}$ and

$$(f_1 - f_2)(u) = f_1(p) + f_2(I-p) = 2.$$

On the other hand, it is assumed that $\mathbf{V} = \mathbf{W}'$. We can pick a non-zero linear functional $f \in \mathbf{W}'$ with $f(e) = 0$. Using a similar proof of [1, Theorem 4], we can find a positive linear functional F on $\mathbf{W} \times \mathbf{W}$ such that

$$F(e,e) = \|f\|, \quad F(e,-e) = f(e) = 0 \quad \text{and} \quad f(x) = F(x,0) - F(0,x).$$

Let $f_1(x) := F(x,0)$ and $f_2(x) := F(0,x)$. Then f_1 and f_2 are positive linear functionals on $\mathbf{W}$ such that $f = f_1 - f_2$. Finally, it is evident that

$$\|f_1\| + \|f_2\| = f_1(e) + f_2(e) = F(e,e) = \|f\|$$

and that

$$\|f_1\| - \|f_2\| = f_1(e) - f_2(e) = F(e,-e) = f(e) = 0.$$

Thus $\|f_1\| = \|f_2\| = \frac{1}{2}\|f\|$.

Next, we will prove that $n_{cb}^+(\mathbf{V}) \leq \frac{1}{2}$. Set $x = x_1 - x_2$ and $x_1, x_2 \in \mathbf{V}_+$ satisfying

$$\|x_1\| = \|x_2\| = \frac{1}{2}\|x\|.$$

Then for each $n \in \mathbb{N}$ and $\varphi \in \mathcal{Q}_n^{\mathbf{V}}$,

$$\|\varphi(x)\| = \|\varphi(x_1) - \varphi(x_2)\| \leq \max\{\|\varphi(x_1)\|, \|\varphi(x_2)\|\} \leq \frac{1}{2}\|x\|.$$

It follows that $\gamma_1^+(x) \leq \frac{1}{2}\|x\|$, and hence $n_{cb}^+(\mathbf{V}) \leq \frac{1}{2}$. Since $\mathbf{V}$ is a matrix regular operator space, Theorem 3.1 implies that $n_{cb}^+(\mathbf{V}) = \frac{1}{2}$. □

For computing the matrix numerical index on the Schatten class spaces $S_p(\mathbf{H})$ and the commutative L_p spaces, we need to introduce some more results. Let $\mathbf{V}$ be a matrix pre-ordered operator space. For each $x \in M_n(\mathbf{V})$, the modified numerical radius is defined by

$$\nu_n(x) := \sup \left\{ |f \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}| : f \in \mathcal{Q}_1^{M_{2n}(\mathbf{V})} \right\}.$$

Lemma 3.4. *Let* $\mathbf{V}$ *be a matrix pre-ordered operator space. Then*

$$n_{cb}^+(\mathbf{V}) = \inf \{\nu_n(x) : x \in M_n(\mathbf{V}), \|x\|_n = 1, n \in \mathbb{N}\}.$$

Proof. Let $n \in \mathbb{N}$, $x \in M_n(\mathbf{V})$. By [10, Lemma 2.4 (b)] for each $f \in \mathcal{Q}_1^{M_{2n}(\mathbf{V})}$, there exist $k \leq 2n$, $\varphi \in \mathcal{Q}_k^{\mathbf{V}}$ as well as a unit vector $\eta \in \mathbb{C}^{2kn}$ satisfying

$$f(v) = \langle \varphi_{2n}(v)\eta, \eta \rangle \quad \text{for each } v \in M_{2n}(\mathbf{V}).$$

Then we obtain $\nu_n(x) \leq \gamma_n(x)$. Conversely, for each $\varphi \in \mathcal{Q}_m^{\mathbf{V}}$, there exists a unit vector $\xi \in \mathbb{C}^{2mn}$ such that

$$\|\varphi_n(x)\| = |\langle \varphi_{2n} \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \xi, \xi \rangle|.$$

Now, consider the functional $f \in \mathcal{Q}_1^{M_{2n}(\mathbf{V})}$ given by

$$f(v) := \langle \varphi_{2n}(v)\xi, \xi \rangle \quad \text{for any } v \in M_{2n}(\mathbf{V}).$$

It is clear that

$$f(\begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}) = \langle \varphi_{2n} \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \xi, \xi \rangle,$$

and thus $\nu_n(x) \geq \gamma_n(x)$. We have proved that $\nu_n(x) = \gamma_n(x)$ for all $n \in \mathbb{N}$, $x \in M_n(\mathbf{V})$. By the definition of $n_{\rm cb}^+(\mathbf{V})$, we conclude that

$$n_{cb}^+(\mathbf{V}) = \inf \{\nu_n(x) : x \in M_n(\mathbf{V}), \|x\|_n = 1, n \in \mathbb{N}\}. \qquad \square$$

If $\mathbf{M}$ is a semi-finite von Neumann algebra, then there exists a normal faithful semi-finite trace τ on $\mathbf{M}$. In this case, the non-commutative L_p-space $L_p(\mathbf{M})$ is defined to be the norm closure

$$L_p(\mathbf{M}, \tau) := cl\{x \in \mathbf{M}, \tau(|x|^p) < \infty\}^{\|\cdot\|_p}$$

with the norm given by

$$\|x\|_p := (\tau((x^*x)^{\frac{p}{2}}))^{\frac{1}{p}}.$$

The matrix order is given by the positive cones $L_p(M_n(\mathbf{M}))^+$, and the operator space structure is given by the complex interpolation

$$M_n(L_p(\mathbf{M})) := (M_n(\mathbf{M}), M_n(\mathbf{M}_*^{op}))_{\frac{1}{p}}.$$

It follows from Lemma 3.4 and [4, Theorem 2.6] that $n_{cb}^+(L_p(\mathbf{M})) \geq 2^{-\frac{1}{p}}$.

Theorem 3.5. *Let* $\mathbf{H}$ *be Hilbert space and* $\mathbf{M}$ *be a finite von Neumann algebra. If* $\mathbf{H}$ *and* $\mathbf{M}$ *have dimension greater than one, then for* $1 \leq p < \infty$,

$$n_{cb}^+(S_p(\mathbf{H})) = n_{cb}^+(L_p(\mathbf{M})) = 2^{-\frac{1}{p}}.$$

In particular, if (Ω, Σ, μ) *is a* σ*-finite measure space such that* $\dim(L_\infty(\mu)) > 1$, *then* $n_{cb}^+(L_p(\mu)) = 2^{-\frac{1}{p}}$.

Proof. Let $\mathbf{N}$ be $\mathcal{L}(\mathbf{H})$ or $\mathbf{M}$. Then $\mathbf{N}$ has two non-zero orthogonal projections $P, Q \in L_p(\mathbf{N})$. Putting $x = P/\|P\|_p - Q/\|Q\|_p$, we get $\|x\|_p^p = 2$ and

$$\|\varphi(x)\| \leq \max\{\varphi(\frac{P}{\|P\|_p}), \varphi(\frac{Q}{\|Q\|_p})\} \leq 1$$

for all $n \in \mathbb{N}$, $\varphi \in \mathcal{Q}_n^{L_p(\mathbf{N})}$. It follows that $\gamma_1^+(x) \leq 1$ and thus

$$n_{cb}^+(L_p(\mathbf{N})) \leq \frac{\gamma_1^+(x)}{\|x\|_p} \leq 2^{-\frac{1}{p}}.$$

Since $\mathbf{N} = \mathcal{L}(\mathbf{H})$ or $\mathbf{M}$ is a semi-finite von Neumann algebra, we have

$$n_{cb}^+(L_p(\mathbf{N})) \geq 2^{-\frac{1}{p}}.$$

It is known that an abelian von Neumann algebra is a finite von Neumann algebra. This completes the proof. □

It is worth noting that for any Hilbert space $\mathbf{H}$ and σ-finite measure space (Ω, Σ, μ), $S_p(\mathbf{H})$ and $L_p(\mu)$ are matrix regular operator spaces [13]. Theorem 3.5, together with Theorem 3.3, gives the following corollary.

Corollary 3.6. *For every $t \in [\frac{1}{2}, 1]$, there is a matrix regular operator space $\mathbf{V}$ such that $n_{cb}^+(\mathbf{V}) = t$.*

Proposition 3.7. *Let $\mathbf{W}$ be a matrix pre-ordered operator space. Then for any matrix regular operator space $\mathbf{V}$, we have $n_{cb}^+(CB(\mathbf{V}, \mathbf{W})) \geq \frac{1}{2} n_{cb}^+(\mathbf{W})$.*

Proof. Let $\varphi \in M_k(\mathrm{CB}(\mathbf{V}, \mathbf{W}))$. Since $\mathbf{V}$ is a matrix regular operator space, for every $\epsilon > 0$, by Proposition 2.2(c), we may find $x \in M_n(\mathbf{V})_+$ and $\|x\| \leq 1$ such that

$$\|\varphi_n(x)\| > \frac{1}{2}\|\varphi\| - \epsilon.$$

It follows from the definition of $n_{cb}^+(\mathbf{W})$ that there exist $m \in \mathbb{N}$ and $g \in \mathcal{Q}_m^{\mathbf{W}}$ such that

$$\|g_{nk}(\varphi_n(x))\| > n_{cb}^+(\mathbf{W})(\|\varphi_n(x)\| - \epsilon).$$

We define the operator

$$\theta : \mathrm{CB}(\mathbf{V}, \mathbf{W}) \to M_{mn}$$

by $\theta(\psi) := g_n(\psi_n(x))$ for each $\psi \in \mathrm{CB}(\mathbf{V}, \mathbf{W})$. So we conclude that $\theta \in \mathcal{Q}_{mn}^{\mathrm{CB}(\mathbf{V},\mathbf{W})}$ and

$$\|\theta_k(\varphi)\| = \|g_{nk}(\varphi_n(x))\| > n_{cb}^+(\mathbf{W})(\|\varphi_n(x)\| - \epsilon) > \frac{1}{2} n_{cb}^+(\mathbf{W})\|\varphi\| - 2\epsilon.$$

The desired inequality $n_{cb}^+(\mathrm{CB}(\mathbf{V}, \mathbf{W})) \geq \frac{1}{2} n_{cb}^+(\mathbf{W})$ follows. □

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