PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 9, September 2012, Pages 3179–3191
S 0002-9939(2012)11157-8
Article electronically published on January 18, 2012

A KAPLANSKY THEOREM FOR JB*-TRIPLES

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(Communicated by Marius Junge)

Abstract. Let $T : E \to F$ be a not necessarily continuous triple homomorphism from a (complex) JB*-triple (respectively, a (real) J*B-triple) to a normed Jordan triple. The following statements hold:
(1) T has closed range whenever T is continuous.
(2) T is bounded below if and only if T is a triple monomorphism.
This result generalises classical theorems of I. Kaplansky and S.B. Cleveland in the setting of C*-algebras and of A. Bensebah and J. Pérez, L. Rico and A. Rodríguez Palacios in the setting of JB*-algebras.

1. Introduction

A celebrated result of I. Kaplansky (cf. [13, Theorem 6.2]) establishes that any algebra norm on a commutative C*-algebra dominates the C*-norm. Subsequently, S.B. Cleveland (see [8]) generalised this result to the noncommutative case by showing that every (not necessarily complete nor continuous) algebra norm on a C*-algebra generates a topology stronger than the topology of the C*-norm. In other words, every not necessarily continuous monomorphism from a C*-algebra to an associative normed algebra is bounded below. Alternative proofs to Cleveland's result were given by H.G. Dales [9] and A. Rodríguez Palacios [22] (see also [16, Theorem 6.1.16]).

The arguments presented by A. Rodríguez Palacios in [22] were adapted by A. Bensebah [3] and J. Pérez, L. Rico and A. Rodríguez Palacios [17] to extend Kaplansky's theorem to the more general setting of JB*-algebras. The results established in [3] and [17] show that every not necessarily continuous Jordan monomorphism from a JB*-algebra to a normed Jordan algebra is bounded below. This result was proved again by S. Hejazian and A. Niknam in [11].

Every C*-algebra, A, admits a triple product defined by

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a). \tag{1}$$

Let us suppose that $\|.\|_2$ is another (not necessarily complete nor continuous) norm on A which makes continuous the triple product of A. It is natural to ask whether this norm generates a topology stronger than the topology of the C*-norm.

Received by the editors February 23, 2010 and, in revised form, September 20, 2010 and March 28, 2011.
2010 *Mathematics Subject Classification.* Primary 46K70, 46L05, 46L10, 46L70; Secondary 17C65.
The authors were partially supported by D.G.I. project No. MTM2008-02186 and Junta de Andalucía grants FQM0199 and FQM3737.

Every C*-algebra, A, equipped with its C*-norm and the triple product defined in (1) can be regarded as an element in the wider category of (complex) JB*-triples (see §2 for the detailed definitions). The question posed in the above paragraph also makes sense in the (larger) categories of (complex) JB*-triples and real J*B-triples. In this setting, the problem can be reformulated in the following terms:

Problem (P). *Let E be a (complex) JB*-triple or a (real) J*B-triple whose norm is denoted by $\|.\|$, and let $\|.\|_2$ be another (not necessarily complete nor $\|.\|$-continuous) norm on the vector space E which makes continuous the triple product of E. Does $\|.\|_2$ generate a topology stronger than the topology generated by the JB*-triple norm $\|.\|$?*

Equivalently, is every (not necessarily continuous) triple monomorphism T from E to a normed Jordan triple bounded below?

Under the additional hypothesis of T being $\|.\|$-continuous (resp., $\|.\|_2$ being $\|.\|$-continuous), Problem (P) was solved by K. Bouhya and A. Fernández López in the case of (complex) JB*-triples [4, Corollary 14].

In this paper we solve Problem (P) without any additional assumptions on the triple monomorphism T (resp., on $\|.\|_2$). When particularized to C*-algebras, our main result shows that every not necessarily continuous triple monomorphism from a real or complex C*-algebra to a normed Jordan triple is bounded below.

Section 2 is devoted to presenting the basic facts and definitions needed in the paper. We shall also survey the results on the property of minimality of norm topology in the setting of Banach algebras and Jordan-Banach triples. We shall adapt the arguments given by K. Bouhya and A. Fernández López [4] to obtain their result in the setting of real J*B-triples.

In Section 3 we present our main results (Theorem 17 and Corollary 18). This section contains a deep study of the separating spaces associated with a triple homomorphism between normed Jordan triples. Among the tools developed here, we mention a main boundedness theorem type for Jordan-Banach triples (see Theorem 12), which is the Jordan triple version of a classical result in the setting of Banach algebras due to W.G. Bade and P.C. Curtis [1].

2. Minimality of norm topology for JB*-triples

A normed algebra A has *minimality of algebraic norm topology (MOANT)* if any other (not necessarily complete) algebra norm dominated by the given norm yields an equivalent topology. It is part of the folklore that C*-algebras have MOANT (compare [8, Lemma 5.3]).

In this section, we study the minimality of norm topology in the setting of normed Jordan triples. We recall that a complex (resp., real) *normed Jordan triple* is a complex (resp., real) normed space E equipped with a nontrivial, continuous triple product

$$E \times E \times E \to E,$$
$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called *"Jordan Identity"*:

$$L(a,b)L(x,y) - L(x,y)L(a,b) = L(L(a,b)x,y) - L(x,L(b,a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$. If E is complete with respect to the norm (i.e. if E is a Banach space), then it is called a complex (resp., real) *Jordan-Banach triple*. Every normed Jordan triple can be completed in the usual way to become a Jordan-Banach triple. Unless specified otherwise, the term "normed Jordan triple" (resp., "Jordan-Banach triple") will always mean a real or complex normed Jordan triple (resp., "Jordan-Banach triple").

For each Jordan-Banach triple E, the constant $N(E)$ or $N(E, \|.\|)$ will denote the supremum of the set $\{\| \{x, y, z\} \| : \|x\|, \|y\|, \|z\| \leq 1\}$.

A real (resp., complex) *Jordan algebra* is a (not necessarily associative) algebra over the real (resp., complex) field whose product is abelian and satisfies $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$. A normed Jordan algebra is a Jordan algebra A equipped with a norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$, a, b in A. A *Jordan-Banach algebra* is a normed Jordan algebra whose norm is complete.

Every real or complex associative Banach algebra (resp., Jordan Banach algebra) is a real Jordan-Banach triple with respect to the product $\{a, b, c\} = \frac{1}{2}(abc + cba)$ (resp., $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$).

A JB*-algebra is a complex Jordan Banach algebra A equipped with an algebra involution * satisfying that $\| \{a, a^*, a\} \| = \|2(a \circ a^*) \circ a - a^2 \circ a^*\| = \|a\|^3$, a in A.

Every JB*-algebra has MOANT (compare [17, Theorem 10]).

We shall say that a normed Jordan triple E has *minimality of triple norm topology (MOTNT)* if any other (not necessarily complete) triple norm dominated by the norm of E defines an equivalent topology.

Remark 1. Let A be a real or complex associative normed algebra whose norm is denoted by $\|.\|$. The symbol A^+ will stand for the normed Jordan algebra A equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the original norm. Let $\|.\|_1$ be a norm on the space A. Since the Jordan product is $\|.\|_1$-continuous whenever the associative product is, we deduce:

$$(A^+, \|.\|) \text{ has MOANT} \implies (A, \|.\|) \text{ has MOANT}.$$

However, we do not know if the reciprocal statement is, in general, true. By [7, Proposition 3], there exists an associative normed algebra $\mathcal{B}$ such that there exists a norm $\|.\|_1$ on $\mathcal{B}$ for which the Jordan product is continuous but the associative product is discontinuous. In particular, $(\mathcal{B}^+, \|.\|_1)$ doesn't have MOANT.

When A is simple and has a unit, every norm on A making the Jordan product continuous also makes continuous the associative product (compare [7, Theorem 3]). Under this additional hypothesis, we have

$$(A^+, \|.\|) \text{ has MOANT} \iff (A, \|.\|) \text{ has MOANT}.$$

Suppose that J is a real or complex normed Jordan algebra, whose norm is denoted by $\|.\|$. When J is regarded as a real or complex normed Jordan triple with respect to the product $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$, every Jordan algebra norm on J makes continuous the triple product. Therefore J has MOANT whenever it has MOTNT.

When J has a unit, the Jordan and the triple product of J are mutually determined, and hence

$$(J, \|.\|) \text{ has MOANT} \iff (J, \|.\|) \text{ has MOTNT}.$$

A *(complex) JB*-triple* is a complex Jordan-Banach triple E satisfying the following axioms:

(JB*1) For each a in E the map $L(a,a)$ is a hermitian operator on E with nonnegative spectrum.

(JB*2) $\|\{a,a,a\}\| = \|a\|^3$ for all a in A.

The following theorem is a celebrated result of I. Kaplansky (see [13, Theorem 6.2] or [23, Theorem 1.2.4]).

Theorem 2. *Let A be a commutative C*-algebra with a norm $\|.\|$ and let $\|.\|_1$ be another norm on A under which A is a normed algebra. Then $\|a\| \leq \|a\|_1$, for every a in A. Further, for any algebra norm, $\|.\|_1$, on A_{sa}, the inequality $\|a\| \leq \|a\|_1$ holds for every a in A_{sa}.* □

Every C*-algebra is a JB*-triple with respect to the product $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. It seems natural to ask whether in the above Theorem 2 the norm $\|.\|_1$ can be replaced with another norm $\|.\|_2$ under which A is a normed Jordan triple. The complex statement in the following result was established by K. Bouhya and A. Fernández López in [4, Proposition 13]. A detailed proof is included here for completeness.

Lemma 3. *Let $L \subset \mathbb{R}_0^+$ be a subset of nonnegative real numbers satisfying that $L \cup \{0\}$ is compact. Let $C_0(L)$ denote the Banach algebra of all real- or complex-valued continuous functions on $L \cup \{0\}$ vanishing at zero (equipped with the supremum norm $\|.\|_\infty$). Suppose that $\|.\|_2$ is a $\|.\|_\infty$-continuous norm on $C_0(L)$ under which $C_0(L)$ is a normed Jordan triple. Then $\|.\|_2$ is equivalent to an algebra norm on $C_0(L)$, and consequently $\|.\|_\infty$ and $\|.\|_2$ are equivalent norms. More concretely, writing $M = \sup\{\|x\|_2 : \|x\|_\infty \leq 1\}$ we have $\|a\|_\infty \leq MN(C_0(L), \|.\|_2)\ \|a\|_2$, for all $a \in C_0(L)$.*

Proof. Since $\|.\|_2$ is $\|.\|_\infty$-continuous, there exists a positive M such that $\|x\|_2 \leq M\|x\|_\infty$, for all $x \in C_0(L)$.

When L is compact, $C_0(L)$ coincides with $C(L)$, the C*-algebra of all complex-valued continuous functions on L or with the selfadjoint part of that C*-algebra. Let 1 denote the unit element in $C(L)$. Take a, b in $C(L)$. Applying the fact that $\|.\|_2$ is a triple norm we have

$$\begin{aligned} \|a\ b\|_2 = \|\ \{a,1,b\}\ \|_2 &\leq N(C_0(L), \|.\|_2)\ \|a\|_2\ \|1\|_2\ \|b\|_2 \\ &\leq N(C_0(L), \|.\|_2)\ M\ \|a\|_2\ \|b\|_2. \end{aligned}$$

This shows that $\|.\|_2$ is equivalent to $MN(C_0(L), \|.\|_2)\ \|.\|_2$, and the latter is an algebra norm on $C_0(L)$.

Suppose that L is not compact. Take a and b in $C_0(L)$. For each natural n, let p_n, a_n and b_n be the functions in $C_0(L)$ defined by

$$a_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2n}] \cap L; \\ \text{affine}, & \text{if } t \in [\frac{1}{2n}, \frac{1}{n}] \cap L; \\ a(t), & \text{if } t \in [\frac{1}{n}, \infty) \cap L; \end{cases} \qquad b_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2n}] \cap L; \\ \text{affine}, & \text{if } t \in [\frac{1}{2n}, \frac{1}{n}] \cap L; \\ b(t), & \text{if } t \in [\frac{1}{n}, \infty) \cap L; \end{cases}$$

$$\text{and } p_n(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{4n}] \cap L; \\ \text{affine}, & \text{if } t \in [\frac{1}{4n}, \frac{1}{2n}] \cap L; \\ 1, & \text{if } t \in [\frac{1}{2n}, \infty) \cap L. \end{cases}$$

Since $a_n\ b_n = \{a_n, p_n, b_n\}$ and $\|p_n\|_\infty \leq 1$, we deduce that

$$\begin{aligned}\|a_n\ b_n\|_2 = \|\ \{a_n, p_n, b_n\}\ \|_2 &\leq N(C_0(L), \|.\|_2)\ \|a_n\|_2\ \|p_n\|_2\ \|b_n\|_2 \\ &\leq N(C_0(L), \|.\|_2)\ M\ \|a_n\|_2\ \|b_n\|_2.\end{aligned}$$

Having in mind that $\|a_n - a\|_\infty \to 0$, $\|b_n - b\|_\infty \to 0$, it follows, from the $\|.\|_\infty$-continuity of the norm $\|.\|_2$, that

$$\|a\ b\|_2 \leq N(C_0(L), \|.\|_2)\ M\ \|a\|_2\ \|b\|_2,$$

which shows that $\|.\|_2$ is equivalent to $MN(C_0(L), \|.\|_2)\ \|.\|_2$, and the latter is an algebra norm on $C_0(L)$. The final statement is a direct consequence of Kaplansky's theorem (see Theorem 2). □

Remark 4. Let K be a compact Haussdorff space. Suppose that $\|.\|_2$ is a norm on $C(K)$ under which $C(K)$ is a normed Jordan triple ($\|.\|_\infty$-continuity of $\|.\|_2$ is not assumed). Let us write $N = N(C(K), \|.\|_2)$. Following the argument given in the proof of Lemma 3, we deduce that

$$\|a\ b\|_2 = \|\ \{a, 1, b\}\ \|_2 \leq N\ \|1\|_2\ \|a\|_2\ \|b\|_2,$$

for all $a, b \in C(K)$, which shows that $\|.\|_2$ is equivalent to $\|1\|_2\ N\ \|.\|_2$, and the latter is an algebra norm on $C(K)$. It follows by Kaplansky's theorem, that $\|a\|_\infty \leq \|1\|_2\ N\ \|a\|_2$, for all $a \in C(K)$.

S.B. Cleveland applied Kaplansky's theorem to prove that every continuous monomorphism from a C*-algebra to a normed algebra is bounded below (cf. [8, Lemma 5.3]); equivalently, every C*-algebra has MOANT. It follows as a consequence of [3, Theorem 1] or [17, Theorem 10] or [11] that JB*-algebras have MOANT. In the setting of (complex) JB*-triples, K. Bouhya and A. Fernández López proved the following result:

Proposition 5 ([4, Corollary 14])**.** *Let $T : E \to F$ be a continuous triple monomorphism from a JB*-triple to a normed complex Jordan triple. Then T is bounded below. That is, every JB*-triple has MOTNT.* □

We shall complete this section by proving a generalization of the above result to the setting of (real) J*B-triples.

We recall that a *real JB*-triple* is a norm-closed real subtriple of a complex JB*-triple (compare [12]). A *J*B-triple* is a real Banach space E equipped with a structure of a real Banach-Jordan triple which satisfies (JB*2) and the following additional axioms:

(J*B1) $N(E) = 1$;
(J*B2) $\sigma^{\mathbb{C}}_{L(E)}(L(x, x)) \subset [0, +\infty)$ for all $x \in E$;
(J*B3) $\sigma^{\mathbb{C}}_{L(E)}(L(x, y) - L(y, x)) \subset i\mathbb{R}$ for all $x, y \in E$.

Every closed subtriple of a J*B-triple is a J*B-triple (cf. [10, Remark 1.5]). The class of J*B-triples includes real (and complex) C*-algebras and real (and complex) JB*-triples. Moreover, in [10, Proposition 1.4] it is shown that complex JB*-triples are precisely those complex Jordan-Banach triples whose underlying real Banach space is a J*B-triple.

T. Dang and B. Russo established a Gelfand theory for J*B-triples in [10, Theorem 3.12]. This Gelfand theory can be refined to show that given an element a in a J*B-triple E, there exists a bounded set $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact such that the smallest (norm) closed subtriple of E containing a, E_a, is isometrically isomorphic to

$$C_0(L,\mathbb{R}) := \{f \in C_0(L), f(L) \subseteq \mathbb{R}\},$$

(see [6, page 14]). The argument given in the proof of Corollary 14 in [4] can be adapted to prove the following result. The proof is included here for completeness.

Proposition 6. *Let $T : E \to F$ be a continuous triple monomorphism from a (real) J*B-triple to a normed Jordan triple. Then T is bounded below. Equivalently, every J*B-triple has MOTNT.* □.

Proof. Take an arbitrary element a in E. Let E_a denote the smallest (norm) closed subtriple of E containing a. By [6, page 14], there exists a subset $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact satisfying that E_a is isometrically J*B-triple isomorphic to $C_0(L,\mathbb{R})$, when the latter is equipped with the supremum norm $\|.\|_\infty$. We shall identify E_a and $C_0(L,\mathbb{R})$. The mapping $T|_{E_a} : E_a \cong C_0(L,\mathbb{R}) \to F$ is a continuous triple monomorphism. Therefore the mapping $x \mapsto \|x\|_2 := \|T(x)\|$ defines a $\|.\|_\infty$-continuous norm on $C_0(L,\mathbb{R})$ under which $C_0(L,\mathbb{R})$ is a normed Jordan triple.

Noticing that $N(E_a, \|.\|_2) \leq N(F)$ and

$$M = \sup\{\|x\|_2 : x \in E_a, \|x\|_\infty \leq 1\} \leq \|T\|,$$

Lemma 3 assures that $\|a\| \leq N(F)\ \|T\|\ \|T(a)\|$, for every $a \in E$. □

We recall that a subspace I of a normed Jordan triple E is a *triple ideal* if $\{E,E,I\} + \{E,I,E\} \subseteq I$. The quotient of a normed Jordan triple by a closed triple ideal is a normed Jordan triple. It is also known that the quotient of a JB*-triple (resp., a J*B-triple) by a closed triple ideal is a JB*-triple (resp., a J*B-triple) (compare [14]).

Let $T : E \to F$ be a continuous triple homomorphism from a (real) J*B-triple to a normed Jordan triple. The kernel of T, $ker(T)$, is a norm-closed triple ideal of E and the linear mapping $\widetilde{T} : E/ker(T) \to F$ given by $\widetilde{T}(a + ker(T)) = T(a)$ is a continuous triple monomorphism from a (real) J*B-triple to a normed Jordan triple and $\widetilde{T}(E) = T(E)$. Proposition 6 assures that $\widetilde{T}$ is bounded below, and hence it has closed range.

A real JB*-algebra is a closed *-invariant real subalgebra of a (complex) JB*-algebra. Real C*-algebras (i.e., closed *-invariant real subalgebras of C*-algebras), equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, are examples of real JB*-algebras.

Corollary 7. *Every continuous triple homomorphism from a (real) J*B-triple to a normed Jordan triple has closed range. In particular, every continuous triple homomorphism from a real or complex C*-algebra to a normed Jordan triple has closed range.* □

Corollary 8. *Let A be a real JB*-algebra and let B be a real Jordan Banach algebra (or a real Jordan-Banach triple). Then every continuous triple monomorphism from A to B is bounded below. That is, A has MOTNT and MOANT.* □

Corollary 9. *Let A be a real or complex C*-algebra and let B be a real Banach algebra (or a real Jordan-Banach triple). Then every continuous triple monomorphism from A to B is bounded below. That is, A has MOTNT and MOANT.* □

3. Separating spaces for triple homomorphisms

We have seen in the previous section that real and complex C*-algebras and real and complex JB*-algebras have MOTNT and MOANT. Equivalently, if A denotes a real or complex C*-algebra (resp., a real or complex JB*-algebra) every continuous (triple) monomorphism T from A to a Banach algebra (resp., a Jordan-Banach algebra) is bounded below. C*-algebras and JB*-algebras satisfy a stronger property: when A is a C*-algebra (resp., a JB*-algebra) every not necessarily continuous monomorphism from A to a Banach algebra (resp., a Jordan-Banach algebra) is bounded below (compare [8, Theorem 5.4] and [3, Theorem 1] or [17, Theorem 10] or [11]).

The question clearly is whether every not necessarily continuous triple monomorphism from a complex JB*-triple (resp., from a real J*B-triple) to a normed Jordan triple is bounded below. In this section we provide a positive answer to this question. Following a classical strategy, we shall study the *separating ideals* associated with a triple homomorphism.

Under additional geometric assumptions, triple homomorphisms are automatically continuous. More concretely, every triple homomorphism between two JB*-triples is automatically continuous (compare [2, Lemma 1]). In this setting the problem reduces to the question of minimality of triple norm topology treated in Section 2. However, when the codomain space is not a JB*-triple, the continuity of a triple homomorphism does not follow automatically. We shall derive a new strategy to solve Problem (P) without any additional geometric hypothesis on the codomain space.

The following definitions and results are inspired by classical ideas developed by C. Rickart [19], B. Yood [26], W.G. Bade and P.C. Curtis [1] and S.B. Cleveland [8]. Let $T : X \to Y$ be a linear mapping between two normed spaces. Following [20, page 70], the *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$. The *separating space*, $\sigma_X(T)$, of T in X is defined by $\sigma_X(T) := T^{-1}(\sigma_Y(T))$. For each element y in Y, $\Delta(y)$ is defined as the infimum of the set $\{\|x\|+\|y-T(x)\| : x \in X\}$. The mapping $x \mapsto \Delta(x)$, called the *separating* function of T, satisfies the following properties:

a) $\Delta(y_1 + y_2) \leq \Delta(y_1) + \Delta(y_2)$,
b) $\Delta(\lambda y) = |\lambda|\ \Delta(y)$,
c) $\Delta(y) \leq \|y\|$ and $\Delta(T(x)) \leq \|x\|$,

for every y, y_1 and y_2 in Y, x in X and λ scalar (compare [20, page 71] or [8, Proposition 4.2]).

A straightforward application of the closed graph theorem shows that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$ (cf. [8, Proposition 4.5]).

It is not hard to see that $\sigma_Y(T) = \{y \in Y : \Delta(y) = 0\}$, while $\sigma_X(T) = \{x \in X : \Delta(T(x)) = 0\}$. Therefore $\sigma_X(T)$ and $\sigma_Y(T)$ are closed linear subspaces of X and Y, respectively. The assignment

$$x + \sigma_X(T) \mapsto \widetilde{T}(x + \sigma_X(T)) = T(x) + \sigma_Y(T)$$

defines an injective linear operator from $X/\sigma_X(T)$ to $Y/\sigma_Y(T)$. Moreover, $\widetilde{T}$ is continuous whenever X and Y are Banach spaces.

The separating subspaces of a triple homomorphism enjoy additional algebraic structure.

Lemma 10. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two normed Jordan triples. Then $\sigma_E(T)$ is a norm-closed triple ideal of E and $\sigma_F(T)$ is a norm-closed triple ideal of the norm closure of $T(E)$ in the completion of F.*

Proof. Let us fix $z \in \sigma_F(T)$. In this case there exists a sequence $(x_n) \subseteq E$ with $x_n \to 0$ and $T(x_n) \to z$. Given x, y in E, the sequences $(\{x_n, x, y\})$ and $(\{x, x_n, y\})$ are norm-null,

$$T(\{x_n, x, y\}) = \{T(x_n), T(x), T(y)\} \to \{z, T(x), T(y)\}$$

and

$$T(\{x, x_n, y\}) = \{T(x), T(x_n), T(y)\} \to \{T(x), z, T(y)\}.$$

This shows that $\sigma_F(T)$ is a norm-closed triple ideal of $\overline{T(E)}^{\|\cdot\|}$.

We have already proved that

$$\{\sigma_F(T), T(E), T(E)\} \subseteq \sigma_F(T)$$

and

$$\{T(E), \sigma_F(T), T(E)\} \subseteq \sigma_F(T).$$

This implies that

$$T(\{\sigma_E(T), E, E\}) \subseteq \{\sigma_F(T), T(E), T(E)\} \subseteq \sigma_F(T)$$

and

$$T(\{E, \sigma_E(T), E\}) \subseteq \{T(E), \sigma_F(T), T(E)\} \subseteq \sigma_F(T),$$

which shows that $\{\sigma_E(T), E, E\}, \{E, \sigma_E(T), E\} \subseteq \sigma_E(T)$. □

The following result follows from Lemma 10 and the basic properties of the separating spaces.

Proposition 11. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two Jordan-Banach triples. Then the mapping $\widetilde{T} : E/\sigma_E(T) \to F/\sigma_F(T)$, defined by $\widetilde{T}(a + E/\sigma_E(T)) = T(a) + F/\sigma_F(T)$, is a continuous triple monomorphism.* □

Let E be a normed Jordan triple. Two elements a and b in E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = L(b, a) = 0$. A direct application of the Jordan identity yields that, for each x in E,

$$a \perp \{b, x, b\} \text{ whenever } a \perp b. \tag{2}$$

The following theorem is a "*main boundedness theorem*" type result for Jordan-Banach triples (compare [1, Theorem 2.1]; see also [8, Theorem 3.1]).

Theorem 12. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between Jordan-Banach triples and let (x_n), (y_n) be two sequences of nonzero elements in E such that $x_n \perp x_m, y_m$ for every $n \neq m$. Then*

$$\sup\left\{\frac{\|T(\{x_n, x_n, y_n\})\|}{\|x_n\|^2\|y_n\|}, n \in \mathbb{N}\right\} < \infty.$$

Proof. Suppose that $\sup\left\{\frac{\|T(\{x_n,x_n,y_n\})\|}{\|x_n\|^2\|y_n\|}, n\in\mathbb{N}\right\}=\infty$. Under this assumption, we may find a subsequence $(a_{p,q})_{p,q\in\mathbb{N}}$ of (x_n) formed by mutually orthogonal elements such that

$$\|T\{a_{p,q},a_{p,q},b_{p,q}\}\| > 4^p 8^q \|a_{p,q}\|^2\|b_{p,q}\|, \quad p,q\in\mathbb{N},$$

where $b_{p,q}=y_m$ whenever $a_{p,q}=x_m$. Now, for each $p\in\mathbb{N}$, we define

$$z_p=\sum_{k=1}^{\infty}\frac{a_{p,k}}{2^k\|a_{p,k}\|}.$$

It is easy to see that, for each natural q, $b_{l,q}\perp z_p$ whenever $l\neq p$. The equality

$$\{z_p,z_p,b_{p,q}\}=\frac{1}{4^q\|a_{p,q}\|^2}\{a_{p,q},a_{p,q},b_{p,q}\},\ q\in\mathbb{N},$$

follows from the (joint) continuity of the triple product and the orthogonality hypothesis. Thus, $T(z_p)\neq 0,\ \forall p\in\mathbb{N}$.

For each p in $\mathbb{N}$ choose $n(p)$ in $\mathbb{N}$ with $2^{n(p)}>\|T(z_p)\|^2$ and define $y=\sum_{k=1}^{\infty}\frac{b_{k,n(k)}}{2^k\|b_{k,n(k)}\|}$. It follows that

$$\{z_p,z_p,y\}=\frac{1}{2^p4^{n(p)}\|b_{p,n(p)}\|\|a_{p,n(p)}\|^2}\ \{a_{p,n(p)},a_{p,n(p)},b_{p,n(p)}\}.$$

Therefore,

$$N(F)\ \|T(y)\|\|T(z_p)\|^2>\|T(\{z_p,z_p,y\})\|>2^p2^{n(p)}>2^p\|T(z_p)\|^2.$$

This implies that $N(F)\ \|T(y)\|>2^p$ for every positive integer p, which is impossible. $\square$

Given an element a in a normed Jordan triple E, we denote $a^{[1]}=a$, $a^{[3]}=\{a,a,a\}$ and $a^{[2n+1]}:=\left\{a,a^{[2n-1]},a\right\}$ $(\forall n\in\mathbb{N})$. The Jordan identity implies that $a^{[5]}=\left\{a,a,a^{[3]}\right\}$, and by induction, $a^{[2n+1]}=L(a,a)^n(a)$ for all $n\in\mathbb{N}$. The element a is called *nilpotent* if $a^{[2n+1]}=0$ for some n.

A Jordan-Banach triple E for which the vanishing of $\{a,a,a\}$ implies that a itself vanishes is said to be *anisotropic*. It is easy to check that E is anisotropic if and only if zero is the unique nilpotent element in E.

Let a and b be two elements in an anisotropic normed Jordan triple E. If $L(a,b)=0$, then, for each x in E, the Jordan identity implies that

$$\{L(b,a)x,L(b,a)x,L(b,a)x\}=0,$$

and hence $L(b,a)=0$. Therefore $a\perp b$ if and only if $L(a,b)=0$.

In the setting of (complex) JB*-triples, every element admits 3rd- and 5th- square roots. In fact, a continuous functional calculus can be derived from the Gelfand representation for abelian JB*-triples (cf. [14, §1]). Let a be an element in a JB*-triple E. Denoting by E_a the JB*-subtriple generated by the element a, it is known that E_a is JB*-triple isomorphic (and hence isometric) to $C_0(L)=C_0(L,\mathbb{C})$ for some locally compact Hausdorff space $L\subseteq(0,\|a\|]$, such that $L\cup\{0\}$ is compact. It is also known that there exists a triple isomorphism Ψ from E_a onto $C_0(L)$ satisfying $\Psi(a)(t)=t$ $(t\in L)$ (compare [14, Lemma 1.14] or [15, Proposition 3.5]). Having in mind this identification we can always find a (unique) element z in E_a such that $z^{[5]}=a$. The element z will be denoted by $a^{[\frac{1}{5}]}$.

When E is a (real) J*B-triple, we have already commented that the norm closed subtriple generated by a single element a is triple isomorphic (and isometric) to

$C_0(L,\mathbb{R}) := \{f \in C_0(L), f(L) \subseteq \mathbb{R}\}$, for some locally compact subset $L \subseteq (0, \|a\|]$ with $L \cup \{0\}$ compact. Therefore there exists a (unique) element $z = a^{[\frac{1}{5}]}$ in E_a such that $z^{[5]} = a$.

It should be noticed here that, in the setting of J*B-triples (resp., JB*-triples) orthogonality is a "local concept" (compare Lemma 1 in [5] whose proof remains valid for J*B-triples). Indeed, two elements a and b in a J*B-triple E are orthogonal if and only if one of the following equivalent statements holds:

$$\text{(a) } \{a,a,b\} = 0, \quad \text{(b) } E_a \perp E_b, \quad \text{(c) } \{b,b,a\} = 0,$$
$$\text{(d) } a \perp b \text{ in a subtriple of } E \text{ containing both elements.}$$

It can be easily seen that $a \perp b$ if and only if $a^{[\frac{1}{5}]} \perp b^{[\frac{1}{5}]}$.

Lemma 13. *Let $T : E \to F$ be a not necessarily continuous triple homomorphism between two Jordan-Banach triples and let (x_n) be a sequence of mutually orthogonal norm-one elements in $\sigma_E(T)$. Then, except for a finite number of values of n, $T(x_n)^{[5]} = 0$. Further, if E is a JB*-triple or a (real) J*B-triple or F is an anisotropic Jordan-Banach triple, then $T(x_n) = 0$, except for finitely many $n \in \mathbb{N}$.*

Proof. We shall argue by contradiction, supposing that $T(x_n)^{[5]} \neq 0$ for infinitely many n in $\mathbb{N}$. By passing to a subsequence, we may assume $T(x_n)^{[5]} \neq 0$ for every $n \in \mathbb{N}$. Since (x_n) is a sequence in $\sigma_E(T)$, for each $n \in \mathbb{N}$, there is a sequence $(a_{n,k})_k \subseteq E$ such that $\lim_k a_{n,k} = 0$ and $\lim_k T(a_{n,k}) = T(x_n)$. Thus, for each n in $\mathbb{N}$, $\lim_k \{x_n, a_{n,k}, x_n\} = 0$. The continuity of the triple product in F implies that

$$\lim_k T(\{x_n, x_n, \{x_n, a_{n,k}, x_n\}\})$$
$$= \lim_k \{T(x_n), T(x_n), \{T(x_n), T(a_{n,k}), T(x_n)\}\} = T(x_n)^{[5]} \neq 0.$$

We observe that, for each $n \in \mathbb{N}$, the set

$$\{k \in \mathbb{N} : \{x_n, a_{n,k}, x_n\} \neq 0\}$$

is infinite. Passing to a subsequence of $(a_{n,k})$ we may assume that

$$\{x_n, a_{n,k}, x_n\} \neq 0, \forall (n,k) \in \mathbb{N} \times \mathbb{N}.$$

Therefore,

$$\lim_k \frac{\|T(\{x_n, x_n, \{x_n, a_{n,k}, x_n\}\})\|}{\|\{x_n, a_{n,k}, x_n\}\|} = \infty.$$

For each positive integer n, pick $m(n)$ such that

$$\frac{\|T(\{x_n, x_n, \{x_n, a_{n,m(n)}, x_n\}\})\|}{\|\{x_n, a_{n,m(n)}, x_n\}\|} > n\|x_n\|^2. \tag{3}$$

Writing $y_n = \{x_n, a_{n,m(n)}, x_n\}$, it follows by (2) that $y_n \perp x_m$ for $n \neq m$. The inequality (3) yields $\frac{\|T(x_n, x_n, y_n)\|}{\|x_n\|^2 \|y_n\|} > n, \forall n \in \mathbb{N}$, which contradicts the main boundeness theorem (compare Theorem 12).

If E is a JB*-triple (resp., a J*B-triple), by Lemma 10, $\sigma_E(T)$ is a norm closed ideal of E and hence a JB*-triple (resp., a J*B-triple). Therefore, the sequence of mutually orthogonal elements $(z_n) = (x_n^{[\frac{1}{5}]})$ lies in $\sigma_E(T)$. Since $T(z_n)^{[5]} = T(z_n^{[5]}) = T(x_n)$, we have $T(x_n) = 0$ except for finitely many n in $\mathbb{N}$.

Finally, when F is anisotropic the final statement follows straightforwardly. $\square$

An element e in a normed Jordan triple E is called *tripotent* if $\{e,e,e\}=e$. Every tripotent e induces a decomposition $E=E_2(e)\oplus E_1(e)\oplus E_0(e)$ into the corresponding *Peirce spaces* where $E_j(e)$ is the $\frac{j}{2}$ eigenspace of $L(e,e)$. Furthermore, the following Peirce rules are satisfied:

$$\{E_2(e),E_0(e),E\}=\{E_0(e),E_2(e),E\}=0,$$
$$\{E_i(e),E_j(e),E_k(e)\}\subseteq E_{i-j+k}(e),$$

where $E_{i-j+k}(e)=0$ whenever $i-j+k\notin\{0,1,2\}$ (compare [24, Proposition 21.9]). The projection $P_j(e):E\to E_j(e)$ is called the *Peirce-j* projection induced by e.

The Peirce-2 subspace, $E_2(e)$, associated with a tripotent e is a normed Jordan *-algebra with respect to the product and involution defined by $x\circ_e y:=\{x,e,y\}$ and $x^{\sharp_e}:=\{e,x,e\}$, respectively (compare [24, Lemma 21.11]).

Lemma 14. *Let $T:E\to F$ be a not necessarily continuous triple homomorphism between two Jordan-Banach triples. Then for each tripotent e in $\sigma_E(T)$ we have $T(e)=0$.*

Proof. Suppose that there exists a tripotent e in $\sigma_E(T)$ with $T(e)\neq 0$. The linear mapping $T_{|E_2(e)}:E_2(e)\to F_2(T(e))$ is a (unital) triple homomorphism between (unital) Jordan-Banach algebras. Then T is a (unital) Jordan homomorphism. Let (x_n) be a sequence in E such that $x_n\to 0$ and $T(x_n)\to T(e)$. Then $P_2(e)(x_n)\to 0$ and $T(P_2(e)(x_n))=P_2(T(e))(T(x_n))\to T(e)$. Thus, e is an idempotent in $\sigma_{E_2(e)}(T_{|E_2(e)})$ with $T(e)\neq 0$, which contradicts Theorem 3.12 or Corollary 3.13 in [18]. □

Lemma 15. *Let $T:E\to F$ be a not necessarily continuous triple monomorphism from a JB*-triple (resp., a J*B-triple) to a Jordan-Banach triple. Then $\sigma_E(T)=0$.*

Proof. Suppose that $\sigma_E(T)\neq 0$. Then, by Lemma 10, $\sigma_E(T)$ is a norm-closed triple ideal of E, and hence a JB*-triple (resp., a J*B-triple). Suppose that a is a nonzero element in $\sigma_E(T)$. We have already seen that E_a is isometrically triple isomorphic to $C_0(L)$ for some subset $L\subseteq(0,\|a\|]$ with $L\cup\{0\}$ compact.

We claim that L is finite. Indeed, if L were infinite, we could find, via Urysohn's lemma, a sequence of mutually orthogonal norm-one elements $(x_n)_{n\in\mathbb{N}}\subseteq E_a\subseteq\sigma_E(T)$. Since T is injective we have $T(x_n)\neq 0,\forall n\in\mathbb{N}$, which contradicts Lemma 13. Therefore L must be finite.

Let $t\in L$. Since L is finite, the function e defined by $e(t)=1$ and $e(L\setminus\{t\})=0$ lies in $C_0(L)$. The element e is a tripotent in $E_a\subseteq\sigma_E(T)$ with $T(e)\neq 0$, which, by Lemma 14, is impossible. □

The following proposition is a direct consequence of Lemma 15 and Proposition 11.

Proposition 16. *Let $T:E\to F$ be a not necessarily continuous triple monomorphism from a (complex) JB*-triple (resp., a (real) J*B-triple) to a Jordan-Banach triple. Then the linear mapping $\widetilde{T}:E\to F/\sigma_F(T)$, $\widetilde{T}(a)=T(a)+F/\sigma_F(T)$, is a continuous triple monomorphism.* □

Theorem 17. *Let $T:E\to F$ be a not necessarily continuous triple monomorphism from a (complex) JB*-triple (resp., a (real) J*B-triple) to a normed Jordan triple. Then T is bounded below.*

Proof. We may assume, without loss of generality, that F is a Jordan-Banach triple; otherwise we can replace F with its canonical completion.

Let π denote the canonical projection of F onto $F/\sigma_F(T)$. Proposition 16 assures that the linear mapping $\widetilde{T} : E \to F/\sigma_F(T)$, $x \mapsto \pi(T(x))$, is a continuous triple monomorphism. By Propositions 6 and 5, there exists a positive constant M satisfying

$$M \|x\| \leq \|\widetilde{T}(x)\| = \|\pi(T(x))\| \leq \|T(x)\|, \ x \in E,$$

which shows that T is bounded below. □

The following corollary is the desired generalisation of a result due to B. Yood [25] and S.B. Cleveland [8].

Corollary 18. *Let $T : E \to F$ be a not necessarily continuous triple monomorphism from a (complex) JB*-triple (resp., a (real) J*B-triple) to a normed Jordan triple. Then the norm closure of $T(E)$ in the canonical completion of F decomposes as the direct sum of $T(E)$ and $\sigma_F(T)$.*

Proof. Let b be an element in the norm closure of $T(E)$ in the completion of F. By assumptions, there exists a sequence (x_n) in E such that $b = \lim T(x_n)$.

Since, by Theorem 17, T is bounded below, the sequence (x_n) is a Cauchy sequence in E. Therefore there exists x_0 in E satisfying $\lim x_n - x_0 = 0$ and $\lim T(x_n - x_0) = b - T(x_0)$. This shows that $b = T(x_0) + (b - T(x_0))$, where $b - T(x_0) \in \sigma_F(T)$. Finally, $T(E) \cap \sigma_F(T) = T(\sigma_E(T)) = \{0\}$, by Lemma 15. □

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