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ON THE RADICAL OF A FREE MALCEV ALGEBRA

I. P. SHESTAKOV AND A. I. KORNEV

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ABSTRACT. We prove that the prime radical $rad\,\mathcal{M}$ of the free Malcev algebra $\mathcal{M}$ of rank more than two over a field of characteristic $\neq 2$ coincides with the set of all universally Engelian elements of $\mathcal{M}$. Moreover, let $T(\mathbb{M})$ be the ideal of $\mathcal{M}$ consisting of all stable identities of the split simple 7-dimensional Malcev algebra $\mathbb{M}$ over F. It is proved that $rad\,\mathcal{M} = J(\mathcal{M}) \cap T(\mathbb{M})$, where $J(\mathcal{M})$ is the Jacobian ideal of $\mathcal{M}$. Similar results were proved by I. Shestakov and E. Zelmanov for free alternative and free Jordan algebras.

An algebra M is called a *Malcev algebra* if it satisfies the identities

$$x^2 = 0,$$
$$J(x, xy, z) = J(x, y, z)x,$$

where $J(x, y, z) = (xy)z + (zx)y + (yz)x$ is the Jacobian of the elements x, y, z [7, 9, 5]. Since for a Lie algebra the Jacobian of any three elements vanishes, Lie algebras fall into the variety of Malcev algebras. Among the non–Lie Malcev algebras, the traceless elements of the octonion algebra with the product given by the commutator $[x, y] = xy - yx$ is one of the most important examples [9, 5, 6].

In 1977 I. P. Shestakov [11] proved that the free Malcev algebra $\mathcal{M}_n$ on $n \geq 9$ free generators is not semiprime; that is, $\mathcal{M}_n$ contains nonzero nilpotent ideals. In 1979, V. T. Filippov [3] extended this result to free Malcev algebras with more than four generators. Therefore, the prime radical $rad\,\mathcal{M}_n \neq 0$ for $n > 4$, and a natural question on the description of this radical arises.

For free alternative algebras, it was proved by Shestakov in [10] that the prime radical coincides with the set of nilpotent elements. A similar fact was established by E. Zel′manov [15] for free Jordan algebras.

In anticommutative algebras, the role of nilpotent elements is played by engelian elements. An element a of an algebra M is called *engelian* if the operator of right multiplication $R_a : x \mapsto xa$ is nilpotent. We will call an element $a \in M$ *universally engelian* if, for every algebra $M' \supseteq M$, the element a is engelian in M'. In other words, the image $\mathcal{R}_a$ of the element a in the (associative) universal multiplicative enveloping algebra $\mathcal{R}(M)$ of M is nilpotent. In the present paper, we prove that the

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prime radical of the free Malcev algebra over a field of characteristic $\neq 2$ coincides with the set of all universally engelian elements.

We consider the algebras over a field F of characteristic $\neq 2$. Denote $[x, y, z] = (xy)z + x(yz)$ and $\{x, y, z\} = [x, y, z] - [x, z, y] = J(x, y, z) - 3(yz)x$, and consider the function

$$h(y, z, t, x) = \{yz, t, x\}x + \{yx, z, x\}t.$$

The variety of Malcev algebras defined by the identity $h = 0$ is denoted by $\mathcal{H}$.

An anticommutative algebra A is called an algebra *without zero divisors* if for any $a, b \in A$ the equality $ab = 0$ implies that a, b are linearly dependent.

Below we sometimes will omit the parentheses in left-normed products; that is, $abcd$ means $((ab)c)d$ and ab^k means aR_b^k.

Lemma 1. *Let A be an algebra of dimension ≥ 2 from the variety $\mathcal{H}$ without zero divisors. If A has a nonzero Engelian element, then A is isomorphic to $sl(2)$ or to the nonabelian two-dimensional Lie algebra.*

Proof. Let $f \in A$, $f \neq 0$ be an engelian element of the algebra A. The operator R_f is nilpotent, and it is well known that A is a direct sum of cyclic subspaces of operator R_f, i.e.

$$A = \bigoplus_i (Fa_i + Fa_iR_f + \cdots + Fa_iR_f^{n_i-1}),$$

where $a_iR_f^{n_i} = 0$. Since $(\bigoplus_i Fa_iR_f^{n_i-1}) \cdot f = \bigoplus_i (Fa_iR_f^{n_i}) = 0$, by the absence of zero divisors we have the inclusion $\bigoplus_i Fa_iR_f^{n_i-1} \subseteq Ff$, and $A = Fa + Faf + \cdots + Faf^{n-1}$, where $af^{n-1} = f$.

If $n = 2$, then $A = Fa + Ff$, $af = f$, and we have the nonabelian two-dimensional Lie algebra.

Let $n \geqslant 3$. Substituting $x = z = f$ in h, we obtain for any $y, t \in A$ the equality $(tf)(yf)f = 0$. This implies $(af)(af^{n-1})f = 0$, i.e. $(af)(af^{n-1}) \in Ff$. Therefore, $af^2 = (af)f = (af)(af^{n-1}) \in Ff$, and $n = 3$. We can choose a such that $af^2 = f$. Consider $a(af) = \alpha a + \beta af + \gamma af^2$ for some $\alpha, \beta, \gamma \in F$. We have $0 = J(a, af, f) = a(af)f + affa = (\alpha - 1)af + \beta af^2$. Since af and af^2 are linearly independent, it follows that $\alpha = 1$ and $\beta = 0$, i.e. $a(af) = a + \gamma af^2$. Now setting $e_1 = a + \frac{\gamma}{2}af^2$ and $e_2 = af^2$ we have $e_1e_2 = (a + \frac{\gamma}{2}af^2)af^2 = af$. $e_1(e_1e_2) = (a + \frac{\gamma}{2}af^2)(af) = a(af) - \frac{\gamma}{2}af^2 = e_1$. Consequently, $A = Fe_1 \oplus Fe_2 \oplus Fe_3$, where $e_1e_2 = e_3$, $e_1e_3 = e_1$, $e_2e_3 = -e_2$, i.e. $A \cong sl(2)$. The lemma is proved. □

Lemma 2. *Let $\mathbb{O}$ be an octonion division algebra over a field F. Then the Malcev algebra $sl(\mathbb{O})$ of traceless octonions has no zero divisors.*

Proof. Assume that $a, b \in \mathbb{O}$, $tr(a) = tr(b) = 0$ and $[a, b] = 0$. Recall that we have the following identity in $\mathbb{O}$ (see [16, identity (2.9)]):

$$xy + yx = tr(x)y + tr(y)x + tr(xy) - tr(x)tr(y).$$

Therefore, we have $0 \neq 2ab = tr(ab) \in F$ and hence $b \in Fa^{-1}$. On the other hand, $0 \neq a^2 = -n(a) \in F$; hence $a^{-1} \in Fa$. Therefore, a and b are linearly dependent. □

Corollary 1. *Let $\mathbb{O}$ be an octonion division algebra over a field F. Then the Malcev algebra $sl(\mathbb{O})$ of traceless octonions has no nonzero engelian elements.*

Proof. In fact, it is well known (see, for instance, [2, p. 104]) that the algebra $sl(\mathbb{O})$ satisfies the identity h. □

Let M be a Malcev algebra and $\mathcal{R}(M)$ be the universal multiplicative enveloping algebra of M. We recall (see [4, Definition 2.3]) that the algebra $\mathcal{R}(M)$ is defined up to isomorphism as the universal object with respect to representations of the algebra M. It can be constructed as the quotient algebra $T(M)/I$, where $T(M)$ is the tensor algebra of the vector space M and I is the ideal of $T(M)$ generated by the set $\{x\otimes y\otimes z - z\otimes x\otimes y - (yz)\otimes x - (xy)z + y\otimes (xz) \mid x, y, z \in M\}$ (see [4, p. 89]). An alternative way to construct $\mathcal{R}(M)$ is the following one. Consider the free product $M' = M * Fx$ of the algebra M and the one-dimensional Malcev algebra Fx in the variety of Malcev algebras (see [1]); then the algebra $\mathcal{R}(M)$ is isomorphic to the subalgebra of the algebra $End_F\,(M')$ generated by all the multiplication operators $R_m,\ m \in M$ (see [14]).

We denote by $\mathcal{R}$ the linear mapping $\mathcal{R} : M \to \mathcal{R}(M)$ which maps $a \in M$ to $\mathcal{R}_a = a + I \in \mathcal{R}(M)$. For an ideal K of M, we denote by $\mathcal{R}_K$ the ideal of the algebra $\mathcal{R}(M)$ generated by the set $\{\mathcal{R}_x \mid x \in K\}$.

Recall that in any Malcev algebra the following identity holds (see [9]):

$$2J(x, y, z)t = J(t, xy, z) + J(t, zx, y) + J(t, yz, x). \tag{1}$$

By $J(M)$ we denote the subspace of M generated by all Jacobians $J(x, y, z)$, $x, y, z \in M$. It follows from (1) that $J(M)$ is an ideal of M.

Lemma 3. *For an element $f \in J(M)$ and for an ideal K of the algebra M, the inclusion $(\mathcal{R}_f)^n \in \mathcal{R}_K$ holds for some $n > 0$ if and only if there exists an $m > 0$ such that $Mf^m \subseteq K$.*

Proof. We note first that if $(\mathcal{R}_f)^n \in \mathcal{R}_K$ for some $n \in \mathbb{N}$, then obviously $Mf^n \subseteq M\mathcal{R}_K \subseteq MK\mathcal{R}(M) \subseteq K$.

Now assume that the inclusion $Mf^m \subseteq K$ holds for some $m \in \mathbb{N}$. Let $f = \sum_i J(a_i, b_i, c_i)$ for some $a_i, b_i, c_i \in M$. Then by (1) we have in the algebra $M' = M * Fx$,

$$2xf = 2\sum_i xJ(a_i, b_i, c_i) = \sum_i J(x, c_i, a_ib_i) + \sum_i J(x, a_i, b_ic_i) + \sum_i J(x, b_i, c_ia_i).$$

Furthermore, using identity (1) again, we have

$$\begin{aligned} 4xf^2 = &\sum_i J(f, xc_i, a_ib_i) + \sum_i J(f, c_i(a_ib_i), x) + \sum_i J(f, (a_ib_i)x, c_i) \\ &+ \sum_i J(f, xa_i, b_ic_i) + \sum_i J(f, a_i(b_ic_i), x) + \sum_i J(f, (b_ic_i)x, a_i) \\ &+ \sum_i J(f, xb_i, c_ia_i) + \sum_i J(f, b_i(c_ia_i), x) + \sum_i J(f, (c_ia_i)x, b_i). \end{aligned}$$

Now, by the defining Malcev identity,

$$
\begin{aligned}
4(-1)^m x f^{m+2}& \\
&= \sum_i J(f, xc_i, a_ib_if^m) + \sum_i J(f, c_i(a_ib_i)f^m, x) + \sum_i J(f, (a_ib_i)x, c_if^m) \\
&\quad + \sum_i J(f, xa_i, b_ic_if^m) + \sum_i J(f, a_i(b_ic_i)f^m, x) + \sum_i J(f, (b_ic_i)x, a_if^m) \\
&\quad + \sum_i J(f, xb_i, c_ia_if^m) + \sum_i J(f, b_i(c_ia_i)f^m, x) + \sum_i J(f, (c_ia_i)x, b_if^m).
\end{aligned}
$$

Since by assumption $Mf^m \subseteq K$, we have

$$
\begin{aligned}
xf^{m+2} \in &\sum_i J(f, xc_i, K) + \sum_i J(f, xa_i, K) + \sum_i J(f, xb_i, K) + J(f, K, x) \\
&+ \sum_i J(f, (a_ib_i)x, K) + \sum_i J(f, (b_ic_i)x, K) + \sum_i J(f, (c_ia_i)x, K) \subseteq x\mathcal{R}_K.
\end{aligned}
$$

Clearly, this implies that $\mathcal{R}_f^{m+2} \in \mathcal{R}_K$. □

Denote by $\mathbb{M} = \mathbb{M}(F)$ the split simple 7-dimensional Malcev algebra over the field F (see [5, 6]) and by $T(\mathbb{M})$ the subset of the free Malcev algebra $\mathcal{M}_\infty$ on a countably infinite set of generators consisting on all *stable* identities of the algebra $\mathbb{M}$, that is, the identities all of whose partial linearizations are also identities of $\mathbb{M}$. By the standard arguments (see, for example, [16, p. 317]), $T(\mathbb{M})$ is a T-ideal of $\mathcal{M}_\infty$; moreover, for every associative commutative F-algebra K, the algebra $\mathbb{M}(K) = K \otimes_F \mathbb{M}$ satisfies all identities from $T(\mathbb{M})$.

The following result follows easily from [2, Theorem 1] by the same arguments as for free alternative algebras in [10]. We give its proof for the sake of completeness.

Theorem 1. *$rad\,\mathcal{M}_\infty = J(\mathcal{M}_\infty) \cap T(\mathbb{M})$.*

Proof. Let $f = f(x_1, \ldots, x_n) \in rad\,\mathcal{M}_\infty$. The free Lie algebra $\mathcal{M}_\infty/J(\mathcal{M}_\infty)$ is evidently semiprime, which implies that $rad\,\mathcal{M}_\infty \subseteq J(\mathcal{M}_\infty)$. In order to prove that $f \in T(\mathbb{M})$, it suffices to prove that f is an identity in the algebra $\mathbb{M}(K)$, where K is an infinite domain containing F. Take $K = F[t]$ and $\mathbb{M}_1 = \mathbb{M}(F[t])$. Assume that f is not the identity in $\mathbb{M}_1$. Then there exist elements $c_1, \ldots, c_n \in \mathbb{M}_1$ such that $f(c_1, \ldots, c_n) \neq 0$. Since $\mathbb{M}_1$ is finitely generated over F, there exists an epimorphism $\varphi : \mathcal{M}_\infty \twoheadrightarrow \mathbb{M}_1$ such that $\varphi(x_i) = c_i$. We have

$$rad\,\mathbb{M}_1 \supseteq \varphi(rad\,\mathcal{M}_\infty) \ni \varphi(f) = f(c_1, \ldots, c_n) \neq 0.$$

But the algebra $\mathbb{M}_1$ is a central order in the simple central 7-dimensional algebra $\mathbb{M}(F(t))$ over the field $F(t)$ and by [8] is prime. Thus $rad\,\mathbb{M}_1 = 0$, a contradiction.

Conversely, let $f \in J(\mathcal{M}_\infty) \cap T(\mathbb{M})$. If $f \notin rad\,\mathcal{M}_\infty$, then there exists a prime Malcev algebra M and an epimorphism $\varphi : \mathcal{M}_\infty \twoheadrightarrow M$ such that $\varphi(f) \neq 0$. In other words, f is not an identity in M. It is clear that M is not a Lie algebra. By [2, Theorem 1], then M is a central order in a simple 7-dimensional Malcev algebra $\widetilde{M}$ over an extension K of F. Clearly, f is not an identity of $\widetilde{M}$. Without loss of generality, we may assume that K is algebraically closed and $\widetilde{M}$ is split. But then $\widetilde{M} \cong K \otimes_F \mathbb{M}$ and $\widetilde{M}$ satisfies all the identities from $T(\mathbb{M})$. Hence f is an identity of $\widetilde{M}$, a contradiction. □

For an algebra M, denote by $Engel\,M$ the set of all universally engelian elements of M.

Theorem 2. *Let $\mathcal{M}$ be the free Malcev algebra on more than two generators over a field F of characteristic $\neq 2$. Then $rad\,\mathcal{M} = Engel\,\mathcal{M}$.*

Proof. Let us first show that $Engel\,\mathcal{M} \subseteq rad\,\mathcal{M}_\infty$. Clearly, every universally engelian element remains so under an epimorphism. Therefore, by Theorem 1, it suffices to show that the quotient algebras $\mathcal{M}/J(\mathcal{M})$ and $\mathcal{M}/(T(\mathbb{M}) \cap \mathcal{M})$ have no nonzero universally engelian elements. For the free Lie algebra $\mathcal{M}/J(\mathcal{M})$ it is clear. Let us prove this for the algebra $\mathcal{M}/(T(\mathbb{M}) \cap \mathcal{M})$. Let $f = f(x_1, \ldots, x_n)$ be a universally engelian element of $\mathcal{M}/(T(\mathbb{M}) \cap \mathcal{M})$. Then this algebra satisfies the identity $x_{n+1}f^k = 0$. Observe that the algebra $\mathcal{M}/(T(\mathbb{M}) \cap \mathcal{M})$ is isomorphic to the free algebra (on the same number of generators) in the variety generated by the algebra $\mathbb{M}(F[t])$. In particular, the identity $x_{n+1}f^k = 0$ is valid in $\mathbb{M}(K)$ for every extension K of F. By [16, Theorem 2.8], there exists an extension K of F and a division octonion algebra $\mathbb{O}$ over K. Let $\widetilde{K}$ be the algebraic closure of K. Then the Malcev algebra $\widetilde{K} \otimes_K sl(\mathbb{O})$ is split and therefore is isomorphic to $\mathbb{M}(\widetilde{K})$. Thus it satisfies the identity $x_{n+1}f^k = 0$ and consequently so does the algebra $sl(\mathbb{O})$. Assume that there exist $a_1, \ldots, a_n \in sl(\mathbb{O})$ such that $a = f(a_1, \ldots, a_n) \neq 0$. Then a is a nonzero engelian element in $sl(\mathbb{O})$, which is impossible according to Corollary 1. This proves the inclusion $Engel\,\mathcal{M} \subseteq rad\,\mathcal{M}_\infty$.

Now, we will prove that $rad\,\mathcal{M} \subseteq Engel\,\mathcal{M}$. Suppose conversely that there exists $f \in rad\,\mathcal{M}$ which is not universally engelian in $\mathcal{M}$. Consider the set of ideals $\Sigma = \{K \lhd \mathcal{M} \mid \forall m \in \mathbb{N}, \mathcal{R}_f^m \notin \mathcal{R}_K\}$. The set Σ is not empty because $\{0\} \in \Sigma$; in addition Σ is partially ordered by inclusions. If $\{K_s \mid s \in S\}$ is a chain of elements of Σ, then $\bigcup_{s\in S} K_s \in \Sigma$. By the Zorn lemma, Σ has a maximal element P. We will prove that P is a prime ideal in $\mathcal{M}$. Suppose that $K, L \lhd \mathcal{M}$ such that $K \supsetneq P$, $L \supsetneq P$, and $KL \subseteq P$. By hypothesis, there exist $l_1 \in \mathbb{N}$ and $l_2 \in \mathbb{N}$ such that $\mathcal{R}_f^{l_1} \in \mathcal{R}_K$ and $\mathcal{R}_f^{l_2} \in \mathcal{R}_L$. Then $\mathcal{R}_f^{l_1+l_2} \in \mathcal{R}_K\mathcal{R}_L$, and we have

$$\mathcal{M}f^{l_1+l_2} \subseteq Kf^{l_2} \subseteq (KL)\mathcal{R}(\mathcal{M}) \subseteq P.$$

Since $f \in J(\mathcal{M})$, by Lemma 3 we have $\mathcal{R}_f^m \subseteq \mathcal{R}_P$ for some m, a contradiction. Therefore the ideal P is prime and $f \notin P$, which is impossible.

We have proved that $rad\,\mathcal{M} \subseteq Engel\,\mathcal{M} \subseteq rad\,\mathcal{M}_\infty \cap \mathcal{M}$. But if A is a subalgebra of B, then $(rad\,B) \cap A \subseteq rad\,A$. Therefore, $rad\,\mathcal{M} = rad\,\mathcal{M}_\infty \cap \mathcal{M} = Engel\,\mathcal{M}$.

The theorem is proved. □

Observe that we have also proved the following result of independent interest:

Corollary 2. *For every $n > 2$, $rad\,\mathcal{M}_n = rad\,\mathcal{M}_\infty \cap \mathcal{M}_n = T(\mathbb{M}) \cap J(\mathcal{M}_n)$.*

We finish the paper with two open questions:

Problem 1. Is $rad\,\mathcal{M}_\infty$ ($rad\,\mathcal{M}_n$) nilpotent?

Problem 2. Is $rad\,\mathcal{M}_4 \neq 0$?

Observe that recently the second author has proved that $rad\,\mathcal{M}_3 = 0$. The proof will appear elsewhere. In [12, 13] it was proved that the radical of a free alternative algebra is nilpotent if the set of generators is finite or the base field has characteristic 0. This easily implies that, under the same conditions, the answer to the first question is positive if the algebra $\mathcal{M}_\infty$ is special, that is, can be embedded into the commutator algebra of an alternative algebra.

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Institute of Mathematics and Statistics, University of São Paulo, Rua do Matao, 1010, Cidade Universitária, São Paulo 05508-090, Brazil

IMECC Cidade Universitária Zeferino Vaz, Campinas, 13083-859 São Paulo, Brazil

Current address: Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Rua Santa Adélia, 166, Blocoa, Bairro Bangu, Santo André, SP, Brazil 09210-170