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POINCARÉ FUNCTIONS WITH SPIDERS' WEBS

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ABSTRACT. For a polynomial p with a repelling fixed point z_0, we consider *Poincaré functions* of p at z_0, i.e. entire functions $\mathfrak{L}$ which satisfy $\mathfrak{L}(0) = z_0$ and $p(\mathfrak{L}(z)) = \mathfrak{L}(p'(z_0) \cdot z)$ for all $z \in \mathbb{C}$. We show that if the component of the Julia set of p that contains z_0 equals $\{z_0\}$, then the (fast) escaping set of $\mathfrak{L}$ is a *spider's web*; in particular, it is connected. More precisely, we classify all linearizers of polynomials with regard to the spider's web structure of the set of all points which escape faster than the iterates of the maximum modulus function at a sufficiently large point R.

1. INTRODUCTION

Let f be a transcendental entire function and denote by f^n the n-th iterate of f. With the fundamental work of Eremenko [4], the *escaping set*

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

has become an intensively studied object in transcendental holomorphic dynamics. Since then, much progress has been achieved in exploring the topological and dynamical properties of the escaping set and some of its subsets (for some results, see [9, 13, 14, 15, 16, 17]).

Rippon and Stallard discovered that the *fast escaping set* $A(f)$, which was originally introduced by Bergweiler and Hinkkanen [2], shares many significant features with $I(f)$. If we set $M(r, f) := \max_{|z|=r} |f(z)|$ and choose any constant R such that

$$M(r, f) > r \text{ whenever } r \geq R, \tag{1.1}$$

the fast escaping set of f can be described as

$$A(f) = \bigcup_{l \in \mathbb{N}} A_R^{-l}(f),$$

where $A_R^l(f)$ are the so-called *level sets*, defined by

$$A_R^l(f) := \{z \in \mathbb{C} : |f^n(z)| \geq M^{n+l}(R, f), n \geq \max\{0, -l\}\}$$

for $l \in \mathbb{Z}$. (Throughout the article M^n denotes the n-th iterate of the maximum modulus function with respect to R.)

Recently, Rippon and Stallard [16, 14] introduced the concept of an *(infinite) spider's web*. This is a connected set $E \subseteq \mathbb{C}$ with the property that there exists a

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sequence of increasing simply connected domains (G_n) whose union is all of $\mathbb{C}$ such that $\partial G_n \subseteq E$ for all n.

In [16], various sufficient criteria are presented such that $I(f)$ and $A(f)$ are spiders' webs. Primarily, this is the case whenever the set

$$A_R(f) := A_R^0(f)$$

is a spider's web for some (and hence all by [16, Lemma 7.1]) R as in (1.1). Functions for which $A_R(f)$ is a spider's web have some strong dynamical properties. For instance, every such function has only bounded Fatou components and there exists no curve to ∞ on which f is bounded (compare [16]). In particular, the set of singular values of f must be unbounded. (For precise definitions, see Section 2.)

In this paper, we present a large and interesting class of functions for which $A_R(f)$, and hence $A(f)$ and $I(f)$, is a spider's web, namely, Poincaré functions of certain polynomials. To make this precise, let p be a polynomial with a repelling fixed point z_0 (i.e. $p(z_0) = z_0$ and $|p'(z_0)| > 1$). Then there exists an entire function $\mathfrak{L}$ called a *Poincaré function* or a *linearizer of p at z_0* which satisfies

$$\mathfrak{L}(0) = z_0 \quad \text{and} \quad p(\mathfrak{L}(z)) = \mathfrak{L}(p'(z_0) \cdot z) \tag{1.2}$$

for all $z \in \mathbb{C}$.

It was conjectured by Rempe that the escaping set of a linearizer of a quadratic polynomial for which the critical point escapes is a spider's web. In this article, we show that this is true; moreover, we classifiy all linearizers of polynomials corresponding to whether the sets $A_R(\mathfrak{L})$ are spiders' webs or not. Before we state the main theorem, recall that the *Fatou set* $\mathcal{F}(f)$ of f is the set of all points that have a neighbourhood in which the iterates of f form a normal family, while the *Julia set* $\mathcal{J}(f)$ is defined to be $\mathbb{C} \setminus \mathcal{F}(f)$. For a point $z \in \mathcal{J}(f)$ let $\mathcal{J}_z(f)$ be the component of $\mathcal{J}(f)$ that contains z.

Theorem 1.1. *Let p be a polynomial of degree $d \geq 2$, let z_0 be a repelling fixed point of p and let $\mathfrak{L}$ be a linearizer of p at z_0. If R satisfies* (1.1)*, then $A_R(\mathfrak{L})$ is a spider's web if and only if $\mathcal{J}_{z_0}(p) = \{z_0\}$.*

Since polynomials for which all critical points converge to ∞ have totally disconnected Julia sets [5, p. 85], we obtain, using [16, Theorem 1.4], the following corollary, which also implies Rempe's conjecture.

Corollary 1.2. *Let p be a polynomial of degree $d \geq 2$ for which all critical points escape, and let $\mathfrak{L}$ be a linearizer of p. Assume that R satisfies* (1.1)*. Then each of the sets $A_R(\mathfrak{L})$, $A(\mathfrak{L})$ and $I(\mathfrak{L})$ is a spider's web. In particular, this is true whenever $p(z) = z^2 + c$ and c lies outside the Mandelbrot set.*

We believe that the dichotomy established in Theorem 1.1 for the sets $A_R(\mathfrak{L})$ also extends to the sets $A(\mathfrak{L})$ and $I(\mathfrak{L})$. However, we were not able to prove this. For the fast escaping set, such a result would follow if every continuum in $A(\mathfrak{L})$ (or every 'loop') must be contained in some level set $A_R^l(\mathfrak{L})$, which we also believe to be true for linearizers of polynomials (compare Questions 2 and 3 in [16]).

In the proof of Theorem 1.1, we establish spiders' webs by proving that the corresponding linearizers grow regularly and that there exist simple closed curves arbitrarily close to 0 on which the minimum modulus grows fast enough.

Since the order of a linearizer of a quadratic polynomial is given by $\log 2/\log|p'(z_0)|$ we obtain for any given $\rho \in (0, \infty)$ a linearizer of order ρ whose escaping set is a spider's web.

The functional equation (1.2) also has a solution $\mathfrak{L}$ when p is an *arbitrary* entire map. While much is known about linearizers of polynomials, there seem to be many unsolved problems regarding linearizers of transcendental entire maps.

The analysis of a linearizer strongly depends on the dynamical properties of p, as is already indicated by the fact that p can be iterated in the functional equation (1.2). However, $\mathfrak{L}$ does not only depend on p but also on z_0 and $p'(z_0)$, which makes linearizers good candidates for constructing functions with various interesting analytical properties. Furthermore, they are naturally good candidates for constructing gauge functions to estimate the Hausdorff measure of escaping sets and Julia sets of exponential functions (see [11]).

The appendix of this article addresses various analytic and dynamical properties of linearizers, beyond the connection to spiders' webs and including the case when p is transcendental entire.

2. Preliminaries

The complex plane is denoted by $\mathbb{C}$, the Euclidean circle centred at 0 with radius r is denoted by $\mathbb{S}_r$, and we write $\mathbb{D}_r(z)$ for the Euclidean disk of radius r centred at z.

Unless stated otherwise, we will assume throughout the article that $f : \mathbb{C} \to \mathbb{C}$ is a non-constant, non-linear entire function; so f is either a polynomial of degree ≥ 2 or a transcendental entire map.

Let $C \subset \mathbb{C}$ be a compact set. The *maximum modulus* $M(C, f)$ and the *minimum modulus* $m(C, f)$ of f relative to C are defined to be

$$M(C,f) := \max_{z\in C} |f(z)| \quad \text{and} \quad m(C,f) := \min_{z\in C} |f(z)|.$$

Note that $M(\mathbb{S}_r, f) = M(r, f)$, as defined in the introduction. In analogy, we write $m(r, f)$ for $m(\mathbb{S}_r, f)$. Finally, recall that the *order* of f is defined as

$$\rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

If z is a periodic point of f of period n, we call $\mu(z) := (f^n)'(z)$ the *multiplier* of z. A periodic point z is called *attracting* if $|\mu(z)| < 1$, *indifferent* if $|\mu(z)| = 1$ and *repelling* if $|\mu(z)| > 1$; we say that z is *superattracting* if $\mu(z) = 0$.

Let z_0 be a repelling fixed point of f with multiplier λ. By the Kœnigs Linearization Theorem [10, Theorem 8.2], there exists a holomorphic function l defined in a neighbourhood of 0 such that $l(0) = z_0$ and, locally, $l^{-1} \circ f \circ l(z) = \lambda z$. It was observed already by Poincaré that l can be analytically continued to give a holomorphic function $\mathfrak{L}$ on the entire complex plane; that is, there exists an entire map $\mathfrak{L}$ such that

$$\mathfrak{L}(0) = z_0 \quad \text{and} \quad f(\mathfrak{L}(z)) = \mathfrak{L}(\lambda z) \tag{2.1}$$

for all $z \in \mathbb{C}$. Every such map is called a *linearizer* or *Poincaré function* of f at z_0.

A linearizer is unique up to a constant. More precisely, if $\mathfrak{L}$ satisfies (2.1), then so does $\mathfrak{L}_c : z \mapsto \mathfrak{L}(cz)$ for every $c \in \mathbb{C}^*$, and every solution of the equation (2.1) is of this form.

Note that one can iterate f inside the functional equation and obtain

$$f^n \circ \mathfrak{L}(z) = \mathfrak{L} \circ \lambda^n(z) \tag{2.2}$$

as an iterated version of (2.1), where λ^n denotes the function $z \mapsto \lambda^n z$.

The growth of the function f and a linearizer $\mathfrak{L}$ are related in the following sense: If f is transcendental entire, then $\mathfrak{L}$ has infinite order. If f is a polynomial of degree d, then $\rho(\mathfrak{L}) = \log d/\log|\lambda|$.

If p is a polynomial, the Julia set of p is compact and $I(p)$ is an open connected subset of $\mathcal{F}(p)$; moreover, $I(p)$ is simply connected if and only if $\mathcal{J}(p)$ is connected.

Near ∞, the iterates of a polynomial behave in the following simple way.

Proposition 2.1. *Let $p(z) = \sum_{n=0}^{d} a_n z^n$ be a polynomial of degree $d \geq 2$. Then for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for every z with $|z| > R_\varepsilon$, we have*

$$(1-\varepsilon)|a_d| \cdot |z|^d \leq |p(z)| \leq (1+\varepsilon)|a_d| \cdot |z|^d,$$

and $R_\varepsilon \to \infty$ as $\varepsilon \to 0$.

If ε is chosen small enough such that $(1-\varepsilon)|a_d|R_\varepsilon^{d-1} > 1$, then

$$((1-\varepsilon)|a_d|)^{q_n(d)} \cdot |z|^{d^n} \leq |p^n(z)| \leq ((1+\varepsilon)|a_d|)^{q_n(d)} \cdot |z|^{d^n}$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C}$ with $|z| > R_\varepsilon$, where $q_n(z) := (z^n - 1)/(z-1) = z^{n-1} + \cdots + z + 1$.

Proof. The first statement is elementary and well-known.

Note that we have chosen ε sufficiently small such that $|z| > R_\varepsilon$ implies $|p(z)| > R_\varepsilon$. We will prove the statement inductively. So for $n = 1$ we have $q_1(z) = 1$, and the claim follows from the first part. For the iterate $p^{n+1}(z) = p(p^n(z))$ we then obtain

$$\begin{aligned} |p(p^n(z))| &\leq (1+\varepsilon)|a_d||p^n(z)|^d \leq (1+\varepsilon)|a_d| \left[((1+\varepsilon)|a_d|)^{q_n(d)}|z|^{d^n}\right]^d \\ &= ((1+\varepsilon)|a_d|)^{d\cdot q_n(d)+1}|z|^{d^{n+1}} = ((1+\varepsilon)|a_d|)^{q_{n+1}(d)}|z|^{d^{n+1}} \end{aligned}$$

as well as

$$\begin{aligned} |p(p^n(z))| &\geq (1-\varepsilon)|a_d||p^n(z)|^d \geq (1-\varepsilon)|a_d| \left[((1-\varepsilon)|a_d|)^{q_n(d)}|z|^{d^n}\right]^d \\ &= ((1-\varepsilon)|a_d|)^{d\cdot q_n(d)+1}|z|^{d^{n+1}} = ((1-\varepsilon)|a_d|)^{q_{n+1}(d)}|z|^{d^{n+1}}. \end{aligned}$$

□

Near a point in the Julia set of p, we can make the following statement about the escaping set $I(p)$.

Proposition 2.2. *If p is a polynomial of degree ≥ 2 and $z_0 \in \mathcal{J}(p)$, then the following statements are equivalent:*

- (*i*) *For every $\delta > 0$ there exists a continuum $\gamma_\delta \subset \mathbb{D}_\delta(z_0) \cap I(p)$ separating z_0 and ∞.*
- (*ii*) *$\mathcal{J}_{z_0}(p) = \{z_0\}$.*

If (i) holds, then one can choose γ_δ to be a simple closed curve.

Proof. Let us first assume that for every $\delta > 0$ there exists a continuum $\gamma_\delta \subset I(p)$ around z_0 such that $\max_{w\in\gamma_\delta}|z_0 - w| < \delta$. Then for every δ, $\mathcal{J}_{z_0}(p)$ is contained in the bounded component of $\mathbb{C} \setminus \gamma_\delta$; hence it must consist of a single point.

If $\mathcal{J}_{z_0}(p) = \{z_0\}$, then for every $\delta > 0$ there exist open, non-empty disjoint sets U_δ and V_δ such that U_δ is connected, $\mathcal{J}(p) \subset U_\delta \cup V_\delta$, $\mathcal{J}_{z_0}(p) \subset U_\delta$ and $\max_{w\in U_\delta}|z_0 - w| < \delta/2$. By the Plane Separation Theorem [19, Chapter VI, Theorem 3.1], there exists a simple closed curve γ_δ which separates z_0 from $\mathcal{J}(p) \cap V_\delta$ such that $\gamma_\delta \cap \mathcal{J}(p) = \emptyset$ and every point in $\mathcal{J}(p) \cap U_\delta$ is at distance less than $\delta/2$

from γ_δ. Hence, $\max_{w\in\gamma_\delta}|z_0-w|<\delta$ and $\gamma_\delta\subset\mathcal{F}(p)$. Moreover, the component of $\mathcal{F}(p)$ which contains γ_δ must be $I(p)$ since every bounded component of the Fatou set is simply connected. □

3. Maximum and minimum modulus estimates

From now on, we consider an arbitrary but fixed polynomial p of degree $d\geq 2$, which we write as

$$p(z)=\sum_{i=0}^{d}a_iz^i=a_0+a_1z+\dots+a_dz^d,\quad a_d\neq 0.$$

For every $\varepsilon\in(0,1)$ we pick a constant $R_\varepsilon\geq 1$ for which the conclusion of Proposition 2.1 is satisfied and such that $\varepsilon_1<\varepsilon_2$ implies $R_{\varepsilon_1}>R_{\varepsilon_2}$. We assume that p has a repelling fixed point z_0 with multiplier λ, and we denote by $\mathfrak{L}$ a linearizer of p at z_0. We also pick a constant $R_{\mathfrak{L}}\geq 1$ such that $M(s,\mathfrak{L})>s$ for all $s\geq R_{\mathfrak{L}}$.

Lemma 3.1 (Regularity of growth)**.** *Let $\varepsilon>0$, $r>\max\{R_\varepsilon,R_{\mathfrak{L}}\}$ and define $k_\varepsilon:=\log((1-\varepsilon)|a_d|)$ and $K_\varepsilon:=\log((1+\varepsilon)|a_d|)$. Then*

$$\prod_{i=0}^{n-1}\left(d+\frac{k_\varepsilon}{\log M(|\lambda|^ir),\mathfrak{L}}\right)\leq\frac{\log M(|\lambda|^nr,\mathfrak{L})}{\log M(r,\mathfrak{L})}\leq\prod_{i=0}^{n-1}\left(d+\frac{K_\varepsilon}{\log M(|\lambda|^ir,\mathfrak{L})}\right)$$

holds for all $n\in\mathbb{N}$.

Proof. Let r be as assumed, and let $\tilde{z}\in\mathbb{S}_r$ be a point for which $\mathfrak{L}(\tilde{z})=M(R,\mathfrak{L})$. Let $\tilde{w}:=\mathfrak{L}(\tilde{z})$. Then $|\tilde{w}|=M(r,\mathfrak{L})$ and it follows from the functional equation (2.1) and Proposition 2.1 that

$$\begin{aligned}\log M(|\lambda|r,\mathfrak{L}) &= \log M(r,p\circ\mathfrak{L})=\log M(\mathfrak{L}(\mathbb{S}_r),p)\geq\log p(\tilde{w})\\ &\geq \log\left((1-\varepsilon)|a_d|\cdot|\tilde{w}|^d\right)=k_\varepsilon+d\cdot\log M(r,\mathfrak{L})\end{aligned}$$

and

$$\begin{aligned}\log M(|\lambda|r,\mathfrak{L}) &= \log M(\mathfrak{L}(\mathbb{S}_r),p)\leq\log M(M(r,\mathfrak{L}),p)\\ &\leq \log\left((1+\varepsilon)|a_d|\cdot M(r,\mathfrak{L})^d\right)=K_\varepsilon+d\cdot\log M(r,\mathfrak{L}).\end{aligned}$$

Hence,

$$\left(d+\frac{k_\varepsilon}{\log M(r,\mathfrak{L})}\right)\leq\frac{\log M(|\lambda|r,\mathfrak{L})}{\log M(r,\mathfrak{L})}\leq\left(d+\frac{K_\varepsilon}{\log M(r,\mathfrak{L})}\right).$$

The statement now follows immediately from the fact that

$$\frac{\log M(|\lambda|^nr,\mathfrak{L})}{\log M(r,\mathfrak{L})}=\frac{\log M(|\lambda|^nr,\mathfrak{L})}{\log M(|\lambda|^{n-1}r,\mathfrak{L})}\cdot\ \dots\ \cdot\frac{\log M(|\lambda|r,\mathfrak{L})}{\log M(r,\mathfrak{L})}.$$

□

Lemma 3.2. *For every $k\in\mathbb{N}$ there exists $R_k>0$ such that for all $R>R_k$, $m\leq d^k$ and $n>k$,*

$$M(r_n,\mathfrak{L})>r_{n+1}^m,$$

where the sequence (r_n) is defined by $r_n:=|\lambda|^n\cdot M^n(R,\mathfrak{L})$.

Proof. Let $\varepsilon \in (0, 1/2)$ be arbitrary but fixed, and let $R > \max\{R_{\mathfrak{L}}, R_\varepsilon\}$. It follows from Lemma 3.1 with $r = M^n(R, \mathfrak{L})$ (and $k_\varepsilon := \log((1-\varepsilon)|a_d|)$) that

$$\begin{aligned}\log M(r_n, \mathfrak{L}) &= \log M(|\lambda|^n M^n(R, \mathfrak{L}), \mathfrak{L}) \\ &\geq \prod_{i=0}^{n-1}\left(d + \frac{k_\varepsilon}{\log M(|\lambda|^i M^n(R, \mathfrak{L}), \mathfrak{L})}\right) \cdot \log M(M^n(R, \mathfrak{L}), \mathfrak{L}) \\ &\geq \left(d - \frac{|k_\varepsilon|}{\log R}\right)^n \cdot \log M^{n+1}(R, \mathfrak{L}).\end{aligned}$$

By definition,

$$\begin{aligned}\log r_{n+1}^m &= m \log(|\lambda|^{n+1} M^{n+1}(R, \mathfrak{L})) \\ &= m(n+1)\log|\lambda| + m \log M^{n+1}(R, \mathfrak{L}).\end{aligned}$$

Define $c_R := \frac{|k_\varepsilon|}{\log R}$. We want to show that there exists R_k such that when $R > R_k$, $m \leq d^k$ and $n > k$, then

$$\log M^{n+1}(R, \mathfrak{L}) \cdot ((d - c_R)^n - m) > m(n+1)\log|\lambda|.$$

Obviously, it is sufficient if R_k satisfies

$$\log R_k \cdot ((d - c_{R_k})^n - d^k) > d^k(n+1)\log|\lambda|$$

for all $n \geq k+1$, and this is certainly true when we choose R_k sufficiently large. We will omit the details since they follow from elementary calculus; however, one can prove inductively that every R_k with

$$\log R_k > \max\{2|k_\varepsilon|, \frac{2k}{d}|k_\varepsilon|, \frac{\sqrt{\mathrm{e}}\log|\lambda|}{(2-\sqrt{\mathrm{e}})(k+2)}\}$$

is sufficiently large. □

Lemma 3.3 (Growth of minimum modulus). *If $\mathcal{J}_{z_0}(p) = \{z_0\}$, then for every $m \in \mathbb{N}_{>1}$ there exists $S_m > 0$ with the following property:*

For every $r > S_m$ there is a simple closed curve Γ^r separating $\mathbb{S}_r$ and $\mathbb{S}_{r^m}$ such that

$$m(\Gamma^r, \mathfrak{L}) > M(r, \mathfrak{L}).$$

Proof. Let $D \subset \mathbb{D}$ be a disk around 0 such that $\mathfrak{L}|_D$ is conformal. Let $\delta > 0$ be sufficiently small such that $\mathbb{D}_\delta(z_0) \subset \mathfrak{L}(D)$. By Proposition 2.2 there exists a simple closed curve $\gamma_\delta \subset \mathbb{D}_\delta(z_0) \cap I(p)$ which surrounds z_0. Let $\Gamma_\delta = \mathfrak{L}^{-1}(\gamma_\delta) \cap D$. Then Γ_δ is a simple closed curve surrounding 0. Define

$$s := \min_{z \in \Gamma_\delta}|z| = \mathrm{dist}(0, \Gamma_\delta) \quad \text{and} \quad t := \max_{z \in \Gamma_\delta}|z|.$$

Let $r > \left(\frac{|\lambda| \cdot t}{s}\right)^{\frac{1}{m-1}}$ be an arbitrary but fixed number. We define l_1 and l_2 to be the unique integers for which

$$|\lambda|^{l_1 - 1} \leq r < |\lambda|^{l_1} \quad \text{and} \quad t \cdot |\lambda|^{l_2} < r^m \leq t \cdot |\lambda|^{l_2+1}. \tag{3.1}$$

Note that the lower bound for r and the inequality for r^m imply that

$$s \cdot |\lambda|^{l_2} > r. \tag{3.2}$$

By taking logarithms in (3.1) we obtain the equivalent equations

$$l_1 - 1 \le \frac{\log r}{\log|\lambda|} < l_1 \quad \text{and} \quad l_2 < \frac{m \cdot \log r - \log t}{\log|\lambda|} \le l_2 + 1.$$

A combination of these two inequalities yields

$$m \cdot l_1 - \left(\frac{\log t}{\log|\lambda|} + m + 1\right) \le l_2 < m \cdot l_1 - \frac{\log t}{\log|\lambda|}. \tag{3.3}$$

Let us fix an $\varepsilon \in (0, 1/2)$. Let $j \in \mathbb{N}$ be minimal with the property that $p^j(\gamma_\delta) \subset \{z : |z| > R_\varepsilon\}$. Note that there is a unique integer j with this property since γ_δ is a compact subset of $I(p)$. We define

$$\Gamma^r := \{z \in \mathbb{C} : \lambda^{-l_2} \cdot z \in \Gamma_\delta\}.$$

It follows from (3.1) and (3.2) that Γ^r separates $\mathbb{S}_r$ and $\mathbb{S}_{r^m}$. Using Proposition 2.1, the logarithms of the minimum and maximum modulus can be estimated as follows:

$$\begin{aligned}
\log m(\mathit{\Gamma}^r, \mathfrak{L}) &= \log m(\mathit{\Gamma}_\delta, p^{l_2} \circ \mathfrak{L}) = \log m(\gamma_\delta, p^{l_2}) \\
&\ge \log m(R_\varepsilon, p^{l_2 - j}) \\
&\ge \log\{((1-\varepsilon)|a_d|)^{q_{l_2-j}(d)} \cdot R_\varepsilon^{d^{l_2-j}}\} \\
&= q_{l_2-j}(d) \cdot \log((1-\varepsilon)|a_d|) + d^{l_2-j} \cdot \log R_\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\log M(r, \mathfrak{L}) &= \log M(\mathfrak{L}(\mathbb{S}_{r\cdot|\lambda|^{-l_1}}), p^{l_1}) \le \log M(R_\varepsilon, p^{l_1}) \\
&\le \log\{((1+\varepsilon)|a_d|)^{q_{l_1}(d)} \cdot R_\varepsilon^{d^{l_1}}\} \\
&= q_{l_1}(d) \cdot \log((1+\varepsilon)|a_d|) + d^{l_1} \cdot \log R_\varepsilon.
\end{aligned}$$

Equation (3.3) yields the relation $m \cdot l_1 - C \le l_2 < m \cdot l_1 - c$ with the constants $C := \log t / \log|\lambda| + m + 1$ and $c := \log t / \log|\lambda|$. Furthermore, by Proposition 2.1 we can estimate the polynomials $q_{n+1}(d) = d^n + \cdots + d + 1 = (d^{n+1} - 1)/(d-1)$ by $d^n \le q_{n+1}(d) \le d^{n+1}$. Together, we obtain

$$\begin{aligned}
\log m(\mathit{\Gamma}^r, \mathfrak{L}) &\ge d^{m\cdot l_1 - C - j - 1} \cdot \log((1-\varepsilon)|a_d|) + d^{m\cdot l_1 - C - j} \cdot \log R_\varepsilon \\
&= d^{m\cdot l_1} \cdot \frac{\log((1-\varepsilon)|a_d|R_\varepsilon^d)}{d^{C+j+1}},
\end{aligned}$$

$$\begin{aligned}
\log M(r, \mathfrak{L}) &\le d^{l_1} \cdot \log((1+\varepsilon)|a_d|) + d^{l_1} \cdot \log R_\varepsilon \\
&= d^{l_1} \cdot \log((1+\varepsilon)|a_d|R_\varepsilon)
\end{aligned}$$

as new lower and upper bounds for the minimum and maximum modulus, respectively. Since $m \ge 2$, it is sufficient to find a constant S_m such that for all $r > S_m$,

$$\begin{aligned}
& d^{2l_1} \cdot \frac{\log((1-\varepsilon)|a_d|R_\varepsilon^d)}{d^{C+j+1}} > d^{l_1} \cdot \log((1+\varepsilon)|a_d|R_\varepsilon) \\
\Longleftrightarrow \qquad & d^{l_1} > \frac{\log((1+\varepsilon)|a_d|R_\varepsilon)}{\log((1-\varepsilon)|a_d|R_\varepsilon^d)} \cdot d^{C+j+1} =: l_\varepsilon.
\end{aligned}$$

Hence

$$S_m := \max\left\{ \left(\frac{|\lambda| \cdot t}{s}\right)^{\frac{1}{m-1}}, |\lambda|^{\frac{\log l_\varepsilon}{\log d}} \right\}$$

is sufficiently large. □

Proof of Theorem 1.1. Let us start with the case when $\mathcal{J}_{z_0}(p) \neq \{z_0\}$. Let $K > 0$ be the radius of the smallest closed disk around 0 which contains the (filled) Julia set of p. Recall from the introduction that it was shown in [16] that $A_R(\mathfrak{L})$ is a spider's web for all R as in (1.1) whenever it is a spider's web for *one* such R. So assume that $A_R(\mathfrak{L})$ is a spider's web for some sufficiently large $R > K$. By definition, there exists a bounded, simply connected domain G containing 0 such that $\partial G \subset A_R(\mathfrak{L})$. Fix $\delta > 0$ such that Proposition 2.2 applies; i.e., every continuum in $\mathbb{D}_\delta(z_0)$ separating z_0 and ∞ intersects the filled Julia set of p. Let m be the smallest integer such that $\partial G/\lambda^m$ is contained in the component of $\mathfrak{L}^{-1}(\mathbb{D}_\delta(z_0))$ that contains 0. When we now apply the m-times iterated functional equation to ∂G, we obtain that $\mathfrak{L}(\partial G) = p^m(\mathfrak{L}(\partial G/\lambda^m))$ intersects the filled Julia set of p. So there exists a point $w \in \partial G$ such that $|\mathfrak{L}(w)| \leq K$. But this contradicts the assumption that all points $z \in \partial G$ satisfy $|\mathfrak{L}(z)| \geq M(R, \mathfrak{L}) \geq R > K$.

Let us now consider the situation when $\mathcal{J}_{z_0}(p) = \{z_0\}$. By [16, Theorem 8.1] it is sufficient to find a sequence of bounded simply connected domains G_n such that for all sufficiently large n,

$$G_n \supset \{z \in \mathbb{C} : |z| < M^n(R, \mathfrak{L})\}$$

and

$$G_{n+1} \text{ is contained in a bounded component of } \mathbb{C} \setminus \mathfrak{L}(\partial G_n).$$

Let R_1 be the constant from Lemma 3.2, and set $R := \max\{R_{\mathfrak{L}}, R_1\}$. For $n \in \mathbb{N}$ let $r_n := |\lambda|^n M^n(R, \mathfrak{L})$. By Lemma 3.3, when n is large enough, there exists a simple closed curve Γ^{r_n} separating $\mathbb{S}_{r_n}$ and $\mathbb{S}_{r_n^d}$ such that $m(\Gamma^{r_n}, \mathfrak{L}) > M(r_n, \mathfrak{L})$. So for n large enough, we define G_n to be the interior of Γ^{r_n}. Then every G_n is a bounded simply connected domain with

$$G_n \supset \{z \in \mathbb{C} : |z| < r_n\} \supset \{z \in \mathbb{C} : |z| < M^n(R, \mathfrak{L})\}.$$

Furthermore, it follows from Lemma 3.2 with $m = d$ that

$$m(\partial G_n, \mathfrak{L}) = m(\Gamma^{r_n}, \mathfrak{L}) > M(r_n, \mathfrak{L}) > r_{n+1}^d > \max_{z \in \partial G_{n+1}} |z|;$$

hence G_{n+1} is contained in a bounded component of $\mathbb{C} \setminus \mathfrak{L}(\partial G_n)$ and the claim follows. □

Note that Corollary 1.2 is an immediate consequence of Theorem 1.1.

4. Appendix: On general properties of linearizers

The main purpose of this section is to study certain sets of points for a linearizer that are relevant from a function-theoretic and dynamical point of view, such as omitted, exceptional and singular values.

For an entire function f, we denote by $\mathrm{Crit}(f) := \{z \in \mathbb{C} : f'(z) = 0\}$ the set of *critical points*, by $\mathcal{C}(f) := f(\mathrm{Crit}(f))$ the set of *critical values*, and by $\mathcal{A}(f)$ the set of all *(finite) asymptotic values* of f. The elements of $\mathcal{S}(f) = \overline{\mathcal{C}(f) \cup \mathcal{A}(f)}$ are called *singular values* of f, and $\mathcal{S}(f)$ can be characterized as the smallest closed subset of $\mathbb{C}$ such that $f : \mathbb{C} \setminus f^{-1}(\mathcal{S}(f)) \to \mathbb{C} \setminus \mathcal{S}(f)$ is a covering map. If f is a polynomial, then $\mathcal{A}(f) = \emptyset$ and $\mathcal{C}(f)$ is finite, so in this case, $\mathcal{S}(f) = \mathcal{C}(f)$. The *postsingular set* of f is defined to be $\mathcal{P}(f) := \overline{\bigcup_{n \geq 0} f^n(\mathcal{S}(f))}$.

A point $w \in \mathbb{C}$ is said to be *exceptional under* f if its backward orbit, i.e., the set of all points z which are mapped to w by some f^n, is finite. It is well-known that the set $\mathcal{E}(f)$ of all exceptional values of f contains at most one point. We write $\mathcal{O}(f)$ for the set of all (finite) *omitted values* of f.

In what follows, let f be an entire map, z_0 a repelling fixed point of f with multiplier λ, and $\mathfrak{L}$ a linearizer of f at z_0.

Proposition 4.1. *The sets $\mathcal{O}(\mathfrak{L})$ and $\mathcal{E}(f) \setminus \{z_0\}$ are equal.*

Proof. By (2.1), $z_0 \notin \mathcal{O}(\mathfrak{L})$. Since $\mathfrak{L}(0) = z_0$, the point z_0 is never an omitted value of $\mathfrak{L}$. Let a be a non-exceptional point of f. Since $\mathfrak{L}$ omits at most one finite value, the backward orbit of a under f intersects $\mathfrak{L}(\mathbb{C})$; i.e., there exist $n \in \mathbb{N}$ and $w \in \mathbb{C}$ with $\mathfrak{L}(w) \in f^{-n}(a)$. Thus $a = f^n(\mathfrak{L}(w)) = \mathfrak{L}(\lambda^n w)$.

Now let $a \in \mathbb{C} \setminus \mathcal{O}(\mathfrak{L})$. If $a = z_0$, then we are done, so suppose that $a \neq z_0$. Then there exists $z \neq 0$ with $\mathfrak{L}(z) = a$. By the iterated functional equation, $\mathfrak{L}(z/\lambda^j) \in f^{-j}(a)$. Since $z \neq 0$ and $\mathfrak{L}$ is injective in a neighborhood of 0, the backward orbit of a under f has infinitely many elements. □

It is a well-known and often used fact that the postsingular set of f equals the set of singular values of $\mathfrak{L}$. However, we could not find a reference, which is why we include a proof. The main parts of what follows have been presented to us by A. Epstein.

Proposition 4.2. *The following relations are true:*

(i) $\mathcal{C}(\mathfrak{L}) \subseteq \bigcup_{n\geq 0} f^n(\mathcal{C}(f)) \setminus \mathcal{E}(f)$.

(ii) $\mathcal{S}(\mathfrak{L}) = \mathcal{P}(f)$.

Proof. Let $w = \mathfrak{L}(z) \in \mathcal{C}(\mathfrak{L})$; in particular, $w \notin \mathcal{O}(\mathfrak{L})$. It follows from Proposition 4.1 that $w \notin \mathcal{E}(f) \setminus \{z_0\}$. Since $\mathfrak{L}'(0) \neq 0$, we have $w \neq z_0$, so $w \notin \mathcal{E}(f)$. Differentiating the iterated functional equation yields

$$0 = (f^n)'(\mathfrak{L}(z/\lambda^n)) \cdot \mathfrak{L}'(z/\lambda^n) \cdot \frac{1}{\lambda^n}.$$

For large n, $\mathfrak{L}'(z/\lambda^n) \neq 0$, so it follows that $\mathfrak{L}(z/\lambda^n) \in \operatorname{Crit}(f^n)$. Since $\operatorname{Crit}(f^n) = \bigcup_{k=0}^{n-1} f^k(\operatorname{Crit}(f))$ by the chain rule, there exist $n \in \mathbb{N}$ and $k \leq n-1$ with $\mathfrak{L}(z/\lambda^n) = f^k(y)$, where $y \in \operatorname{Crit}(f)$. It follows that

$$w = \mathfrak{L}(z) = f^n(\mathfrak{L}(z/\lambda^n)) = f^n(f^k(y)) = f^{n+k}(y),$$

i.e., $w \in \bigcup_{n\geq 0} f^n(\mathcal{C}(f))$.

We now prove (ii). For the composition $f \circ \mathfrak{L}$ one obtains

$$\mathcal{S}(f \circ \mathfrak{L}) = S(f|_{f(\mathbb{C})}) \cup \overline{f(\mathcal{S}(\mathfrak{L}))} = \mathcal{S}(f) \cup f(\mathcal{S}(\mathfrak{L})),$$

since every Picard value of f is also a singular value of f. Let us abbreviate $S := \mathcal{S}(f) \cup f(\mathcal{S}(\mathfrak{L}))$. Since the composition

$$\mathbb{C} \setminus \mathfrak{L}^{-1}(f^{-1}(S)) \xrightarrow{\mathfrak{L}} \mathbb{C} \setminus f^{-1}(S) \xrightarrow{f} \mathbb{C} \setminus S$$

is a covering map, it follows from (2.1) that

$$\mathbb{C} \setminus \lambda^{-1} \cdot \mathfrak{L}^{-1}(f^{-1}(S)) \xrightarrow{\lambda} \mathbb{C} \setminus \mathfrak{L}^{-1}(S) \xrightarrow{\mathfrak{L}} \mathbb{C} \setminus S$$

must be a covering map as well. Hence

$$\mathcal{S}(f) \cup f(\mathcal{S}(\mathfrak{L})) = S \supset \mathcal{S}(\mathfrak{L} \circ \lambda) = \mathcal{S}(\mathfrak{L}).$$

The argument is commutative with respect to (2.1), so we obtain the opposite inclusion, yielding the equality $\mathcal{S}(\mathfrak{L}) = \mathcal{S}(f) \cup f(\mathcal{S}(\mathfrak{L}))$. But for a point $w \in \mathcal{S}(f)$, this implies that $w \in \mathcal{S}(\mathfrak{L})$, and so $f(w) \in f(\mathcal{S}(\mathfrak{L})) \subseteq \mathcal{S}(\mathfrak{L})$. By proceeding inductively, it follows for every $n \in \mathbb{N}$ that $f^n(w) \in \mathcal{S}(\mathfrak{L})$; hence $\mathcal{P}(f) \subset \mathcal{S}(\mathfrak{L})$.

Let $w \in \mathbb{C} \setminus \mathcal{P}(f)$. Then there exists a disk $D \ni w$ such that all inverse branches of all iterates of f exist in D. Let $v \in D$ and $z \in \mathfrak{L}^{-1}(v)$, and define $z_n := z/\lambda^n$ and $v_n := \mathfrak{L}(z_n)$. Let g_n be the branch of $(f^n)^{-1}$ such that $g_n(v) = v_n$ and let $D_n := g_n(D)$. By the Shrinking Lemma in [8], it follows that the diameters of the domains D_n converge to 0. (Actually, the statement in [8] is not phrased such that it completely covers our setting, but the proof gives what we require.) We choose a domain U containing 0 in which $\mathfrak{L}$ is injective. Then for n large enough, D_n lies in $\mathfrak{L}(U)$. Let T be the branch of $\mathfrak{L}^{-1}$ that maps D_n into U. Then we have

$$\mathfrak{L} \circ (\lambda^n \circ T \circ g_n)(z) = f^n \circ \mathfrak{L} \circ \underbrace{(T \circ g_n)(z)}_{\in U} = (f^n \circ g_n)(z) = z.$$

Since z is an arbitrarily chosen preimage of an arbitrary point in D, all inverse branches of $\mathfrak{L}$ can be defined in D. Hence $w \in \mathbb{C} \setminus \mathcal{S}(\mathfrak{L})$. □

If f is a polynomial, then the sets in Proposition 4.2(i) are in fact equal, as was shown by Drasin and Okuyama [3, Theorem 2.10]. However, if f is transcendental entire this is not true in general, as the following example shows: Let $f(z) = \mathrm{e}^{z^2}$; then $w = 1 = f(0)$ is a critical value but not an exceptional value of f. Now let $\mathfrak{L}$ be any linearizer of f. (Note that by [7, Theorem 2], the function f has infinitely many fixed points, and since $S(f)$ is finite, infinitely many of them must be repelling.) Assume that $\mathfrak{L}$ has a critical point, say z. Differentiation of the functional equation then yields

$$0 = \mathfrak{L}^{'}(z) = \frac{1}{\lambda} \cdot f^{'}\left(\mathfrak{L}\left(\frac{z}{\lambda}\right)\right) \cdot \mathfrak{L}^{'}\left(\frac{z}{\lambda}\right) = \frac{2}{\lambda} \cdot \mathfrak{L}\left(\frac{z}{\lambda}\right) \cdot f\left(\mathfrak{L}\left(\frac{z}{\lambda}\right)\right) \cdot \mathfrak{L}^{'}\left(\frac{z}{\lambda}\right).$$

Since 0 is an omitted value of both f and $\mathfrak{L}$, this implies that $\mathfrak{L}^{'}(z/\lambda) = 0$. Repeating this argument shows that every z/λ^n must be a critical point of $\mathfrak{L}$, contradicting the Identity Theorem, since $\mathfrak{L}$ is non-constant. Hence $\mathcal{C}(\mathfrak{L}) = \emptyset$.

Let us continue with the consideration of asymptotic values of linearizers. If f is a polynomial, then $\mathcal{A}(\mathfrak{L})$ is contained in the union of attracting and parabolic periodic cycles and the accumulation points of recurrent critical points in $\mathcal{J}(f)$ [3, Theorem 1]. Depending on the location of the repelling fixed point z_0 relative to $\mathcal{F}(f)$, we can exclude certain attracting cycles of f as asymptotic values for $\mathfrak{L}$.

Proposition 4.3. *Let f be a polynomial and let $w \in \mathcal{A}(\mathfrak{L})$. If w is an attracting periodic point of f, then z_0 lies in the boundary of the immediate attracting basin of w.*

Proof. Let w be an attracting periodic point of f of period k and assume that w is an asymptotic value of $\mathfrak{L}$. Then there exists a path γ to ∞ for which $\lim_{t\to\infty} \mathfrak{L}(\gamma(t)) = w$. Since $w \in \mathcal{F}(f)$ and $\mathcal{F}(f)$ is open, we can assume that $\mathfrak{L}(\gamma) \subset \mathcal{F}(f)$. It follows from (2.2) that every path $\gamma_n(t) := \lambda^{-n} \cdot \gamma(t)$ is again an asymptotic path for $\mathfrak{L}$. Moreover, the limit of $\mathfrak{L}$ along γ_{nk} is contained in $f^{-nk}(w)$; hence it follows from [3, Theorem 1] that $\lim_{t\to\infty} \mathfrak{L}(\gamma_{nk}(t)) = w$. By construction, the distance between γ_{nk} and 0 tends to 0 as $n \to \infty$; hence for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, the curve $\mathfrak{L}(\gamma_{nk})$ intersects $\mathbb{D}_\varepsilon(z_0)$. Hence $z_0 = \mathfrak{L}(0)$ is contained in the boundary of the immediate attracting basin of w. □

Recall that a point $z \in \mathcal{J}(f)$ is called a *buried point* if it does not belong to the boundary of any Fatou component (other than $I(f)$).

Corollary 4.4. *If f is a polynomial and z_0 is a buried point (of f), then $\mathfrak{L}$ has no asymptotic values.*

Linearizers can be very useful to construct an entire or meromorphic function whose set of singular values satisfies certain conditions. For instance, in [9], an example was given of an entire function of finite order with no asymptotic values and only finitely many critical values such that the ramification degree on its Julia set was unbounded; the function was a linearizer of a hyperbolic polynomial in the spirit of Proposition 4.3.

Another interesting application of linearizers shows the following example: Let $f(z) := \mu \exp(z)$, where $\mu \in \mathbb{C}$ is chosen such that $\bigcup_{n\geq 0} f^n(0)$ is dense in $\mathbb{C}$; the existence of such parameters is well-known. Since f has infinitely many fixed points and $\mathcal{S}(f) = \{0\}$, at most one fixed point is non-repelling, so we can pick a repelling fixed point z_0 of f. Let $\mathfrak{L}$ be a linearizer of f at z_0. It follows from the functional equation that 0 is an omitted value of $\mathfrak{L}$, and hence every point $w_n := f^n(0)$ must be an asymptotic value of $\mathfrak{L}$. It is also not hard to check that $\mathfrak{L}$ has a direct singularity lying over each of the points w_n. (For a clarification of terminology, see e.g. [3]; our last claim also follows from [3, Theorem 1.4], which is formulated for linearizers of rational maps only, but extends to linearizers of transcendental entire maps with the same proof.) Hence $\mathfrak{L}$ is a map for which the set of projections of direct singularities (or direct asymptotic values) is dense in $\mathbb{C}$. This is optimal, since by a theorem of Heins [6], the set of projections of direct singularities is always countable.

In many dynamical settings, conformal conjugacies produce no relevant dynamical consequences; hence it is natural to ask the following: Assume that f_1 and f_2 are entire functions and that there exists a conformal map $\varphi(z) = az + b$ such that

$$f_2 \circ \varphi = \varphi \circ f_1 \tag{4.1}$$

everywhere in $\mathbb{C}$. Let $\mathfrak{L}_1$ be a linearizer of f_1. Does there exist a linearizer $\mathfrak{L}_2$ of f_2 which is conformally conjugate to $\mathfrak{L}_1$ (and hence has the same dynamics)? In general, the answer is no. The first step towards showing this is the following observation.

Proposition 4.5. *Let f_1, f_2 be entire functions, and let $\varphi(z) = az+b$ be such that $f_2 \circ \varphi(z) = \varphi \circ f_1(z)$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are linearizers of f_1 and f_2 at z_1 and $z_2 = \varphi(z_1)$, respectively, with $\mathfrak{L}_1'(0) = \mathfrak{L}_2'(0)$, then*

$$\mathfrak{L}_2 \circ (\varphi - b) = \varphi \circ \mathfrak{L}_1, \tag{4.2}$$

where $(\varphi - b)(z) := \varphi(z) - b = az$.

Proof. Let $\lambda := f_1'(z_1)$. Then $\widetilde{\mathfrak{L}}(z) := \varphi \circ \mathfrak{L}_1 \circ (\varphi^{-1} + b/a)(z)$ satisfies

$$\begin{aligned} f_2 \circ \widetilde{\mathfrak{L}}(z/\lambda) &= f_2 \circ \varphi \circ \mathfrak{L}_1 \circ (\varphi^{-1} + b/a)(z/\lambda) = \varphi \circ f_1 \circ \mathfrak{L}_1(z/(a\lambda)) \\ &= \varphi \circ \mathfrak{L}_1(z/a) = \widetilde{\mathfrak{L}}(z). \end{aligned}$$

Since f_1 and f_2 are conformally conjugate, the multipliers at z_1 and z_2 coincide; hence $\widetilde{\mathfrak{L}}$ is a linearizer of f_2 at $\varphi(z_1) = z_2$. Furthermore, $\widetilde{\mathfrak{L}}'(0) = \mathfrak{L}_1'(0)$, yielding $\mathfrak{L}_2 = \widetilde{\mathfrak{L}}$. □

Now let us assume that there exists a linearizer $\mathfrak{L}_2$ of f_2 and a conformal map $\psi : \mathbb{C} \to \mathbb{C}$ such that

$$\mathfrak{L}_2 \circ \psi = \psi \circ \mathfrak{L}_1. \tag{4.3}$$

A comparison of the equations (4.2) and (4.3) already indicates that ψ only exists under very restrictive conditions. Let us make this more precise.

The set $\mathcal{S}(\mathfrak{L}_1)$ is mapped by ψ bijectively onto $\mathcal{S}(\mathfrak{L}_2)$. By Proposition 4.2, this is equivalent to the condition

$$\psi(\mathcal{P}(f_1)) = \mathcal{P}(f_2). \tag{4.4}$$

By equation (4.1), φ already satisfies $\varphi(\mathcal{P}(f_1)) = \mathcal{P}(f_2)$, so in particular, the map $\psi^{-1} \circ \varphi$ is a conformal automorphism of $\mathbb{C}$ that fixes the set $\mathcal{P}(f_1)$. Recall that in general, this set can be arbitrarily large. In particular, in most cases it contains at least two elements (e.g. whenever f_1 is transcendental entire).

Now if Z is an arbitrary finite subset of $\mathbb{C}$ with at least two elements, then $G_Z := \{h(z) = \alpha z + \beta : \alpha \in \mathbb{C} \setminus \{0\}, \beta \in \mathbb{C}, h(Z) = Z\}$ is a finite group, and one can easily check that the map $G_Z \to \mathbb{C} \setminus \{0\}, \alpha z + \beta \mapsto \alpha$ is an injective group homomorphism. Hence G_Z is isomorphic to a finite subgroup of $\mathbb{C} \setminus \{0\}$, which must be a cyclic group generated by a root of unity. So every such G_Z is generated by a map of the form $z \mapsto \exp(2\pi ik/n)z + \beta$ with coprime k and n and $n \leq |Z|$. This allows us to phrase necessary geometric conditions on a finite set Z such that G_Z is not trivial. It is clear that such conditions are rather strong; e.g., if $z \mapsto \exp(2\pi ik/n)z + \beta$ is a generator of G_Z and p its (unique) fixed point in $\mathbb{C}$, then all elements of Z must lie on r circles centred at p, where $r \cdot n \leq |Z \setminus \{p\}|$. To give an explicit dynamical example, one can consider the unique real parameter c, for which $f(z) := z^2 + c$ has a superattracting cycle of period three; one easily sees that $G_{\mathcal{P}(f)}$ is trivial.

However, triviality of $G_{\mathcal{P}(f_1)}$ implies that $\psi(z) = \varphi(z) = az + b$ everywhere in $\mathbb{C}$. So by the equations (4.2) and (4.3), we have

$$\mathfrak{L}_2(z) = \psi \circ \mathfrak{L}_1 \circ d(\psi^{-1} + b/a) = \psi \circ \mathfrak{L}_1 \circ \psi^{-1}$$

for some $d \in \mathbb{C} \setminus \{0\}$, which can only be true when $b = 0$ and $d = 1$.

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