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A NEW CHARACTERIZATION OF CONVEXITY IN FREE CARNOT GROUPS

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Abstract. A characterization of convex functions in $\mathbb{R}^N$ states that an upper semicontinuous function u is convex if and only if $u(Ax)$ is subharmonic (with respect to the usual Laplace operator) for every symmetric positive definite matrix A. The aim of this paper is to prove that an analogue of this result holds for *free* Carnot groups $\mathbb{G}$ when considering convexity in the viscosity sense. In the subelliptic context of Carnot groups, the linear maps $x \mapsto Ax$ of the Euclidean case must be replaced by suitable group isomorphisms $x \mapsto T_A(x)$, whose differential preserves the first layer of the stratification of Lie($\mathbb{G}$).

1. Introduction

A characterization of convexity in $\mathbb{R}^N$ states that an upper semicontinuous (u.s.c., for short) function $u : \mathbb{R}^N \to [-\infty, \infty)$ is convex (in the classical sense) if and only if, for every positive definite matrix A, the function $x \mapsto u(Ax)$ is subharmonic, with respect to the usual Laplace operator in $\mathbb{R}^N$ (see [20]).

The aim of this paper is to establish an analogue on Carnot groups $\mathbb{G}$ for *v-convex functions*, i.e., for functions which are convex in the viscosity sense first introduced by Lu, Manfredi and Stroffolini in [22] and by Juutinen, Lu, Manfredi and Stroffolini in [21]. Roughly speaking, these authors call v-convex any function which is subharmonic with respect to every sub-Laplacian of the stratified group $\mathbb{G}$. On the other hand, if $\mathbb{G}$ is *free*, we know that every sub-Laplacian can be reduced to a single one via a "linear" change of variables; see [6].

These ideas have led us to show that, in the free setting, a u.s.c. function $u : \mathbb{G} \to [-\infty, \infty[$ is v-convex if and only if a suitable family of functions $\{u \circ \Theta_B\}_B$ is subharmonic with respect to *a fixed sub-Laplacian* on $\mathbb{G}$. More precisely, if m denotes the dimension of the horizontal layer H of Lie($\mathbb{G}$) and B varies over the cone of the positive definite $m \times m$ matrices, then Θ_B is the (unique) group automorphism of $\mathbb{G}$ whose differential coincides (when restricted to H) with the linear endomorphism of H associated to B. See Theorem 3.8 below for the precise statement. We explicitly remark that the well-posedness of the above Θ_B is a consequence of the freeness of $\mathbb{G}$. In the Euclidean case $\mathbb{G} = (\mathbb{R}^N, +)$, this gives back the classical result cited in the introduction, since, in this case, Θ_B coincides with the linear map associated to B.

The idea of characterizing the v-convexity in terms of a fixed sub-Laplacian is, moreover, in the spirit of a characterization given in [21], where it is proved that

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u is v-convex if and only if $(D^2_{\mathrm{h}}u)^* \geq 0$ in the viscosity sense. Here $(D^2_{\mathrm{h}}\phi)^* = \big(\frac{1}{2}(X_iX_j\phi + X_jX_i\phi)\big)_{i,j}$ denotes the symmetrized horizontal second derivative matrix of $\phi \in C^2$, with respect to the *fixed* horizontal frame[1] $\{X_1,\dots,X_m\}$.

Our main argument here is the following one. Assuming any fixed sub-Laplacian $\mathcal{L}_0 = \sum_{i=1}^m Z_i^2$ on $\mathbb{G}$, any other sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ is of the form $\mathcal{L} = \sum_{i=1}^m X_i^2$, where $X_i = \sum_{j=1}^m b_{i,j}Z_j$ (for $i = 1,\dots,m$) and $B = (b_{i,j})$ is a non-singular matrix. In other words, setting $H := \mathrm{span}\{Z_1,\dots,Z_m\}$, the system of vector fields $\{X_1,\dots,X_m\}$ is obtained from $\mathcal{Z} := \{Z_1,\dots,Z_m\}$ via the linear isomorphism $\varphi_B : H \to H$ naturally associated (w.r.t. the basis $\mathcal{Z}$) to the matrix B^T. Since $\mathrm{Lie}(\mathbb{G})$ is free and it is Lie-generated by $\mathcal{Z}$, φ_B extends to a unique Lie-algebra isomorphism of $\mathrm{Lie}(\mathbb{G})$, say $\widehat{\varphi}_B$. If we denote by $\Theta_B : \mathbb{G} \to \mathbb{G}$ the (unique) isomorphism of the Lie group $\mathbb{G}$ whose differential is $\widehat{\varphi}_B$, then it is not difficult to recognize that, in the new coordinates defined by $x \mapsto y = \Theta_B(x)$, the sub-Laplacian $\mathcal{L}_0$ turns into $\mathcal{L}$, or, more precisely, $(\mathcal{L}u)\circ\Theta_B = \mathcal{L}_0(u\circ\Theta_B)$, for every smooth u on $\mathbb{G}$ (see Lemma 3.1).

Our main task is then to show that the $\mathcal{L}$-subharmonicity of u is equivalent to the $\mathcal{L}_0$-subharmonicity of $u\circ\Theta_B$. Whereas this is easily seen when u is of class C^2, for arbitrary u.s.c. functions u this is more concealed. We shall prove it by making use of a suitable submean characterization of subharmonic functions on Carnot groups, as given in our previous paper [3]. We shall see that the mean-integral operators $M^{\mathcal{L}}$ and $M^{\mathcal{L}_0}$ related, respectively, to the sub-Laplacians $\mathcal{L}$ and $\mathcal{L}_0$ are (roughly speaking) Θ_B-related, i.e., $M^{\mathcal{L}}(u)\circ\Theta_B = M^{\mathcal{L}_0}(u\circ\Theta_B)$ for every u.s.c. function u (see Lemma 3.4 for the precise statement). Since the mean-integral operators on Carnot groups which are different from the Euclidean one *always* involve non-identically 1 density kernels, this result seems to have some interest in its own right. Moreover, we hope that the above idea of changing coordinates via the maps Θ_B may be fruitful, in the context of convexity, as it has already been in other contexts (see [4, 6]).

Before closing the introduction, we would like to recall some works on convexity in Carnot groups. Due to the relevance of the notion of convexity in theoretical and applied areas of mathematics, several notions of convexity have been recently proposed in the context of Carnot groups. The papers by Danielli, Garofalo and Nhieu [10] and by Lu, Manfredi and Stroffolini [22] opened the way for appropriate definitions of convexity in this context: they respectively introduced the classes of h-convex (horizontally convex) and v-convex (viscosity convex) functions. On the Heisenberg groups, these notions coincide (Balogh and Rickly [1]), whereas, generally, the v-convex functions are the u.s.c. h-convex functions (see Magnani [23], Wang [32]). In this paper we are concerned with the notion of v-convexity.

Along with the mentioned references, contributions to the understanding of convexity in stratified groups come from many authors. The problem of fine regularity for convex functions has especially been investigated. See Capogna and Maldonado [7]; Capogna, Pauls, and Tyson [8]; Danielli, Garofalo, Nhieu, and Tournier [11]; Garofalo [14]; Garofalo and Tournier [15]; Gutiérrez and Montanari [17, 18]; Juutinen, Lu, Manfredi, and Stroffolini [21]; Monti and Rickly [24], Rickly [25]; Sun and Yang [28, 29]. (For the notion of "r-convex" function in Carnot groups, see Dah-Yan [9] and Sun and Yang [26, 27].)

[1] This is any basis of the first layer of the stratification of the Lie algebra of $\mathbb{G}$.

2. Notation and definitions

A Carnot group is a couple (G, V_1) where $(G, \cdot)$ is a connected and simply connected (real) Lie group and V_1 is a subspace of the Lie algebra $\mathrm{Lie}(G)$ of G such that $\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_r$, where $V_i = [V_1, V_{i-1}]$, for $2 \le i \le r$ and $[V_1, V_r] = \{0\}$. Throughout the paper, we set the positions

$$H := V_1, \quad m := \dim(V_1). \tag{2.1}$$

Vector fields in H are usually called *horizontal*. The integer $Q := \sum_{i=1}^r i \dim(V_i)$ is called the homogeneous dimension of $\mathbb{G}$. Throughout this paper we assume that $Q \ge 3$. A sub-Laplacian on $\mathbb{G}$ is a differential operator of the form $\sum_{i=1}^m X_i^2$, where $\{X_1, \ldots, X_m\}$ is any basis of H.

We next recall the definition of *free* Carnot group. To this aim we first need the notion of free nilpotent Lie algebra $\mathfrak{f}_{m,r}$ of step r and m generators. By definition, $\mathfrak{f}_{m,r}$ is the unique (up to isomorphism) nilpotent Lie algebra of step r Lie-generated by m (≥ 2) of its elements, say $F_1, \ldots, F_m$, with the following property: for every Lie algebra $\mathfrak{n}$, nilpotent of step $\le r$, and for every map L from $\{F_1, \ldots, F_m\}$ to $\mathfrak{n}$, there exists a (unique) Lie algebra morphism from $\mathfrak{f}_{m,r}$ to $\mathfrak{n}$ extending L. The construction of such a Lie algebra $\mathfrak{f}_{m,r}$ is classical (see, e.g., [30, 31]; the reader is also referred to [19, 16] for the construction of a basis for $\mathfrak{f}_{m,r}$). We say that a Carnot group G is a *free Carnot group* if its Lie algebra is isomorphic to $\mathfrak{f}_{m,r}$, for some m and r. Notice that, in this case, m necessarily equals the dimension of H (as in (2.1)) and r is the step of nilpotency of G.

In order to simplify computations, we fix suitable coordinates on G. Since a Carnot group is nilpotent, the exponential map $\mathrm{Exp} : \mathrm{Lie}(G) \to G$ is an (analytic) diffeomorphism. If the V_i's are as in the introduction of this section, we fix a basis $\mathcal{E}$ for $\mathrm{Lie}(G)$ which is "adapted" to the stratification, i.e.,

$$\mathcal{E} = \big\{E_1^{(1)}, \ldots, E_m^{(1)}; E_1^{(2)}, \ldots, E_{N_2}^{(2)}; \ldots; E_1^{(r)}, \ldots, E_{N_r}^{(r)}\big\} =: \{E_1, \ldots, E_N\}, \tag{2.2}$$

where (for any $i = 1, \ldots, r$) $N_i := \dim(V_i)$, $N = m + N_2 + \cdots + N_r$ and $E_1^{(i)}, \ldots, E_{N_i}^{(i)}$ is a basis for V_i. Via the exponential map and via this choice of basis for $\mathrm{Lie}(G)$, we can identify G to $\mathbb{R}^N$. This amounts to fixing a global coordinate system

$$\varphi : G \to \mathbb{R}^N, \quad g \mapsto \varphi(g) =: (x_1, \ldots, x_N),$$

where $\varphi^{-1}(x_1, \ldots, x_N) = \mathrm{Exp}\big(x_1 E_1 + \cdots + x_N E_N\big)$. This identification is particularly useful since the Haar measure on G simply boils down to the Lebesgue measure on $\mathbb{R}^N$. Also, the multiplication $\cdot$ on G "pushes forward" to the following composition law on $\mathbb{R}^N$:

$$x * y := \varphi\big(\varphi^{-1}(x) \cdot \varphi^{-1}(y)\big), \quad x, y \in \mathbb{R}^N.$$

(Due to the definition of φ, we recognize that the analogue of $*$ on $\mathrm{Lie}(\mathbb{G})$ is simply the Campbell-Baker-Hausdorff-Dynkin multiplication.) Form now on, we will tacitly identify $(G, \cdot)$ with the group $\mathbb{G} = (\mathbb{R}^N, *)$. We further equip the latter group with the family of non-isotropic "dilations"

$$\begin{aligned} &\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x_1, \ldots, x_N) \\ &\qquad := \big(\lambda x_1, \ldots, \lambda x_m; \lambda^2 x_1^{(2)}, \ldots, \lambda^2 x_{N_2}^{(2)}; \ldots; \lambda^r x_1^{(r)}, \ldots, \lambda^r x_{N_r}^{(r)}\big). \end{aligned} \tag{2.3}$$

Here we applied the obvious notation suggested by (2.2). Following the definition of [5, §1.4], we say that $(\mathbb{R}^N, *, \delta_\lambda)$ is a *homogeneous Carnot group* of step r and m generators. As shown in [5], a selected basis for $\mathrm{Lie}(\mathbb{G})$ is given by the vector fields

$Z_1, \dots, Z_N$, whose component functions (w.r.t. $\partial_{x_1}, \dots, \partial_{x_N}$) are given by the N column vectors of $\mathcal{J}_{\tau_x}(0)$, the Jacobian matrix at 0 of the left translations $\tau_x(y) = x * y$. Then (see [5, Proposition 2.2.22]) the exponential map of $\mathbb{G}$ is given by

$$\mathrm{Exp} : \mathrm{Lie}(\mathbb{G}) \to \mathbb{G}, \quad x_1 Z_1 + \cdots + x_N Z_N \mapsto (x_1, \dots, x_N). \tag{2.4}$$

In the sequel, we shall consider the following "canonical" sub-Laplacian on $\mathbb{G}$:

$$\Delta_{\mathbb{G}} := \sum_{i=1}^m Z_i^2. \tag{2.5}$$

(This choice, along with the above choice of coordinates, is obviously immaterial, and it is only meant to simplify computations throughout and to fix the notation.) Moreover, we denote by $\mathcal{M}$ the vector space of the real $m \times m$ matrices and we set

$$\mathcal{A} := \{A \in \mathcal{M} \,|\, A \text{ is symmetric and positive definite}\}. \tag{2.6}$$

For every $A = (a_{i,j})_{i,j \le m}$ in $\mathcal{A}$, we consider the differential operator

$$\mathcal{L}_A := \sum_{i,j=1}^m a_{i,j}\, Z_i Z_j. \tag{2.7}$$

It is easily seen that the family $\{\mathcal{L}_A\}_{A \in \mathcal{A}}$ is precisely the family of all sub-Laplacians on $\mathbb{G}$. Indeed, a sub-Laplacian on $\mathbb{G}$ is of the form $\mathcal{L} = \sum_{i=1}^m X_i^2$, where

$$X_i = \sum_{j=1}^m b_{i,j}\, Z_j, \quad i = 1, \dots, m, \tag{2.8}$$

where $B = (b_{i,j})_{i,j \le m} \in \mathcal{M}$ is non-singular, so that $\mathcal{L} = \mathcal{L}_A$ with $A = B^T \cdot B$ (which obviously belongs to $\mathcal{A}$). Vice versa, if $A \in \mathcal{A}$ and if $B \in \mathcal{M}$ is any non-singular matrix satisfying $A = B^T \cdot B$ (for example, $B = A^{1/2}$, the symmetric positive definite square root of A), we have $\mathcal{L}_A = \sum_{i=1}^m X_i^2$ where the X_i's are as in (2.8), whence $\mathcal{L}_A$ is indeed a sub-Laplacian on $\mathbb{G}$. This is a consequence of the following computation, to which we will return later in this paper:

$$\begin{aligned}\sum_{i=1}^m X_i^2 &\overset{(2.8)}{=} \sum_{j,k=1}^m \Big(\sum_{i=1}^m b_{i,j} b_{i,k}\Big) X_j X_k = \sum_{j,k=1}^m (B^T \cdot B)_{j,k} X_j X_k \\ &= \sum_{j,k=1}^m a_{j,k}\, X_j X_k = \mathcal{L}_A.\end{aligned}$$

We next turn to fundamental solutions. If $\mathcal{L}$ is any sub-Laplacian on $\mathbb{G}$, we say that Γ is the *fundamental solution* of $\mathcal{L}$ if the following facts hold:

(i) $\Gamma : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ and $\Gamma \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$;
(ii) Γ vanishes at infinity;
(iii) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and, in the weak sense of distributions, $-\mathcal{L}\Gamma$ is the Dirac measure supported at $\{0\}$.

By the weak maximum principle for $\mathcal{L}$, it is easily seen that the above Γ (whose existence follows, e.g., from the results in [12]) is unique. In the sequel, we set

$$d_{\mathcal{L}} := \Gamma^{1/(2-Q)}$$

(with the convention $d_{\mathcal{L}}(0) = 0$) and we say that $d_{\mathcal{L}}$ is the $\mathcal{L}$-*gauge*. The definition of $d_{\mathcal{L}}$ is well-posed since $\Gamma(x) > 0$ for every $x \neq 0$, and it turns out that $d_{\mathcal{L}} \in C(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$ and that $d_{\mathcal{L}}(x^{-1}) = d_{\mathcal{L}}(x)$, $d_{\mathcal{L}}(\delta_\lambda(x)) = \lambda\, d_{\mathcal{L}}(x)$, for every $\lambda > 0$ and

$x \in \mathbb{R}^N$. For a general Carnot group $\mathbb{G}$, we cannot expect the fundamental solutions of different sub-Laplacians on $\mathbb{G}$ to have much in common. Instead, for *free* Carnot groups, all fundamental solutions are related to the fundamental solution of $\Delta_{\mathbb{G}}$ by a strikingly simple formula (see Section 3).

We next turn to the definition of $\mathcal{L}$-subharmonicity. Among plenty of possible equivalent definitions (see [3] and [5, Chapter 8]), we choose the following one, since we shall employ the mean integral operator M_r below, again in this paper. Let $\mathbb{G}$ be a Carnot group, $\mathcal{L} = \sum_{i=1}^m X_i^2$ a sub-Laplacian on $\mathbb{G}$, Γ its fundamental solution and $d_{\mathcal{L}}$ the relevant $\mathcal{L}$-gauge. If $\Omega \subseteq \mathbb{G}$ is open, $u : \Omega \to [-\infty, \infty)$ is an upper semi-continuous (u.s.c., in short) function, and $x \in \Omega$ and $r > 0$ are such that

$$D_{\mathcal{L}}(x, r) := \{y \in \mathbb{R}^N \,:\, d_{\mathcal{L}}(x^{-1} * y) < r\}$$

has closure contained in Ω, we set

$$\begin{aligned} &M_r^{\mathcal{L}}(u)(x) := \frac{Q(Q-2)}{r^Q} \int_{d_{\mathcal{L}}(x^{-1}*y)<r} u(y)\, \Psi_{\mathcal{L}}(x^{-1} * y)\, \mathrm{d}y, \\ &\text{where } \Psi_{\mathcal{L}} := \textstyle\sum_{i=1}^m |X_i d_{\mathcal{L}}|^2. \end{aligned} \tag{2.9}$$

With the above notation, we say that a u.s.c. function $u : \Omega \to [-\infty, \infty)$ is *$\mathcal{L}$-subharmonic in* Ω if

$$u(x) \le M_r^{\mathcal{L}}(u)(x), \quad \text{whenever } \overline{D_{\mathcal{L}}(x, r)} \subset \Omega. \tag{2.10}$$

Finally, we recall the notion of v-convexity (convexity in the viscosity sense; see [22]). Given an open set $\Omega \subseteq \mathbb{G}$, a u.s.c. function $u : \Omega \to [-\infty, \infty)$ is *v-convex in* Ω if u is $\mathcal{L}$-subharmonic with respect to *all* sub-Laplacians $\mathcal{L}$ on $\mathbb{G}$. As we said in the introduction, this definition can be compared to other remarkable definitions available in the literature (as that of horizontal convexity; see [10]). Thanks to the remarks made a few paragraphs above, we explicitly observe that u is v-convex if and only if it is $\mathcal{L}_A$-subharmonic *for every* $A \in \mathcal{A}$, where $\mathcal{L}_A$ is as in (2.7).

3. A new characterization of v-convexity in free groups

Throughout the sequel, $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ is a fixed *free* (homogenous) Carnot group and the notation in Section 2 is tacitly employed. We begin with the following central result. It improves (both in the statement and in the proof) Theorem 16.1.2 of [5] and Theorem 2.2 of [6].

Lemma 3.1. *Let $A \in \mathcal{A}$. There exists a group isomorphism $T_A : \mathbb{G} \to \mathbb{G}$ such that*

$$(\mathcal{L}_A u) \circ T_A = \Delta_{\mathbb{G}}(u \circ T_A), \quad \forall\, u \in C^\infty(\mathbb{G}, \mathbb{R}). \tag{3.1}$$

More precisely, T_A can be constructed as follows. If $B = (b_{i,j})_{i,j \le m}$ is any (non-singular) real matrix satisfying $A = B^T \cdot B$ (for example $B = A^{1/2}$) and if we denote by $\{X_1, \dots, X_m\}$ the basis of H defined in (2.8)*, then*

$$T_A = \mathrm{Exp} \circ \varphi_B \circ \mathrm{Log}, \tag{3.2}$$

where $\varphi_B : \mathrm{Lie}(\mathbb{G}) \to \mathrm{Lie}(\mathbb{G})$ is the unique Lie-algebra isomorphism mapping Z_i into X_i, for any $i = 1, \dots, m$.

Actually, with respect to the logarithmic coordinates previously fixed on $\mathbb{G}$, T_A is linear and it coincides with the linear map related to the matrix representing (w.r.t. the basis $\{Z_1, \dots, Z_N\}$ of $\mathrm{Lie}(\mathbb{G})$) the linear function $\varphi_B : \mathrm{Lie}(\mathbb{G}) \to \mathrm{Lie}(\mathbb{G})$.

Remark 3.2. The notation $T_A = \mathrm{Exp} \circ \varphi_B \circ \mathrm{Log}$ may be somewhat misleading: in fact, given $A \in \mathcal{A}$ there may exist many matrices $B \in \mathcal{M}$ such that $A = B^T \cdot B$. In the sequel, given $A \in \mathcal{A}$, when we write T_A we mean any map of the form $\mathrm{Exp} \circ \varphi_B \circ \mathrm{Log}$, where $B \in \mathcal{M}$ is any matrix satisfying $A = B^T \cdot B$, whereas φ_B is the unique Lie-algebra morphism described in the assertion of Lemma 3.1. We hope that the reader may accept this little abuse of notation up until the statement of our main result, Theorem 3.8 below, where all will be made unambiguous.

Proof. Let $A \in \mathcal{A}$ and let $B = (b_{i,j})_{i,j \le m} \in \mathcal{M}$ satisfy $A = B^T \cdot B$. Notice that B is non-singular. By definition of Carnot group, the system of vector fields $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ Lie-generates $\mathfrak{g} := \mathrm{Lie}(\mathbb{G})$. Thus, if we set

$$X_i := \textstyle\sum_{j=1}^m b_{i,j}\, Z_j, \qquad i = 1, \dots, m,$$

$\mathfrak{g}$ being a free nilpotent Lie algebra, there exists a unique Lie-algebra morphism $\varphi_B : \mathfrak{g} \to \mathfrak{g}$ extending the map on $\mathcal{Z}$ defined by

$$Z_i \mapsto X_i, \qquad i = 1, \dots, m.$$

It is easily seen that φ_B is a Lie-algebra isomorphism.

Since $\mathbb{G}$ is simply connected, there exists a unique Lie-group morphism $T_A : \mathbb{G} \to \mathbb{G}$ whose differential is φ_B. It is not difficult to see that this is precisely given by the position $T_A := \mathrm{Exp} \circ \varphi_B \circ \mathrm{Log}$. Notice that T_A is an isomorphism since its differential φ_B is invertible. If we show that

$$(X_i u) \circ T_A = Z_i(u \circ T_A), \quad \text{for all } u \in C^\infty(\mathbb{G}, \mathbb{R}) \text{ and all } i \in \{1, \dots, m\}, \tag{3.3}$$

then (3.1) will follow, since $\Delta_{\mathbb{G}} = \sum_{i=1}^m Z_i^2$ and $\mathcal{L}_A = \sum_{i=1}^m X_i^2$ (the latter being a direct consequence of $A = B^T \cdot B$ and the definition of the X_i's). Now, (3.3) is a consequence of the following computation: for every smooth u and any $x \in \mathbb{G}$,

$$\begin{aligned} Z_i(u \circ T_A)(x) &= Z_i|_x(u \circ T_A) = (\mathrm{d}_x T_A(Z_i|_x))u = (\mathrm{d}T_A Z_i)|_{T_A(x)} u \\ &= X_i|_{T_A(x)} u = (X_i u)(T_A(x)). \end{aligned}$$

Here we applied the fact that $\mathrm{d}T_A = \varphi_B$, $\varphi_B(Z_i) = X_i$ and the properties of differentials. The last assertion of the statement follows by collecting together the very definition of T_A in (3.2) and the normalization resulting from our previous choice of coordinates (see (2.4)). This ends the proof. □

Lemma 3.3. *Let $A \in \mathcal{A}$. Let $T_A : \mathbb{G} \to \mathbb{G}$ be the group isomorphism as defined by* (3.2) *of Lemma* 3.1*, where $B \in \mathcal{M}$ is any matrix satisfying $A = B^T \cdot B$.*

Let Γ_A and Γ denote respectively the fundamental solutions of the sub-Laplacians $\mathcal{L}_A$ and $\Delta_{\mathbb{G}}$. Then, there exists a constant $c_A > 0$ such that

$$\Gamma_A = c_A \cdot \Gamma \circ T_A^{-1}, \quad \text{on } \mathbb{G}. \tag{3.4}$$

As a consequence, the $\mathcal{L}_A$-gauge $d_{\mathcal{L}_A}$ is given by

$$d_{\mathcal{L}_A} = c_A^{1/(2-Q)} \cdot d_{\Delta_{\mathbb{G}}} \circ T_A^{-1}, \tag{3.5}$$

where $d_{\Delta_{\mathbb{G}}}$ is the $\Delta_{\mathbb{G}}$-gauge.

Proof. Let all the notation in the assertion hold. By the last statement of Lemma 3.1, the Jacobian matrix $\mathcal{J}_{T_A}(x)$ is independent of x. We let

$$c_A := 1/|\det(\mathcal{J}_{T_A})(x)|.$$

We are left to show that $\Gamma_A := c_A \cdot \Gamma \circ T_A^{-1}$ is the fundamental solution of the sub-Laplacian $\mathcal{L}_A$. By definition, this amounts to proving the following facts:

(i) It holds that $\Gamma_A : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ and $\Gamma_A \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$; this is obviously true, thanks to the analogous properties of Γ and the fact that T_A^{-1} is smooth, injective and null only at $x = 0$.

(ii) Γ_A *vanishes at infinity*; this derives from the analogous property of Γ and the fact that $\lim_{|x|\to\infty} |T_A^{-1}(x)| = \infty$, T_A^{-1} being a non-singular linear map.

(iii) $\Gamma_A \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *and, in the weak sense of distributions,* $-\mathcal{L}_A\Gamma_A$ *is the Dirac measure supported at* $\{0\}$; we only prove the last assertion since the L^1_{loc} property is an obvious consequence of $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

To this aim, let $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$. Then we have[2]

$$\int \Gamma_A\, \mathcal{L}_A\varphi = c_A \int \Gamma(T_A^{-1}(x))\, \mathcal{L}_A\varphi(x)\,\mathrm{d}x \quad \text{(substituting } x = T_A(y))$$

$$= c_A \int \Gamma(y)\, (\mathcal{L}_A\varphi)(T_A(y))\, |\det(\mathcal{J}_{T_A})(y)|\,\mathrm{d}y$$

(recall that T_A is linear and use the definition of c_A)

$$= \int \Gamma(y)\, (\mathcal{L}_A\varphi)(T_A(y))\,\mathrm{d}y \quad \text{(apply (3.1))}$$

$$= \int \Gamma(y)\, \Delta_{\mathbb{G}}(\varphi \circ T_A)(y)\,\mathrm{d}y = -(\varphi \circ T_A)(0) = -\varphi(0).$$

In the second-to-last equality we invoked the fact that Γ is the fundamental solution of $\Delta_{\mathbb{G}}$ (and the fact that $\varphi \circ T_A \in C_0^\infty$). The last equality is a consequence of $T_A(0) = 0$. This completes the proof. □

Lemma 3.4. *Let $A \in \mathcal{A}$. Let $T_A : \mathbb{G} \to \mathbb{G}$ be the group isomorphism as defined by* (3.2) *of Lemma* 3.1*, where $B \in \mathcal{M}$ is any matrix satisfying $A = B^T \cdot B$.*

Consider the mean-integral operator defined in (2.9)*. Then for any u.s.c. function $u : \mathbb{G} \to [-\infty, \infty)$ and for every $r > 0$ and $x \in \mathbb{G}$, we have*

$$M_r^{\mathcal{L}_A}(u)(x) = M_{\widetilde{r}}^{\Delta_{\mathbb{G}}}(u \circ T_A)(T_A^{-1}(x)), \quad \textit{with } \widetilde{r} = r\, c_A^{1/(Q-2)}. \tag{3.6}$$

Here, as usual, $c_A = 1/|\det(\mathcal{J}_{T_A})(0)|$.

Proof. For the sake of brevity, we drop any prefix or suffix '$\Delta_{\mathbb{G}}$'. Let $A = B^T \cdot B$ and let $T_A = \mathrm{Exp} \circ \varphi_B \circ \mathrm{Log}$ be as constructed in the proof of Lemma 3.1. Then $\mathcal{L}_A = \sum_{i=1}^m X_i^2$, where the X_i's are as in (2.8). Consequently, the relevant kernel $\Psi_{\mathcal{L}_A}$ in (2.9) is given by the following computation (see also Lemma 3.3):

$$\Psi_{\mathcal{L}_A} = \textstyle\sum_{i=1}^m |X_i d_{\mathcal{L}_A}|^2 \overset{(3.5)}{=} c_A^{2/(2-Q)} \sum_{i=1}^m |X_i(d \circ T_A^{-1})|^2$$
$$\overset{(3.3)}{=} c_A^{2/(2-Q)} \textstyle\sum_{i=1}^m |(Z_i d) \circ T_A^{-1}|^2 = c_A^{2/(2-Q)} \cdot \Psi \circ T_A^{-1}.$$

[2] Here we implicitly use the fact that any sub-Laplacian is selfadjoint, since the formal adjoint of any field X in H equals $-X$ (see, e.g., [5, §1.5]).

Here, for brevity, d is the $\Delta_{\mathbb{G}}$-gauge, where $\Delta_{\mathbb{G}} = \sum_{i=1}^m Z_i^2$. We next write the set $D_{\mathcal{L}_A}$ in terms of d: since T_A^{-1} is a group morphism, one has

$$d_{\mathcal{L}_A}(x^{-1} * y) \overset{(3.5)}{=} c_A^{1/(2-Q)} \cdot d(T_A^{-1}(x^{-1} * y)) = c_A^{1/(2-Q)} \cdot d\big((T_A^{-1}(x))^{-1} * T_A^{-1}(y)\big).$$

This gives $D_{\mathcal{L}_A}(x,r) = T_A\big(D(T_A^{-1}(x), r\, c_A^{1/(Q-2)})\big)$. Summing up, we infer

$$\begin{aligned}
&M_r^{\mathcal{L}_A}(u)(x) \overset{(2.9)}{=} \frac{Q(Q-2)}{r^Q} \int_{D_{\mathcal{L}_A}(x,r)} u(y)\, \Psi_{\mathcal{L}_A}(x^{-1} * y)\, \mathrm{d}y \\
&= \frac{Q(Q-2)}{r^Q} \int_{T_A\left(D(T_A^{-1}(x), r\, c_A^{1/(Q-2)})\right)} u(y)\, c_A^{2/(2-Q)} \cdot \Psi\big((T_A^{-1}(x))^{-1} * T_A^{-1}(y)\big)\, \mathrm{d}y \\
&\text{(use the substitution } y = T_A(z) \text{ and recall that } c_A = 1/|\det \mathcal{J}_{T_A}|) \\
&= \frac{Q(Q-2)}{r^Q} \int_{D(T_A^{-1}(x), r\, c_A^{1/(Q-2)})} u(T_A(z))\, c_A^{2/(2-Q)} \cdot \Psi\big((T_A^{-1}(x))^{-1} * z\big)\, c_A^{-1}\, \mathrm{d}z \\
&= \frac{Q(Q-2)}{\widetilde{r}^Q} \int_{D(\widetilde{x},\widetilde{r})} u(T_A(z))\, \Psi\big(\widetilde{x}^{-1} * z\big)\, \mathrm{d}z = M_{\widetilde{r}}(u \circ T_A)(\widetilde{x}),
\end{aligned}$$

where we have set $\widetilde{x} := T_A^{-1}(x)$ and $\widetilde{r} = r\, c_A^{1/(Q-2)}$. This is precisely (3.6). □

We are ready for the following characterization of subharmonicity on a free group $\mathbb{G}$. (Obviously, *mutatis mutandis*, one can state an analogous characterization for subharmonicity on an arbitrary open set of the free group $\mathbb{G}$.)

Proposition 3.5. *Let $\mathbb{G}$ be a free Carnot group. Let $u : \mathbb{G} \to [-\infty, \infty)$ be upper semicontinuous. Also let $A \in \mathcal{A}$ and let $\mathcal{L}_A$ be the sub-Laplacian defined in* (2.7).

Then, u is $\mathcal{L}_A$-subharmonic if and only if $u \circ T_A$ is $\Delta_{\mathbb{G}}$-subharmonic, where T_A is any isomorphism of the form (3.2) *(B being any matrix such that $A = B^T \cdot B$).*

Remark 3.6. More precisely, we have two slightly stronger results (it being understood that the notation of the assertion holds):

(1) if u is $\mathcal{L}_A$-subharmonic, then $u \circ T_A$ is $\Delta_{\mathbb{G}}$-subharmonic *for every* B such that $A = B^T \cdot B$;
(2) if there exists *at least one* B satisfying $A = B^T \cdot B$ for which $u \circ T_A$ is $\Delta_{\mathbb{G}}$-subharmonic, then u is $\mathcal{L}_A$-subharmonic.

Proof of Proposition 3.5. Suppose that the u.s.c. function u is $\mathcal{L}_A$-subharmonic on $\mathbb{G}$. By the submean condition (2.9) for the sub-Laplacian $\mathcal{L}_A$, this is equivalent to $u(x) \le M_r^{\mathcal{L}_A}(u)(x)$, for every $x \in \mathbb{G}$ and every $r > 0$. Thanks to identity (3.6) in Lemma 3.4, by taking $x = T_A(\xi)$ and $r = \rho\, c_A^{1/(2-Q)}$ (with $\xi \in \mathbb{G}$ and $\rho > 0$ arbitrary), this rewrites as

$$u(T_A(\xi)) \le M_\rho(u \circ T_A)(\xi), \quad \text{for every } \xi \in \mathbb{G} \text{ and } \rho > 0.$$

Since $u \circ T_A : \mathbb{G} \to [-\infty, \infty)$ is clearly u.s.c., the last inequality is equivalent to the subharmonicity of $u \circ T_A$ w.r.t. the sub-Laplacian $\Delta_{\mathbb{G}}$.

The reverse implication can be proved analogously. □

Before proving our main result, Theorem 3.8 below, we seize the opportunity to remove any ambiguity from the notation $A \mapsto T_A$ (which is a multiple-valued map) once and for all by introducing new notation.

Definition 3.7. Let $\mathbb{G}$ be a free homogenous Carnot group $\mathbb{G} = (\mathbb{R}^N, *, \delta_\lambda)$ with m generators. Let $Z_1, \ldots, Z_N$ be the basis of $\mathrm{Lie}(\mathbb{G})$ of the left-invariant vector fields coinciding at the origin with $\partial_{x_1}|_0, \ldots, \partial_{x_N}|_0$, respectively.

Given any real non-singular matrix $B = (b_{i,j})_{i,j\le m}$ we denote by φ_B the unique Lie-algebra isomorphism of $\mathrm{Lie}(\mathbb{G})$ mapping Z_i into $\sum_{j=1}^m b_{i,j}\, Z_j$, for every $i \in \{1, \ldots, m\}$. With this position, we let

$$\Theta_B := \mathrm{Exp} \circ \varphi_B \circ \mathrm{Log};$$

that is, Θ_B is the (unique) Lie-group automorphism of $\mathbb{G}$ whose differential is φ_B.

Theorem 3.8. *Let $\mathbb{G}$ be a free (homogenous) Carnot group with m generators, and fix the sub-Laplacian $\Delta_{\mathbb{G}} = \sum_{i=1}^m Z_i^2$ related to the left-invariant vector fields $Z_1, \ldots, Z_m$ coinciding at the origin with $\partial_{x_1}|_0, \ldots, \partial_{x_m}|_0$.*

Let $u : \mathbb{G} \to [-\infty, \infty)$ be upper semicontinuous. Then the following facts are equivalent (here Θ_B is as in Definition 3.7):

(i) *u is v-convex on $\mathbb{G}$;*
(ii) *$u \circ \Theta_B$ is subharmonic on $\mathbb{G}$ with respect to $\Delta_{\mathbb{G}}$, for every real $m \times m$ non-singular matrix B;*
(iii) *$u \circ \Theta_B$ is subharmonic on $\mathbb{G}$ with respect to $\Delta_{\mathbb{G}}$, for every real $m \times m$ symmetric and positive definite matrix B.*

Proof. It follows by collecting together Proposition 3.5 and the very definition of v-convexity. We have indeed the following implications.

(i)$\Rightarrow$(ii): Let u be v-convex and let $B \in \mathcal{M}$ be non-singular. Consider the matrix $A := B^T \cdot B$. Clearly $A \in \mathcal{A}$ and note that Θ_B is one of the admissible maps T_A introduced in Lemma 3.1. Since u is v-convex, it is in particular $\mathcal{L}_A$-subharmonic. Now, assertion (1) in Remark 3.6 ensures that $u \circ T_A = u \circ \Theta_B$ is $\Delta_{\mathbb{G}}$-subharmonic.

(ii)$\Rightarrow$(iii): Obvious.

(iii)$\Rightarrow$(i): Suppose that hypothesis (iii) holds. We aim to prove that u is v-convex, that is, u is $\mathcal{L}_A$-subharmonic for every $A \in \mathcal{A}$. Given $A \in \mathcal{A}$ there exists a unique symmetric positive definite matrix B such that $A = B^2$, say $B = A^{1/2}$. By hypothesis (iii), $u \circ \Theta_{A^{1/2}}$ is $\Delta_{\mathbb{G}}$-subharmonic. Note that $\Theta_{A^{1/2}}$ is one of the admissible maps T_A introduced in Lemma 3.1, since $B^T \cdot B = B^2 = A$. Now, assertion (2) in Remark 3.6 ensures that u is $\mathcal{L}_A$-subharmonic, as we intended to prove. This completes the proof of the theorem. $\square$

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