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# VOLUME FORMULAS FOR A SPHERICAL TETRAHEDRON

JUN MURAKAMI

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ABSTRACT. The present paper gives two concrete formulas for the volume of an arbitrary spherical tetrahedron that is in a 3-dimensional spherical space of constant curvature +1. One formula is given in terms of dihedral angles, and another one is given in terms of edge lengths.

## INTRODUCTION

The calculation of the volume of an arbitrary tetrahedron in a 3-space of non-zero constant curvature is rather hard, and the first result was given in [1] in 1999 for hyperbolic tetrahedra. The papers [5] and [4] gave another formula for hyperbolic tetrahedra, which is implicitly based on the quantum $6j$-symbol. Moreover, it was stated in [5] that an adequate analytic continuation of the obtained formula is also applicable for a spherical tetrahedron. But the formula is given by multi-valued functions, and it is not described which stratum we should select for actual computation. On the other hand, volumes of spherical tetrahedra of special shapes were given by many people in the past, and the most recent work is [2], which gives a formula for a spherical tetrahedron having a small symmetry.

In the present paper, volume formulas for a spherical tetrahedron $T$ of general shape are given in Theorems 1.1 and 1.2. The formula in Theorem 1.1 is given in terms of dihedral angles, and the formula in Theorem 1.2 is given in terms of edge lengths. These formulas are obtained by improving those in [5] and [4], and, by using the Schläfli differential equality, it is shown that the new formulas actually give the volume of $T$ modulo $2\,\pi^2$. Note that $2\,\pi^2$ is the volume of $S^3$ with radius 1, which is the universal cover of any 3-dimensional spherical space of constant curvature +1. Since $T$ can be included in a 3-dimensional hemisphere, the volume of $T$ is less than $\pi^2$ and so we can actually compute the volume of $T$ from the formulas in Theorems 1.1 and 1.2.

## 1. VOLUME FORMULAS

1.1. **Volume formula in terms of dihedral angles.** Let $T$ be a spherical tetrahedron and let $\theta_1, \theta_2, \cdots, \theta_6$ be its dihedral angles at edges $e_1, e_2, \cdots, e_6$, respectively, given in Figure 1. We assume that $0 < \theta_j < \pi$ for $j = 1, 2, \cdots, 6$. Let

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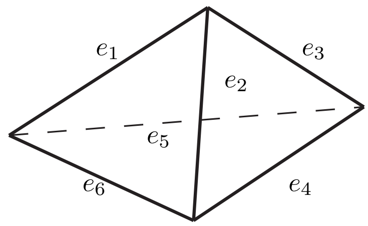


FIGURE 1. Edges of $T$

$a_1 = e^{i\theta_1}$, $a_2 = e^{i\theta_2}$, $\cdots$, $a_6 = e^{i\theta_6}$, and

$$\begin{aligned} &L(a_1, a_2, \cdots, a_6, z) \\ &\quad = \frac{1}{2}\Big(\mathrm{Li}_2(z) + \mathrm{Li}_2(a_1^{-1}\, a_2^{-1}\, a_4^{-1}\, a_5^{-1}\, z) + \mathrm{Li}_2(a_1^{-1}\, a_3^{-1}\, a_4^{-1}\, a_6^{-1}\, z) \\ &\qquad + \mathrm{Li}_2(a_2^{-1}\, a_3^{-1}\, a_5^{-1}\, a_6^{-1}\, z) - \mathrm{Li}_2(-a_1^{-1}\, a_2^{-1}\, a_3^{-1}\, z) - \mathrm{Li}_2(-a_1^{-1}\, a_5^{-1}\, a_6^{-1}\, z) \\ &\qquad - \mathrm{Li}_2(-a_2^{-1}\, a_4^{-1}\, a_6^{-1}\, z) - \mathrm{Li}_2(-a_3^{-1}\, a_4^{-1}\, a_5^{-1}\, z) + \sum_{j=1}^{3} \log a_j \, \log a_{j+3}\Big), \end{aligned}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function defined by analytic continuation of the following integral:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt \qquad \text{for a real number } x < 1. \tag{1.1}$$

The analytic continuation of the right-hand side integral defines a multi-valued complex function $\mathrm{li}_2(z)$, and let $\mathrm{Li}_2(z)$ be the principal branch of $\mathrm{li}_2(z)$ which is the analytic continuation of (1.1) on the region $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 1\}$. We also fix the principal branch of the log function as usual by the branch cut along the negative real axis.

We define the auxiliary parameter $z_0$ as

$$z_0 = \frac{-q_1 + \sqrt{q_1^2 - 4\, q_0\, q_2}}{2\, q_2}, \tag{1.2}$$

where

$$\begin{aligned} q_0 &= a_1\, a_4 + a_2\, a_5 + a_3\, a_6 + a_1\, a_2\, a_6 + a_1\, a_3\, a_5 + a_2\, a_3\, a_4 \\ &\qquad\qquad + a_4\, a_5\, a_6 + a_1\, a_2\, a_3\, a_4\, a_5\, a_6, \\ q_1 &= -(a_1 - a_1^{-1})(a_4 - a_4^{-1}) - (a_2 - a_2^{-1})(a_5 - a_5^{-1}) - (a_3 - a_3^{-1})(a_6 - a_6^{-1}), \\ q_2 &= a_1^{-1}a_4^{-1} + a_2^{-1}a_5^{-1} + a_3^{-1}a_6^{-1} + a_1^{-1}a_2^{-1}a_6^{-1} + a_1^{-1}a_3^{-1}a_5^{-1} \\ &\qquad + a_2^{-1}a_3^{-1}a_4^{-1} + a_4^{-1}a_5^{-1}a_6^{-1} + a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}a_5^{-1}a_6^{-1}. \end{aligned}$$

Then $z_0$ is a solution of

$$\exp\left(2\, z\, \frac{\partial L}{\partial z}\right) = 1, \tag{1.3}$$

where

$$\exp\left(2\, z\, \frac{\partial L}{\partial z}\right) = \frac{(a_1\, a_2\, a_3 + z)\,(a_1\, a_5\, a_6 + z)\,(a_2\, a_4\, a_6 + z)\,(a_3\, a_4\, a_5 + z)}{(1 - z)\,(a_1\, a_2\, a_4\, a_5 - z)\,(a_1\, a_3\, a_4\, a_6 - z)\,(a_2\, a_3\, a_5\, a_6 - z)}.$$

Now we state the main result of this paper.

**Theorem 1.1.** *Let $T$ be a spherical tetrahedron with dihedral angles $\theta_1, \theta_2, \cdots, \theta_6$ at the edges $e_1, e_2, \cdots e_6$ given in Figure 1. Let $a_j = e^{i\theta_j}$ for $j = 1, 2, \cdots, 6$ and let $\mathrm{Vol}(T)$ be the volume of $T$. Then*

$$\mathrm{Vol}(T) = -\mathrm{Re}(L(a_1, a_2, \cdots, a_6, z_0)) + \pi \left(\arg(-q_2) + \frac{1}{2}\sum_{j=1}^{6} \theta_j\right) - \frac{3}{2}\pi^2 \mod 2\pi^2,$$

*where $\mathrm{Re}(z)$ is the real part of $z$ and $z_0$, $q_2$ given by (1.2).*

1.2. **Volume formula in terms of edge lengths.** Let $T$ be a spherical tetrahedron with edge lengths $l_1, l_2, \cdots, l_6$ at the edges $e_1, e_2, \cdots e_6$, respectively, given in Figure 1. Let $b_j = e^{i\,l_j}$ for $j = 1, 2, \cdots, 6$ and let $\widetilde{L}(b_1, b_2, b_3, b_4, b_5, b_6, z) = L(-b_4^{-1}, -b_5^{-1}, -b_6^{-1}, -b_1^{-1}, -b_2^{-1}, -b_3^{-1}, z)$. Then the following formula holds.

**Theorem 1.2.** *For a spherical tetrahedron $T$ as above,*

$$\mathrm{Vol}(T) = \mathrm{Re}\Big(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)\Big) - \pi \arg(-\widetilde{q}_2) - \sum_{j=1}^{6} l_j \left.\frac{\partial\,\mathrm{Re}\big(\widetilde{L}(b_1, b_2, \cdots, b_6, z)\big)}{\partial l_j}\right|_{z=\widetilde{z}_0} - \frac{1}{2}\pi^2 \mod 2\pi^2,$$

*where $\widetilde{z}_0$ and $\widetilde{q}_2$ are obtained from $z_0$ and $q_2$ in (1.2) by substituting $a_j$ with $-b_{j\pm 3}^{-1}$ for $j = 1, 2, \cdots, 6$.*

## 2. Proof of the formulas

2.1. **Gram matrices.** Let $T$ be a spherical tetrahedron with dihedral angles $\theta_1, \cdots, \theta_6$ as before. Let $G$ be the Gram matrix of $T$ defined by

$$G = \begin{pmatrix} 1 & -\cos\theta_1 & -\cos\theta_2 & -\cos\theta_6 \\ -\cos\theta_1 & 1 & -\cos\theta_3 & -\cos\theta_5 \\ -\cos\theta_2 & -\cos\theta_3 & 1 & -\cos\theta_4 \\ -\cos\theta_6 & -\cos\theta_5 & -\cos\theta_4 & 1 \end{pmatrix}.$$

An actual computation shows that the discriminant in (1.2) is given by

$$q_1^2 - 4\,q_0\,q_2 = 16\,\det G, \tag{2.1}$$

which is positive since $T$ is spherical. It is known[1] that

$$\cos l_j = \frac{c_{pq}}{\sqrt{c_{pp}\,c_{qq}}}, \tag{2.2}$$

and so we have

$$\exp(2\,i\,l_j) = \frac{2\,c_{pq}^2 - c_{pp}\,c_{qq} + 2\,i\,c_{pq}\sqrt{\det G}\,\sin\theta_j}{c_{pp}\,c_{qq}} \tag{2.3}$$

[1] Formula (2.2) comes from the formula on p. 7, l.4, of [2] applied to the dual tetrahedron $T^*$. It is a spherical version of the formula just below (5.1) in [6].

by using the formula (5.1) in [6] that is $c_{pq}^2 - c_{pp}\,c_{qq} = -\det G\,\sin^2\theta_j$. Here $p$ and $q$ denote the row and column of $G = (g_{ab})$ such that $g_{p'q'} = -\cos\theta_j$, $\{p, q\} = \{1,2,3,4\} \setminus \{p', q'\}$ and $c_{ab}$ is the cofactor of $G$, i.e. $c_{ab} = (-1)^{a+b}\det G_{ab}$ where $G_{ab}$ is the submatrix obtained from $G$ by deleting its $a$-th row and $b$-th column.

2.2. **Some functions and their properties.** Before proving the formulas, we introduce some functions and investigate their properties. Let $T$ be an abstract tetrahedron, let $\theta_1, \theta_2, \cdots, \theta_6$ be its dihedral angles at the edges $e_1, e_2, \cdots, e_6$ as before, and let

$$D_s = \{(\theta_1, \theta_2, \cdots, \theta_6) \in (0,\pi)^6 \subset \mathbb{R}^6 \mid \theta_1,\, \theta_2,\, \cdots,\, \theta_6 \text{ correspond to the dihedral angles of a spherical tetrahedron}\}.$$

Let $a_j = e^{i\theta_j}$ for $j = 1, 2, \cdots, 6$,

$$\Delta_0(x,y,z) = -\frac{1}{4}\Big(\mathrm{Li}_2(-xy^{-1}z^{-1}) + \mathrm{Li}_2(-x^{-1}yz^{-1}) + \mathrm{Li}_2(-x^{-1}y^{-1}z) + \mathrm{Li}_2(-xyz)\Big),$$

$$\Delta(a_1, a_2, \cdots, a_6) = \Delta_0(a_1,a_2,a_3) + \Delta_0(a_1,a_5,a_6) + \Delta_0(a_2,a_4,a_6) + \Delta_0(a_3,a_4,a_5) - \frac{1}{2}\sum_{j=1}^{6}\big(\log a_j\big)^2,$$

$$U(a_1, a_2, \cdots, a_6, z) = L(a_1, a_2, \cdots, a_6, z) + \Delta(a_1, a_2, \cdots, a_6),$$

and

$$V(a_1,a_2,a_3,a_4,a_5,a_6) = -U(a_1,a_2,a_3,a_4,a_5,a_6,z_0) + \pi\, i\left(\log z_0 - \sum_{j=1}^{6}\log a_j\right) - \frac{13}{6}\,\pi^2.$$

**Lemma 2.1.** *The function $\Delta(a_1, a_2, \cdots, a_6)$ is analytic on $D_s$ and the imaginary part of $4a_j\frac{\partial\Delta}{\partial a_j}$ is given by*

$$\mathrm{Im}\left(4\,a_j\,\frac{\partial\Delta}{\partial a_j}\right) = -2\,\pi.$$

*Proof.* We prove this for the case $j = 1$. For the function $\Delta$,

$$a_1\,\frac{\partial\Delta}{\partial a_1} = a_1\,\frac{\partial\Delta_0(a_1,a_2,a_3)}{\partial a_1} + a_1\,\frac{\partial\Delta_0(a_1,a_5,a_6)}{\partial a_1} - \log a_1$$

and

$$a_1\,\frac{\partial\Delta_0(a_1,a_p,a_q)}{\partial a_1} = \frac{1}{4}\left(\log(1+\frac{a_1}{a_pa_q}) - \log(1+\frac{a_p}{a_1a_q}) - \log(1+\frac{a_q}{a_1a_p}) + \log(1+a_1a_pa_q)\right)$$

for $\{p,q\} = \{2,3\}$, $\{5,6\}$. The imaginary part $\operatorname{Im}\log(1+e^{i\theta})$ is given by

$$\operatorname{Im}\log(1+e^{i\theta}) = \begin{cases} \dfrac{\theta}{2} & \text{if } -\pi < \theta < \pi, \\ \dfrac{\theta}{2} - \pi & \text{if } \pi < \theta < 3\pi. \end{cases}$$

Let $\theta_u$, $\theta_v$, $\theta_w$ be three dihedral angles at three edges having a vertex in common. Then they satisfy

(2.4) $0 < \theta_u + \theta_v - \theta_w,\ \theta_u - \theta_v + \theta_w,\ -\theta_u + \theta_v + \theta_w < \pi, \quad \pi < \theta_u + \theta_v + \theta_w < 3\pi.$

Hence $\Delta_0(a_1, a_p, a_q)$ is analytic on $D_s$ and we have

$$\operatorname{Im}\left(a_1 \frac{\partial \Delta_0(a_1,a_p,a_q)}{\partial a_1}\right) = \frac{\theta_1}{2} - \frac{\pi}{4}, \qquad \operatorname{Im}\left(4\,a_1 \frac{\partial \Delta}{\partial a_1}\right) = -2\,\pi.$$

Moreover, $\Delta$ is analytic on $D_s$ because none of the imaginary parts of the log terms of $\Delta$ attain either $\pi$ or $-\pi$ on $D_s$. □

**Lemma 2.2.** *The function $L(a_1, a_2, \cdots, a_6, z_0(a_1, a_2, \cdots, a_6))$ is analytic on $D_s$, and so $U(a_1, a_2, \cdots, a_6, z_0(a_1, a_2, \cdots, a_6))$ is analytic on $D_s$.*

*Proof.* We know that $|z_0| < 1$ because, for $q_0$, $q_1$, $q_2$ in (1.2), $q_1$ is a real number and $q_0 q_2 = q_0 \overline{q_0} = |q_0|^2$ is a positive real number, and $q_1^2 - 4q_0q_2$ is also a positive real number by (2.1). This implies that, for $w \in \mathbb{C}$ with $|w| = 1$, $|w\,z_0| < 1$. Noting that $\operatorname{Li}_2(z)$ is analytic on the unit open disk $\{z \in \mathbb{C} \mid |z| < 1\}$, all the dilog terms of $L$ are analytic on $D_s$ since $|a_1| = \cdots = |a_6| = 1$. □

**Lemma 2.3.** *The differential $\frac{\partial U}{\partial z}$ satisfies $z_0 \frac{\partial U}{\partial z}\big|_{z=z_0} = \pi\, i$.*

*Proof.* Since $\frac{\partial U}{\partial z} = \frac{\partial L}{\partial z}$ and $z_0$ is a solution of equation (1.3), $z_0 \frac{\partial U}{\partial z}\big|_{z=z_0} = k\,\pi\, i$ for some integer constant $k$ because $U$ is analytic on $D_s$ by the above lemma. Let $T_{\frac{\pi}{2}}$ be the regular spherical tetrahedron with edge lengths $\pi/2$. Then $\theta_j = \pi/2$, $a_j = i$ for $j = 1, \cdots, 6$, $z_0 = (i+1)/2$ and

$$z_0 \left.\frac{\partial U}{\partial z}\right|_{z=z_0} = \frac{1}{2}\left(-4\,\log\frac{1-i}{2} + 4\,\log\frac{1+i}{2}\right) = \pi\, i.$$

Hence $z_0 \frac{\partial U}{\partial z}\big|_{z=z_0} = \pi i$ for all the spherical tetrahedra. □

Now we show the following proposition for $V$ corresponding to the Schläfli differential equality

$$d\,\mathrm{Vol}(T) = \sum_{j=1}^{6} \frac{l_j}{2}\, d\theta_j, \tag{2.5}$$

which is a fundamental tool for analyzing the volume. For example, see [3].

**Proposition 2.4.** *The function $V$ satisfies $\frac{\partial V}{\partial \theta_j} = l_j/2$ for $j = 1, 2, \cdots, 6$.*

*Proof.* Let $\varphi = \exp\left(4\,a_1 \frac{\partial \Delta}{\partial a_1}\right)$ and $\psi = \exp\left(2\,a_1 \frac{\partial L}{\partial a_1}\big|_{z=z_0}\right)$. Then

$$\varphi = \frac{(a_1 + a_2\,a_3)(a_1\,a_2\,a_3 + 1)(a_1 + a_5\,a_6)(a_1\,a_5\,a_6 + 1)}{(a_1\,a_2 + a_3)(a_1\,a_3 + a_2)(a_1\,a_5 + a_6)(a_1\,a_6 + a_5)},$$

$$\psi = \frac{(a_1\,a_2\,a_4\,a_5 - z_0)(a_1\,a_3\,a_4\,a_6 - z_0)}{a_4\,(a_1\,a_2\,a_3 + z_0)(a_1\,a_5\,a_6 + z_0)}.$$

An actual computation and (2.3) show that

$$\exp\left(4a_1\frac{\partial U}{\partial a_1}\right)\Big|_{z=z_0} = \varphi\psi^2 = \frac{2\,c_{34}^2 - c_{33}c_{44} + 2ic_{34}\sqrt{\det G}\,\sin\theta_1}{c_{33}\,c_{44}} = \exp(2\,l_1\,i).$$

Hence we get $a_1\frac{\partial U}{\partial a_1}\Big|_{z=z_0} = i\,(l_1 + k\,\pi)/2$ for some integer constant $k$ because $U$ is analytic on $D_s$ by Lemma 2.2. For the tetrahedron $T_{\frac{\pi}{2}}$ given in the proof of Lemma 2.3, $l_1 = \pi/2$ and $a_1\frac{\partial U}{\partial a_1}\Big|_{z=z_0} = -3\,\pi\,i/4$, which means that $k = -2$ and $a_1\,\frac{\partial U}{\partial a_1}\Big|_{z=z_0} = i\,(l_1 - 2\,\pi)/2$. According to $\frac{\partial U}{\partial \theta_1} = ia_1\frac{\partial U}{\partial a_1}$, we have

$$\frac{\partial U}{\partial \theta_1}\Big|_{z=z_0} = \frac{1}{2}\,(2\,\pi - l_1). \tag{2.6}$$

Therefore

$$\frac{\partial}{\partial \theta_1}\left(-U\big(a_1,\cdots,a_6,z_0(a_1,\cdots,a_6)\big) + \pi\,i\left(\log z_0 - \sum_{j=1}^{6}\log a_j\right)\right) = \frac{l_1}{2} - \frac{\partial z_0}{\partial \theta_1}\,\frac{\partial U}{\partial z}\Big|_{z=z_0} + \pi\,i\,\frac{\partial z_0}{\partial \theta_1}\,\frac{1}{z_0}.$$

Since $\frac{\partial U}{\partial z}\big|_{z=z_0} = i\,\pi/z_0$ by Lemma 2.3, we get $\frac{\partial V}{\partial \theta_1} = l_1/2$. □

2.3. **Proof of the formula in terms of dihedral angles.** We first give a formula using complex analytic functions.

**Proposition 2.5.** *Let $T$ be a spherical tetrahedron with dihedral angles $\theta_1$, $\theta_2$, $\cdots$, $\theta_6$ at the edges $e_1$, $e_2$, $\cdots$ $e_6$ as in Figure 1. Let $a_j = e^{i\theta_j}$ for $j = 1, 2, \cdots, 6$ as before and let $\mathrm{Vol}(T)$ be the volume of $T$. Then*

$$\mathrm{Vol}(T) = V(a_1,a_2,a_3,a_4,a_5,a_6) \mod 2\,\pi^2.$$

*Proof.* For the tetrahedron $T_{\frac{\pi}{2}}$ in the proof of Lemma 2.3, we have $a_j = i$, $z_0 = \frac{1+i}{2}$ and $V(i,i,i,i,i,i) = \pi^2/8 = \mathrm{Vol}(T_{\frac{\pi}{2}})$ since $T_{\frac{\pi}{2}}$ is one-sixteenth of $S^3$ and the volume of $S^3$ with radius 1 is $2\pi^2$. Because $V$ is analytic on some neighborhood $N$ of $T_{\frac{\pi}{2}}$ in $D_s$, two functions $V$ and Vol are identical on $N$ by Proposition 2.4 and Schäfli differential equality. Moreover, Vol is analytic on $D_s$ and so it is given by an adequate analytic continuation of $V$. We already showed in previous lemmas that all the terms in $V$ except $\pi i\log z_0$ are analytic on $D_s$, and the analytic continuation of $\pi i\log z_0$ is $\pi i\log z_0 + 2k\pi^2$ for some integer $k$. Hence we get the proposition. □

*Proof of Theorem* 1.1. We prove Theorem 1.1 by investigating the real part of $V$. For $\theta \in [0,2\,\pi] \subset \mathbb{R}$, the real part of $\mathrm{Li}_2(e^{i\theta})$ is given by $\mathrm{Re}\big(\mathrm{Li}_2(e^{i\theta})\big) = \mathrm{Re}\big(\mathrm{Li}_2(e^{-i\theta})\big) = \theta^2/4 - \pi\,\theta/2 + \pi^2/6$. Substituting this in each dilog function of $\mathrm{Re}(\Delta(a_1,\,a_2,\,\cdots,\,a_6))$, we get $\mathrm{Re}(\Delta(a_1,a_2,\cdots,a_6)) = -2\,\pi^2/3 + \sum_{j=1}^{6}\pi\theta_j/2$ by using (2.4). We also know that $\mathrm{Im}\log z_0 = -\arg(-q_2)$ since the numerator of $z_0$ in (1.2) is a negative real number. Hence we get Theorem 1.1 from Proposition 2.5. □

*Remark* 2.6. The function $V$ is non-continuous at the points where the values of $q_2$ are positive real numbers.

2.4. **Proof of the formula in terms of edge lengths.** We use the notation in Subsection 2.2.

*Proof of Theorem* 1.2. Let $\theta_1, \theta_2, \cdots, \theta_6$ be the dihedral angles at the edges $e_1, e_2, \cdots, e_6$ of $T$ and let $T^*$ be the dual tetrahedron of $T$ given by [3, p. 294]. Then the dihedral angles of $T^*$ are $\pi - l_4$, $\pi - l_5$, $\pi - l_6$, $\pi - l_1$, $\pi - l_2$, $\pi - l_3$ and the edge lengths of $T^*$ are $\pi - \theta_4$, $\pi - \theta_5$, $\pi - \theta_6$, $\pi - \theta_1$, $\pi - \theta_2$, $\pi - \theta_3$. The relation of volumes of $T$ and $T^*$ is given by [3, p. 294] as follows:

$$\mathrm{Vol}(T) + \mathrm{Vol}(T^*) + \frac{1}{2}\sum_{j=1}^{6} l_j\,(\pi - \theta_j) = \pi^2.$$

By Theorem 1.1, we have

$$\mathrm{Vol}(T^*) = -\mathrm{Re}(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)) + \pi\left(\arg(-\widetilde{q}_2) + \frac{1}{2}\sum_{j=1}^{6}(\pi - l_j)\right) - \frac{3}{2}\pi^2 \mod 2\pi^2.$$

Because $\frac{\partial}{\partial(\pi - l_j)} U(-b_4^{-1}, -b_5^{-1}, -b_6^{-1}, -b_1^{-1}, -b_2^{-1}, -b_3^{-1}, z)\Big|_{z=\widetilde{z}_0} = \big(2\pi - (\pi - \theta_j)\big)/2$ by (2.6) and $\frac{\partial}{\partial(\pi - l_j)}\mathrm{Re}\left(\Delta(-b_4^{-1}, -b_5^{-1}, -b_6^{-1}, -b_1^{-1}, -b_2^{-1}, -b_3^{-1})\right) = \pi/2$ by Lemma 2.1, we know that $\frac{\partial\,\mathrm{Re}\widetilde{L}}{\partial l_j}\Big|_{z=\widetilde{z}_0} = -\theta_j/2$. Hence

$$\begin{aligned}\mathrm{Vol}(T) = {} & \mathrm{Re}\big(\widetilde{L}(b_1, b_2, \cdots, b_6, \widetilde{z}_0)\big) - \pi\,\arg(-\widetilde{q}_2) \\ & - \sum_{j=1}^{6} l_j\,\frac{\partial\,\mathrm{Re}\big(\widetilde{L}(b_1, b_2, \cdots, b_6, z)\big)}{\partial l_j}\Bigg|_{z=\widetilde{z}_0} - \frac{1}{2}\pi^2 \mod 2\pi^2,\end{aligned}$$

and we get Theorem 1.2. □

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Department of Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan

*E-mail address*: murakami@waseda.jp