COMPACTNESS ESTIMATES FOR $\Box_b$ ON A CR MANIFOLD

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Abstract. This paper aims to state compactness estimates for the Kohn-Laplacian on an abstract CR manifold in full generality. The approach consists of a tangential basic estimate in the formulation given by the first author in his thesis, which refines former work by Nicoara. It has been proved by Raich that on a CR manifold of dimension $2n - 1$ which is compact pseudoconvex hypersurface type embedded in the complex Euclidean space and orientable, the property named “$(CR-P_1)$” for $1 \leq q \leq \frac{n-2}{n-1}$, a generalization of the one introduced by Catlin, implies compactness estimates for the Kohn-Laplacian $\Box_b$ in any degree $k$ satisfying $q \leq k \leq n - 1 - q$. The same result is stated by Straube without the assumption of orientability. We regain these results by a simplified method and extend the conclusions to CR manifolds which are not necessarily embedded nor orientable. In this general setting, we also prove compactness estimates in degree $k = 0$ and $k = n - 1$ under the assumption of $(CR-P_1)$ and, when $n = 2$, of closed range for $\partial_b$. For $n \geq 3$, this refines former work by Raich and Straube and separately by Straube.

1. Introduction and Statements

Let $M$ be a compact pseudoconvex CR manifold of hypersurface type of real dimension $2n - 1$ endowed with the Cauchy-Riemann structure $T^{1,0} M$. We choose a basis $L_1, ..., L_{n-1}$ of $T^{1,0} M$, the conjugated basis $\bar{L}_1, ..., \bar{L}_{n-1}$ of $T^{0,1} M$, and a transversal, purely imaginary, vector field $T$. We also take a hermitian metric on the complexified tangent bundle in which we get an orthogonal decomposition $\mathbb{C}T M = T^{1,0} M \oplus T^{0,1} M \oplus \mathbb{C}T$. We denote by $\omega_1, ..., \omega_{n-1}, \bar{\omega}_1, ..., \bar{\omega}_{n-1}, \gamma$ the dual basis of 1-forms. We denote by $\mathcal{L}_M$ the Levi form defined by $\mathcal{L}_M(L, L') := \partial \gamma(L, L')$ for $L, L' \in T^{1,0} M$. The coefficients of the matrix $(c_{ij})$ of $\mathcal{L}_M$ in the above basis are described through the Cartan formula as

$$c_{ij} = \langle \gamma, [L_i, \bar{L}_j] \rangle.$$

We denote by $\mathcal{B}^k$ the space of $(0, k)$-forms $u$ with $C^\infty$ coefficients. They are expressed, in the local basis, as $u = \sum_{|J|=k} u_J \bar{\omega}_J$ for $\bar{\omega}_J = \bar{\omega}_{j_1} \wedge ... \wedge \bar{\omega}_{j_k}$. Associated to the Riemannian metric $\langle \cdot, \cdot \rangle_z$, $z \in M$, and to the element of volume $dV$, there is a $L^2$-inner product $\langle u, v \rangle = \int_M \langle u, v \rangle_z dV$. We denote by $(L^2)^k$ the completion of $\mathcal{B}^k$ under this norm. We also use the notation $(H^s)^k$ for the completion under the Sobolev norm $H^s$. The de-Rham exterior derivative induces a complex $\partial_b : \mathcal{B}^k \to \mathcal{B}^{k+1}$. We denote by $\bar{\partial}_b : \mathcal{B}^k \to \mathcal{B}^{k-1}$ the adjoint and set $\Box_b = \partial_b \partial_b^* + \bar{\partial}_b \bar{\partial}_b$. Let $\varphi$ be a smooth function, denote by $(\varphi_{ij})$ the matrix of the
Levi form $L_\phi = \frac{1}{2}(\bar{\partial}_h \bar{\partial}_h - \bar{\partial}_h \bar{\partial}_h)(\phi)$ in the basis above, and by $\lambda_1^{\phi} \leq \ldots \leq \lambda_n^{\phi}$ the ordered eigenvalues of $L_\phi$. Let $L_\phi^2$ be the $L^2$ space weighted by $e^{-\phi}$ and, for $\phi_j := L_j(\phi)$, denote by $L_j^{\phi} = L_j - \phi_j$ the $L_\phi^2$-adjoint of $-L_j$. The following is the tangential version of the celebrated Hörmander-Kohn-Morrey basic estimate. Here we present the refinement by Khanh [Kh10] of a former statement by Nicoara [N06]. Let $z_o \in M$. For a suitable neighborhood $U$ of $z_o$ and a constant $c > 0$, we have

$$\|\bar{\partial}_b u\|^2_\phi + \|\bar{\partial}_{\bar{\partial}_b} u\|^2_\phi + c\|u\|^2_\phi \geq \sum_{|K|=k-1}^{q_o} \sum_{i,j} (\bar{\phi}_{ij} u_{iK}, u_{jK})_\phi - \sum_{|J|=k}^{q_o} \sum_{j=1}^{q_o} (\bar{\phi}_{jj} u_j, u_j)_\phi$$

(1.1)

$$+ \sum_{|K|=k-1}^{q_o} \sum_{i,j} (c_{ij} T_{uiK}, u_{jK})_\phi - \sum_{|J|=k}^{q_o} \sum_{j=1}^{q_o} (c_{jj} T_{uj}, u_j)_\phi$$

$$+ \frac{1}{2} \left( \sum_{j=1}^{q_o} \|L_j^\phi u\|^2_\phi + \sum_{j=q_o+1}^{n-1} \|L_j u\|^2_\phi \right),$$

for any $u \in B_k(U)$ where $q_o$ is any integer with $0 \leq q_o \leq n - 1$. We now introduce a potential-theoretical condition which is a variant of the “$P$-property” by Catlin [C83]. In the present version it has been introduced by Raich [R10].

**Definition 1.1.** Let $z_o$ be a point of $M$ and $q$ an index in the range $1 \leq q \leq n - 1$. We say that $M$ satisfies property $(CR - P_q)$ at $z_o$ if there is a family of weights $\{\phi^x\}$ in a neighborhood $U$ of $z_o$ such that

$$\begin{cases}
|\phi^x(z)| \leq 1, & z \in U, \\
\sum_{j=1}^{q} \lambda_j^{\phi^x}(z) \geq \epsilon^{-1}, & z \in U \text{ and } \ker L_M(z) \neq \{0\}.
\end{cases}$$

(1.2)

It is obvious that $(CR - P_q)$ implies $(CR - P_k)$ for any $k \geq q$.

**Remark 1.2.** Outside a neighborhood $V_\epsilon$ of ker $d\gamma$, the sum $\sum_{j \leq q} \lambda_j^{\phi^x}$ can get negative; let $-b_\epsilon$ be a bound from below. Now, if $c_\epsilon$ is a bound from below for $d\gamma$ outside $V_\epsilon$, by setting $a_\epsilon := \frac{\epsilon^{-1} + b_\epsilon}{q c_\epsilon}$, we have

$$\sum_{j \leq q} \lambda_j^{\phi^x} + a_\epsilon d\gamma = \sum_{j \leq q} \lambda_j^{\phi^x} + q a_\epsilon c_\epsilon \geq \epsilon^{-1} \text{ on the whole } U.$$  

(1.3)

Conversely, (1.3) readily yields the second line of (1.2). This equivalence was already noticed in [S10] and justifies our abuse of notation. In fact, (1.3) is named $(CR - P_q)$ by [S10] in accordance with [R10], whereas (1.2) is named “property $(P_q)$ in the nullspace of the Levi form”.

Again, (1.3) for $q$ implies (1.3) for any $k \geq q$.

We now state one of the two main results of the paper.
Theorem 1.3. Let $M$ be a compact pseudoconvex CR manifold of hypersurface type of dimension $2n - 1$. Assume that $(CR - P_q)$ holds for a fixed $q$ with $1 \leq q \leq \frac{n+1}{2}$ over a covering $\{U\}$ of $M$. Then we have compactness estimates: given $\epsilon$ there is $C_\epsilon$ such that

$$\|u\|^2 \leq \epsilon(\|\bar{\partial}_b u\|^2 + \|\partial_b u\|^2) + C_\epsilon \|u\|^{-1}$$

for any $u \in D^k_{\partial_b} \cap D^k_{\bar{\partial}_b}$ and $k \in [q, n - 1 - q]$,

where $D^k_{\partial_b}$ and $D^k_{\bar{\partial}_b}$ are the domains of $\partial_b$ and $\bar{\partial}_b$ respectively.

By Hodge duality between forms of complementary degree, we need the double constraint $k \geq q$ (for the positive microlocalization) and $k \leq n - 1 - q$ (for the negative one); this forces $q \leq \frac{n+1}{2}$. For $M$ embedded and orientable, Theorem 1.3 is contained in [R10]. The same is proved in [S10] without the assumption of orientability. The proof of this, as well as of the theorem which follows, is given in Section 2. Let $\mathcal{H}_k = \text{ker} \partial_b \cap \text{ker} \bar{\partial}_b^*$ be the space of harmonic forms of degree $k$. As a consequence of (1.4), we have that for $q \leq k \leq n - 1 - q$, the space $\mathcal{H}_k$ is finite-dimensional, $\Delta_b$ is invertible over $\mathcal{H}_k$ (cf. [N06] Lemma 5.3) and its inverse $G_k$ is a compact operator. When $k = 0$ and $k = n - 1$ it is no longer true that it is finite-dimensional. However, if $q = 1$, we have a result analogous to (1.3) also in the critical degrees $k = 0$ and $k = n - 1$.

Theorem 1.4. Let $M$ be a compact, pseudoconvex CR manifold of hypersurface type of dimension $2n - 1$. Assume that property $(CR - P_q)$ holds for $q = 1$ over a covering $\{U\}$ of $M$ and, in case $n = 2$, make the additional hypothesis that $\partial_b$ has closed range. Then for any $\epsilon$ there is $C_\epsilon$ such that

$$\|u\|^2 \leq \epsilon(\|\bar{\partial}_b u\|^2 + \|\partial_b u\|^2) + C_\epsilon \|u\|^{-1}$$

for any $u \in \mathcal{H}_k^\perp$, $k = 0$ and $k = n - 1$.

In particular, $G_k$ is compact for $k = 0$ and $k = n - 1$.

For $n \geq 3$ and $M$ a boundary of a domain in $\mathbb{C}^n$, resp. embedded and orientable, Theorem 1.3 is contained in [RS08] (resp. [S10]).

2. Proofs

Proof of Theorem 1.3. We choose a local patch $U$ where a local frame of vector fields is found for which (1.1) is fulfilled. The key point is to specify the convenient choices of $q_o$ and $\varphi$ in (1.1). Let $1 = \psi^+ + \psi^- + \psi^0$ be a conic, smooth partition of the unity in the space $\mathbb{R}^{2n-1}$ dual to the space in which $U$ is identified in local coordinates. Let $\Psi_{\bar{\xi}}$ be the pseudodifferential operators with symbols $\psi_{\bar{\xi}}$ and let $id = \psi^+ \Psi^+ \ast + \psi^- \Psi^- \ast + \psi^0 \Psi^0 \ast$ be the corresponding microlocal decomposition of the identity. For a cut off function $\zeta \in C^\infty_c(U)$ we decompose a form $u$ as

$$u^\zeta = \zeta \Psi_{\bar{\xi}} u \quad u \in \mathcal{E}_c^\infty(U), \quad \zeta|_{\text{supp}\ u} \equiv 1.$$ 

For $u^+$ we choose $q_o = 0$ and $\varphi = \varphi^+$. We also need to go back to Remark 1.2. Now, if $\alpha$ has been chosen so that (1.3) is fulfilled, we remove $T$ from our scalar products observing that, for large $\xi$, we have $\xi^{2n+1} > a_\epsilon$ over $\text{supp} \psi^+$. In the same
way as in Lemma 4.12 of [N06], we conclude that for \( k \geq q \)
\[
\sum'_{|K|=k-1} \sum_{i,j=1,...,n-1} \left( (c_{ij} T + \varphi^e_{ij}) u_{iK}^+, u_{jK}^+ \right)_{\varphi^e}
\geq \sum'_{|K|=k-1} \sum_{i,j=1,...,n-1} \left( (a_c c_{ij} + \varphi^e_{ij}) u_{iK}^+, u_{jK}^+ \right)_{\varphi^e}
- C \| u^+ \|^2_{\varphi^e} - C_e \| u^+ \|^2_{-1,\varphi^e} - C_e \| \zeta^2 \tilde{\Psi}^0 u^+ \|^2_{\varphi^e}
\geq \epsilon^{-1} \| u^+ \|^2_{\varphi^e} - C_e \| u^+ \|^2_{-1,\varphi^e} - C_e \| \zeta^2 \tilde{\Psi}^0 u^+ \|^2_{\varphi^e},
\]
where \( \tilde{\Psi}^0 \succ \Psi^0 \) and \( \zeta^2 \succ \zeta^1 \) in the sense that \( \tilde{\Psi}^0|_{\supp \Psi^0} \equiv 1 \) and \( \zeta^2|_{\supp \zeta^1} \equiv 1 \) respectively. (Here \( \| u^+ \|_{-1,\varphi^e} = \| \Lambda^{-1} u^+ \|_{\varphi^e} \), where \( \Lambda^{-1} \) is the standard tangential pseudodifferential operator of order \(-1\) in the local patch \( U \).) Note that there is an additional term \(-C_e \| u^+ \|^2_{1,\varphi^e}\) with respect to [N06]. The reason is that \((c_{ij} \xi_{2n-1} + \varphi^e_{ij})\) can get negative values, even on \( \supp \psi^+ \), when \( \xi_{2n-1} < a_c \). Integration in this compact region produces the above error term. It follows that
\[
(\chi(\varphi^e))_{ij} = \chi \varphi^e_{ij} + \tilde{\chi}|\varphi^e_j|^2 \kappa_{ij},
\]
where \( \kappa_{ij} \) is the Kronecker symbol. We also notice that
\[
|\tilde{\partial}_{b,\chi} \varphi^e_j| u_j^2 \leq 2 |\tilde{\partial}_{b}^e| u_j^2 + 2 \chi^2 \sum'_{|K|=k-1} |\sum_{j=1}^{n-1} \varphi^e_j u_{jK}|^2.
\]
Remember that \( \{ \varphi^e \} \) are uniformly bounded by 1. Thus, if we choose \( \chi = \frac{1}{4} e^{(t-1)} \), then we have that \( \tilde{\chi} \geq 2 \chi^2 \) for \( t = \varphi^e \). For this reason, with this modified weight, we can replace the weighted adjoint \( \tilde{\partial}_{b,\varphi}^e \) by the unweighted \( \tilde{\partial}_{b}^e \) in (2.2). By the uniform boundedness of the weights, we can also remove them from the norms and end up with the estimate
\[
(2.3) \quad \| u^+ \|^2 \leq \epsilon \left( \| \tilde{\partial}_{b} u^+ \|^2 + \| \tilde{\partial}_{b}^e u^+ \|^2 \right) + C_e \| u^+ \|^2_{-1} + C_e \| \zeta^2 \tilde{\Psi}^0 u^+ \|^2, \quad k = q, \ldots, n-1.
\]
For \( u^- \), we choose \( q_0 = n-1 \) and \( \varphi = -\varphi^e \). Observe that for \( |\xi| \) large we have \( -\xi_{2n-1} \geq a_c \) over \( \supp \psi^- \) (cf. [N06], Lemma 4.13). Thus, we have in the current case, for \( k \leq n-1-q \),
\[
\sum'_{|K|=k-1} \sum_{i,j=1,...,n-1} \left( (c_{ij} T - \varphi^-_{ij}) u_{iK}^-, u_{jK}^- \right)_{-\varphi^-} - \sum'_{|J|=k} \sum_{j=1}^{n-1} \left( (c_{ij} T - \varphi^-_{ij}) u_{iJ}^-, u_{jJ}^- \right)_{-\varphi^-}
\geq - \sum'_{|K|=k-1} \sum_{i,j=1,...,n-1} \left( (a_c c_{ij} + \varphi^-_{ij}) u_{iK}^-, u_{jK}^- \right)_{-\varphi^-}
+ \sum'_{|J|=k} \sum_{j=1}^{n-1} \left( (a_c c_{ij} + \varphi^-_{ij}) u_{iJ}^-, u_{jJ}^- \right)_{-\varphi^-}
- C \| u^- \|^2_{\varphi^-} - C_e \| u^- \|^2_{-1,\varphi^-} - C_e \| \zeta^2 \tilde{\Psi}^0 u^- \|^2_{\varphi^-}
\geq \epsilon^{-1} \| u^- \|^2_{\varphi^-} - C \| u^- \|^2_{\varphi^-} - C_e \| u^- \|^2_{-1,\varphi^-} - C_e \| \zeta^2 \tilde{\Psi}^0 u^- \|^2_{\varphi^-}.
\]
Thus, we get the analogue of (2.2) for $u^+$ replaced by $u^-$ and, again removing the weight from the adjoint $\bar{\partial}_{b,\phi^*}$ and from the norms, we conclude that
\begin{equation}
\|u^\sim\|_2^2 \leq \epsilon \left( \|\bar{\partial}_b u^\sim\|_2^2 + \|\bar{\partial}_{b,\phi^*} u^\sim\|_2^2 \right) + C_\epsilon \|u^\sim\|_{-1,\phi^*}^2 + C_\epsilon \|\zeta^2 \bar{\Psi}^0 u\|_2^2 , \quad k = 0, \ldots, n - 1 - q.
\end{equation}
In addition to (2.3) and (2.4), we have elliptic estimates for $u^0$:
\begin{equation}
\|u^0\|_2^2 \leq \|\bar{\partial}_b u\|_2^2 + \|\bar{\partial}_{b,\phi^*} u^0\|_2^2 + \|u\|_{-1}^2.
\end{equation}
The same estimate also holds for $u^0$ replaced by $\zeta^2 \bar{\Psi}^0 u$. We put together (2.3), (2.4) and (2.5) and notice that
\begin{equation}
\|\bar{\partial}_b (\zeta^1 \bar{\Psi}^0 u)\|_2^2 \leq \|\zeta^1 \bar{\Psi}^0 \bar{\partial}_b u\|_2^2 + \|\bar{\partial}_b, \zeta^1 \bar{\Psi}^0\| u\|_2^2 \leq \|\zeta^1 \bar{\Psi}^0 \bar{\partial}_b u\|_2^2 + \|\zeta^2 \bar{\Psi}^0 u\|_2^2 + \|\zeta^2 \bar{\Psi}^0 u\|_2^2,
\end{equation}
for $\zeta^2 \supset \zeta^1$ and $\bar{\Psi}^0 \supset \Psi^0$. A similar estimate holds for $\bar{\partial}_b$ replaced by $\bar{\partial}_{b,\phi^*}$. Since $\zeta^1|_{\text{supp } u} \equiv 1$, then
\begin{equation}
\|u\|_2^2 \leq \sum_{+,\ldots, 0} \|\zeta^1 \bar{\Psi}^0 u\|_2^2 + O p^{-\infty} (u) \leq \epsilon \sum_{+,\ldots, 0} \left( \|\bar{\partial}_b u\|_2^2 + \|\bar{\partial}_{b,\phi^*} u\|_2^2 \right) + C_\epsilon \|u\|_{-1}^2,
\end{equation}
and therefore
\begin{equation}
\|u\|_2^2 \leq \epsilon \left( \|\bar{\partial}_b u\|_2^2 + \|\bar{\partial}_{b,\phi^*} u\|_2^2 \right) + C_\epsilon \|u\|_{-1}^2 , \quad q \leq k \leq n - 1 - q.
\end{equation}
We now consider $u$ globally defined on the whole $M$ instead of a local patch $U$. To pass from local to global compactness estimates is immediate (cf. e. g. [10]). We cover $M$ by $\{U_\nu\}$ so that in each patch there is a basis of forms in which the basic estimate holds. In the identification of $U_\nu$ to $\mathbb{R}^{2n-1}$, we suppose that the microlocal decomposition by the operators $\bar{\Psi}^0$ which occur in (2.6) is well defined. We then get (2.7) and apply it to a decomposition $u = \sum_\nu \zeta_\nu u$ for a partition of the unity $\sum_\nu \zeta_\nu = 1$ on $M$. We point out that we first take summation over $+, -, 0$ on each patch $U_\nu$ and then summation over $\nu$; this is why orientability of $M$ is needless.

We observe that $[\bar{\partial}_b, \zeta_\nu]$ and $[\bar{\partial}_{b,\phi^*}, \zeta_\nu]$ are 0-order operators and, since they come with a factor of $\epsilon$, they are absorbed in the left side of (2.7); thus (2.7) holds for any $u \in B^k$. Finally, we use the density of smooth forms $B^k$ into Sobolev forms $(H^1)^k$ of $D^k_{\bar{\partial}_b} \cap D^k_{\bar{\partial}_{b,\phi^*}}$ for the graph norm and get (2.7). The proof is complete.

**Proof of Theorem 1.4** We prove estimates in degree 0 (those in degree $n - 1$ being similar). We first discuss the case $n > 2$. We make repeated use of (2.7) in degree 1. This first implies that $\bar{\partial}_b^* \bar{\partial}_b$ has closed range on 1-forms, that is,
\begin{equation}
\mathcal{H}^{0,1} = (\ker \bar{\partial}_b) \perp = \text{range } \bar{\partial}_b^*.
\end{equation}
(Thus, if $u \in \mathcal{H}^{0,1}$, then there exists a solution $v \in (L^2)^1$ to the equation $\bar{\partial}_b^* v = u$. Moreover, we can choose $v$ belonging to $(Ker(\bar{\partial}_b^*) \perp)$. This is a consequence of the following estimate:
\begin{equation}
\|v\|_0^2 \leq \|\bar{\partial}_b^* v\|_0^2 \quad \text{for any } v \in (\ker \bar{\partial}_b^*) \perp.
\end{equation}
This can be proved by contradiction. If \((2.8)\) is violated, there exists a sequence \(v_\nu \in (\ker \bar{\partial}_b^*)^\perp\) such that \(\|v_\nu\|^2_0 \equiv 1\) and \(\|\bar{\partial}_b^* v_\nu\|_0 \to 0\). Take a subsequential \(L^2\)-weak limit \(v_0\) of \(v_\nu\); it satisfies \(v_0 \in \text{Ker}(\bar{\partial}_b^*) \cap (\text{Ker}(\bar{\partial}_b^*))^\perp\) and in particular \(\|v_0\|_{-1} \to 0\). This violates (2.7) and proves (2.8). We also have

\[(2.9)\]
\[
\|v\|_{-1}^2 \leq \epsilon \|\bar{\partial}_b^* v\|_2^2 + c_\epsilon \|\bar{\partial}_b^* v\|_{-1}^2,
\]
for any \(v \in (\ker \bar{\partial}_b^*)^\perp\).

The argument is similar. If \((2.9)\) is violated, there is a sequence \(v_\nu \in (\ker \bar{\partial}_b^*)^\perp\) such that \(\|v_\nu\|_{-1} \equiv 1\), \(\|\bar{\partial}_b^* v_\nu\| \to 0\) and \(\|\bar{\partial}_b^* v_\nu\|_0 \leq c\). By (2.7), \(\|v_\nu\|_0 \leq C\); hence there is a subsequential \(L^2\)-weak limit \(v_\nu \to v_0 \in (\text{Ker}(\bar{\partial}_b^*))^\perp \cap \text{Ker}(\bar{\partial}_b^*)\); thus \(v_0 = 0\) and \(\|v_\nu\|_{-1} \to 0\), a contradiction.

We now point out that \((\text{Ker}(\bar{\partial})^\dagger) = \text{range}(\bar{\partial}) \subset \text{Ker}(\partial);\) in particular, our solution \(v\) satisfies \(\bar{\partial} v = 0\). We are ready to conclude the proof for \(n > 2\). We use the notation \(\mathcal{H}\) and \(\mathcal{E}\) for a large and small constant respectively. We have for any function \(u \in \mathcal{H}\)

\[
\|u\|^2 = (u, \bar{\partial}_b^* v)
\]
\[
= (\bar{\partial}_b u, v)
\]
\[
\leq \|\bar{\partial}_b u\| \|v\|_2^2 + \|\bar{\partial}_b v\| \|u\|_1^2 + c_\epsilon \|v\|_{-1}^2
\]
(2.10)
\[
\leq \|\bar{\partial}_b u\| \|v\|_2^2 + \|\bar{\partial}_b v\| \|u\|_{-1}^2
\]
\[
\leq c_1 \epsilon^2 |\bar{\partial}_b u|^2 + s c_1 \|u\|^2_2 + l c_2 c_1^2 \|u\|_{-1}^2 + s c_2 \|\bar{\partial}_b u\|^2_2
\]
\[
\leq \epsilon' \|\bar{\partial}_b u\|^2 + c_\epsilon \|u\|_{-1}^2 + s c_1 \|u\|^2_2,
\]
for \(\epsilon' = l c_1 \epsilon^2 + s c_2\) and \(c_\epsilon = l c_2 c_1^2\). By choosing \(c_1\) so that \(s c_1 \|u\|^2_2\) is absorbed in the left, (2.10) yields (2.7) for \(u\) in degree 0. This concludes the proof of the case \(n > 2\) for functions.

Let \(n = 2\). We have only estimates for positively microlocalized 1-forms and for negatively microlocalized functions. We have to show how to get estimates for positively microlocalized functions (the argument for negative 1-forms being similar). We use our extra assumption of closed range for \(\bar{\partial}_b\); thus for any \(u \in (\ker \bar{\partial}_b)^\perp\) there is \(v \in (\ker \bar{\partial}_b^*)^\perp\) such that \(\bar{\partial}_b^* v = u\). On each \(U_\nu\) we consider the positive microlocalization \(\Psi^+\), take a pair of cut-off functions \(\zeta_\nu, \zeta_\nu^0 \in C^\infty_\text{supp}(U_\nu)\) with \(\zeta_\nu^0|_{\text{supp} \zeta_\nu} \equiv 1\), and define \(\Psi^+_\nu := \zeta_\nu^0 \Psi^+ \zeta_\nu\). Note that the commutators \([\bar{\partial}_b^+, \Psi^+_\nu]\) and \([\bar{\partial}_b, \Psi^+_\nu]\) are operators with symbols of types \(\zeta_\nu^0 \zeta_\nu^+\zeta_\nu, \zeta_\nu^0 \zeta_\nu^+ \zeta_\nu^0\) and \(\zeta_\nu^0 \zeta_\nu^+ \zeta_\nu^0\). All these symbols have support contained in the positive half-space \(\xi_{2n-1} > 0\), and hence we have compactness estimates for 1-forms if their coefficients are subjected to the action of the corresponding pseudodifferential operators. We denote by \(\Phi^+\nu\) all these operators coming from commutators. We have

\[
\|\Phi^+\nu \| \leq \epsilon \|\bar{\partial}_b^+ \Phi^+\nu \| + c_\epsilon \|\Phi^+\nu\|_{-1} + c_\epsilon \|\zeta_\nu^0 \zeta_\nu^+ \zeta_\nu\|_1
\]
(2.11)
\[
\leq \epsilon \|\Phi^+\nu \| + c_\epsilon \|\Phi^+\nu\|_{-1} + c_\epsilon \|\zeta_\nu^0 \zeta_\nu^+ \zeta_\nu\|_{1}\]
\[
\leq \epsilon \|u\| + c_\epsilon \|u\|_{-1}.
\]

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The same estimate also holds for $\|\Phi^+_\nu u\|$. It follows that

$$
\|\Phi^+_\nu u\|^2 = (\Phi^+_\nu u, \Phi^+_\nu \bar{\partial} v) 
= (\|\Phi^+_\nu \bar{\partial} u\| + \|\Phi^+_\nu u\|)(\|\Phi^+_\nu v\|)
\leq \left( \|\Phi^+_\nu \bar{\partial} u\| + \|\Phi^+_\nu u\| \right) \epsilon \|u\| + c_\epsilon \|u\|^{-1}
\leq \epsilon \|\Phi^+_\nu \bar{\partial} u\| \|u\| + c_\epsilon \|\Phi^+_\nu u\| \|u\|^{-1} + \epsilon \|u\|^2 + c_\epsilon \|u\|^{-1} \|u\|
\leq l c_1 \epsilon^2 \|\Phi^+_\nu \bar{\partial} b u\|^2 + sc_1 \|u\|^2 + sc_2 \|\Phi^+_\nu \bar{\partial} b u\|^2 + lc_2 \epsilon^2 \|u\|^2_{-1} + \epsilon \|u\|^2 + sc_3 \|u\|^2_{-1} + lc_3 \epsilon^2 \|u\|^2_{-1}
\leq \epsilon' \|\Phi^+_\nu \bar{\partial} b u\|^2 + sc_4 \|u\|^2 + c_{\epsilon'} \|u\|^2_{-1},
$$

where $\epsilon' = l c_1 \epsilon^2 + sc_3$, $c_{\epsilon'} = lc_2 \epsilon^2 + lc_3 \epsilon^2$ and $sc_4 = sc_1 + \epsilon + sc_2$. We have to recall now that the same estimate as (2.12) also holds for $\|\Phi^-_\nu u\|^2$ (the one for $\|\Phi^0_\nu u\|^2$ being trivial by ellipticity). Taking summation over $+, -$ and 0 on each $U_\nu$, we get

$$
\|\zeta_\nu u\|^2 \leq \epsilon (1 + \epsilon') \|\bar{\partial} b u\|^2 + c_\epsilon \|u\|^2_{-1} + sc \|u\|^2.
$$

We now take summation over $\nu$ and choose $sc$ so that the related term is absorbed by $\sum_{\nu} \|\zeta_\nu u\|^2 \sim \|u\|^2$ and end up with

$$
\|u\|^2 \leq \epsilon \|\bar{\partial} b u\|^2 + c_\epsilon \|u\|^2_{-1}
$$
for any function $u$. \hfill \square

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