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A NOTE ON POINCARÉ'S PROBLEM FOR QUASI-HOMOGENEOUS FOLIATIONS

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ABSTRACT. We consider the question of bounding the degree of curves which are invariant by a holomorphic foliation of a given degree on a well-formed weighted projective plane.

1. INTRODUCTION AND STATEMENT OF RESULT

In this paper we consider the question of bounding the degree of curves which are invariant by a holomorphic foliation of a given degree on a well-formed weighted projective plane. This question had been considered by several authors ever since H. Poincaré raised it in a more specific setting, that of algebraic integration of polynomial differential equations in $\mathbb{C}^2$; see [1], [2], [5], [6], [10], [11] and [13].

We recall that, in the usual complex projective situation, the presence of dicritical singularities makes the question meaningless, as is shown by simple examples, such as Example 3 below and, in a much more involved and elucidating context, by the examples given by A. Lins Neto in [8].

However, denoting by d^o and d the degrees of the curve and of the foliation, respectively, in case the invariant curve is nonsingular we get the bound $d^o \leq d+1$, which holds also in higher dimensions [13] regardless of the nature of the singularities of the foliation along the invariant variety. Also, as shown in [1], for curves with only nodal singularities, we get the same estimate in case it is irreducible, and $d^o \leq d+2$ in the reducible case.

We consider this question in $\mathbb{P}_w := \mathbb{P}(w_0, w_1, w_2)$, $w_i \in \mathbb{N}$ two-by-two coprimes, endowed with a singular holomorphic foliation which admits an invariant quasi-smooth curve. Let $Sing(\mathbb{P}_w)$ and $Sing(\mathcal{F})$ denote the singular sets of $\mathbb{P}_w$ and of $\mathcal{F}$, respectively. The result is as follows:

Theorem. *Let $\mathcal{F}$ be a singular holomorphic foliation on $\mathbb{P}_w$, of degree $\mathsf{deg}(\mathcal{F})$, with $Sing(\mathcal{F}) \cap Sing(\mathbb{P}_w) = \emptyset$ and C a quasi-smooth curve of degree $\mathsf{deg}(C)$ which is invariant by $\mathcal{F}$. Then,*

$$\mathsf{deg}(C) \leq \mathsf{deg}(\mathcal{F}) + \frac{w_0 + w_1 + w_2 - 2}{w_0 w_1 w_2}. \tag{1}$$

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This bound cannot be improved, as is shown by

Example 1. Let $f(x,y,z) = x^{mk} + y^{mk} - z^k$ and $g(x,y,z) = ax^m + by^m + cz$, $m, k \in \mathbb{N}$. These are quasi-homogeneous polynomials of type $(1,1,m)$ and degrees km and m, both defining quasi-smooth curves of degrees k and 1, respectively, which avoid the singularity of $\mathbb{P}(1,1,m)$. The 1-form $\eta = k\,f\,dg - g\,df$ defines a singular foliation $\mathcal{F}$ on $\mathbb{P}(1,1,m)$ of degree $\mathsf{deg}(\mathcal{F}) = k - 1/m$. The orbifold $C = (f = 0)$ is $\mathcal{F}$-invariant and

$$\mathsf{deg}(C) = k \leq k - \frac{1}{m} + 1 = \mathsf{deg}(\mathcal{F}) + 1. \tag{2}$$

Example 2. Let $f(x,y,z) = x^{m+1} + y^{m+1} - (x+y)z$ and $g(x,y,z) = ax^m + by^m + cz$, $2 \leq m \in \mathbb{N}$. These are quasi-homogeneous polynomials of type $(1,1,m)$ and degrees $m+1$ and m, both defining quasi-smooth curves of degrees $1 + 1/m$ and 1. Note that the curve $C = (f = 0)$ passes through the singular point $[0:0:1]_{(1,1,m)}$ of $\mathbb{P}(1,1,m)$ and is smooth there. The 1-form $\eta = (1 + 1/m)\,f\,dg - g\,df$ defines a singular foliation $\mathcal{F}$ on $\mathbb{P}(1,1,m)$, of degree $\mathsf{deg}(\mathcal{F}) = 1$, which is nonsingular at $[0:0:1]_{(1,1,m)}$. C is $\mathcal{F}$-invariant and

$$\mathsf{deg}(C) = 1 + 1/m \leq 1 + 1 = \mathsf{deg}(\mathcal{F}) + 1. \tag{3}$$

Example 3. Let k be a positive integer and

$$\eta = \frac{yz}{k}\,dx + \left(m - \frac{1}{k}\right) xz\,dy - xy\,dz. \tag{4}$$

η induces a singular foliation $\mathcal{F}$ on $\mathbb{P}(1,1,m)$ of degree $\mathsf{deg}(\mathcal{F}) = 1/m$. The curve $C = \{z^k - xy^{km-1} = 0\}$ has degree k and is $\mathcal{F}$- invariant. Since k is arbitrary, the theorem does not hold. Here we have $[0:0:1]_{(1,1,m)}$ as a singular point of the foliation and also, $[1:0:0]_{(1,1,m)}$ is a singularity of C and a dicritical singularity of $\mathcal{F}$.

2. Foliations on weighted projective planes

We start by recalling weighted projective planes. Let $w := w_0, w_1, w_2$ be positive integers which are two-by-two coprimes. $\mathbb{C}^*$ acts on $\mathbb{C}^3 \setminus \{0\}$ by $\lambda.(z_0, z_1, z_2) = (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2)$. The quotient space by this action is the weighted projective plane of type w, $\mathbb{P}(w_0, w_1, w_2) := \mathbb{P}_w$. In case $w_i > 1$ for some i, $\mathbb{P}(w)$ is a compact algebraic surface with cyclic quotient singularities. Following [9], we have a natural orbifold map

$$\begin{aligned} f_w : \mathbb{P}^2 &\longrightarrow \mathbb{P}_w \\ [z_0 : z_1 : z_2] &\longmapsto [z_0^{w_0} : z_1^{w_1} : z_2^{w_2}]_w, \end{aligned} \tag{5}$$

which allows us to show that there exists a unique (up to isomorphism) rank 1 complex $\mathbb{Q}$-bundle $\mathcal{O}_{\mathbb{P}_w}(1)$ over $\mathbb{P}_w$, such that

$$f_w^* \quad \mathcal{O}_{\mathbb{P}_w}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1). \tag{6}$$

The Euler sequence carries over to the weighted case, and we have the exact sequence of $\mathbb{Q}$-bundles over $\mathbb{P}_w$:

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_{\mathbb{P}_w}(w_0) \oplus \mathcal{O}_{\mathbb{P}_w}(w_1) \oplus \mathcal{O}_{\mathbb{P}_w}(w_2) \longrightarrow T\mathbb{P}_w \longrightarrow 0, \tag{7}$$

where $\underline{\mathbb{C}}$ is the trivial orbifold bundle and $T\mathbb{P}_w$ is the orbifold tangent bundle of $\mathbb{P}_w$. Also, see [9], the Chern-Weil theory of Chern classes holds as well in $\mathbb{P}_w$ as in projective spaces and, denoting by $\zeta = c_1(\mathcal{O}_{\mathbb{P}_w}(1))$ we have, from (7),

$$c(T\mathbb{P}_w) = (1 + w_0\zeta)(1 + w_1\zeta)(1 + w_2\zeta) \tag{8}$$

and hence

$$c_i(T\mathbb{P}_w) = \sigma_i(w_0, w_1, w_2)\,\zeta^i \tag{9}$$

where σ_i is the i-th elementary symmetric function.

Now, let X be a quasi-homogeneous vector field of type (w_0, w_1, w_2) and degree d in $\mathbb{C}^3$; that is, writing $X = \sum_{i=0}^{2} P_i(z)\frac{\partial}{\partial z_i}$ we have that $P_i(\lambda^{w_0}z_0, \lambda^{w_1}z_1, \lambda^{w_2}z_2) = \lambda^{d+w_i-1}P_i(z_0, z_1, z_2)$. These descend well to $\mathbb{P}_w$. In fact, tensorize (1) by $\mathcal{O}_{\mathbb{P}_w}(d-1)$ to get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_w}(d-1) \longrightarrow \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}_w}(d + w_i - 1) \longrightarrow T\mathbb{P}_w \otimes \mathcal{O}_{\mathbb{P}_w}(d-1) \longrightarrow 0. \tag{10}$$

It follows that a quasi-homogeneous vector field X induces a foliation $\mathcal{F}$ of $\mathbb{P}_w$ and that $g\,R_w + X$ define the same foliation as X, where R_w is the adapted radial vector field $R_w = w_0 z_0 \frac{\partial}{\partial z_0} + w_0 z_1 \frac{\partial}{\partial z_1} + w_2 z_2 \frac{\partial}{\partial z_2}$, with g a quasi-homogeneous polynomial of type (w_0, w_1, w_2) and degree $d-1$.

Dually, noting that $|w| = w_0 + w_1 + w_2$, we have the exact sequence

$$0 \to \Omega^1_{\mathbb{P}_w} \otimes \mathcal{O}_{\mathbb{P}_w}(d+|w|-1) \to \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}_w}(d+|w|-w_i-1) \to \mathcal{O}_{\mathbb{P}_w}(d+|w|-1) \to 0. \tag{11}$$

Hence, a foliation $\mathcal{F}$ of $\mathbb{P}_w$ is also induced by a 1-form $\eta = A_0\,dz_0 + A_1\,dz_1 + A_2\,dz_2$, with A_i a quasi-homogeneous polynomial of type (w_0, w_1, w_2), degree $d+|w|-w_i-1$ and $\imath_{R_w}\eta = w_0 z_0 A_0 + w_1 z_1 A_1 + w_2 z_2 A_2 \equiv 0$.

We shall assume that

$$Sing(\mathcal{F}) \cap Sing(\mathbb{P}_w) = \emptyset. \tag{12}$$

The reason for this assumption is the following argument, given in [4]: around an orbifold point $p_i \in Sing(\mathbb{P}_w)$ the surface is of the type B^2/Γ_i, where B^2 is the unit ball in $\mathbb{C}^2$ and Γ_i is a w_i-cyclic group. Suppose we have a foliation $\mathcal{F}$ on $\mathbb{P}_w \setminus Sing(\mathbb{P}_w)$ and consider the neighbourhood $U = B^2/\Gamma_i$ of p_i. Then we can lift $\mathcal{F}_{|U\setminus p_i}$ to $B^2\setminus\{0\}$, and on this covering the foliation can be defined by a holomorphic vector field which admits a holomorphic extension to 0. $\mathcal{F}$ is nonsingular at p_i if this extension is nonvanishing at 0. In this case the foliation is, after a Γ_i-equivariant holomorphic change of coordinates, the quotient of, say, the vertical foliation on B^2. Hence, the leaves are defined as in the smooth situation, by gluing the local leaves of $\mathcal{F}$ at regular points of the foliation and are therefore orbifolds.

We proceed now to define the "degree", $\deg\mathcal{F}$, of such a foliation. Recall that, in the usual projective situation, $\deg\mathcal{F}$ is the degree of the variety of tangencies of $\mathcal{F}$ with a generic hyperplane.

An analogous geometric interpretation holds in the weighted situation, and we similarly have the corresponding canonical $\mathbb{Q}$-bundles $K_{\mathbb{P}_w}$, $K_{\mathcal{F}}$ and the $\mathbb{Q}$-bundles $T_{\mathcal{F}}$, $N_{\mathcal{F}}$, $N^*_{\mathcal{F}}$, all lying in $\mathsf{Pic}(\mathbb{P}_w) \otimes \mathbb{Q}$. The adjunction formula

$$K_{\mathbb{P}_w} = K_{\mathcal{F}} \otimes N^*_{\mathcal{F}} \tag{13}$$

still holds, and we point out that $K_{\mathbb{P}_w} = \mathcal{O}_{\mathbb{P}_w}(-|w|)$, $K_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}_w}(d-1)$ and $N_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}_w}(d+|w|-1)$.

Let C be a compact connected curve (possibly singular) whose irreducible components are not $\mathcal{F}$-invariant. Then, for $p \in C$, the index $tang(\mathcal{F}, C, p)$ is defined as in [3] and, writing $tang(\mathcal{F}, C) = \sum_{p\in C} tang(\mathcal{F}, C, p)$, we have that

$$tang(\mathcal{F}, C) = K_{\mathcal{F}}.C + C.C \geq 0. \tag{14}$$

We define the degree of $\mathcal{F}$ just as in the usual projective situation, that is,

$$\mathsf{deg}(\mathcal{F}) := tang(\mathcal{F}, H), \tag{15}$$

where H is a curve defined by a generic irreducible quasi-homogeneous polynomial of degree $w_0w_1w_2$.

After the work of I. Satake (see [9] and [12]), Poincaré's duality holds, so that H can be seen as the Poincaré dual of $c_1(\mathcal{O}_{\mathbb{P}_w}(1))$.

Before calculating (15) we recall that

$$\int_{\mathbb{P}_w}^{orb} c_1^2(\mathcal{O}_{\mathbb{P}_w}(1)) = \frac{1}{w_0w_1w_2}. \tag{16}$$

Hence, (15) reads

(17)

$$\begin{aligned}\mathsf{deg}(\mathcal{F}) = K_{\mathcal{F}}.H + H.H &= \int_H^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(d-1)) + \int_H^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(1)) \\ &= \int_{\mathbb{P}_w}^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(d-1)) \wedge c_1(\mathcal{O}_{\mathbb{P}_w}(1)) + \int_{\mathbb{P}_w}^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(1)) \wedge c_1(\mathcal{O}_{\mathbb{P}_w}(1)) \\ &= \frac{d-1}{w_0w_1w_2} + \frac{1}{w_0w_1w_2} = \frac{d}{w_0w_1w_2}.\end{aligned}$$

Now suppose C is a quasi-smooth curve in $\mathbb{P}_w$, that is, is defined by a quasi-homogeneous polynomial $P(z_0, z_1, z_2)$, of degree d^o, whose only singularity is at $0 \in \mathbb{C}^3$. In particular, C is smooth at the orbifold points it goes through. In this case we have Kollár-Mori's adjunction formula since C is a Cartier divisor in $\mathbb{P}_w$ (see [7], Proposition 5.73) and hence

$$K_C = K_{\mathbb{P}_w|C} \otimes N_C. \tag{18}$$

With this at hand we have, using Poincaré's duality,

$$\mathsf{deg}(C) = \int_C^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(1))^{\dim C} = \int_{\mathbb{P}_w}^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(d^o)) \wedge c_1(\mathcal{O}_{\mathbb{P}_w}(1)) = \frac{d^o}{w_0w_1w_2}. \tag{19}$$

3. Proof of the theorem

Suppose C is quasi-smooth, avoids the singularities of $\mathbb{P}_w$ and is $\mathcal{F}$-invariant.

The sum of the Camacho-Sad indices, $CS(\mathcal{F}, C)$, over $C \cap Sing(\mathcal{F})$ satisfies (see [3])

$$CS(\mathcal{F}, C) = \sum_{p \in C \cap Sing(\mathcal{F})} CS(\mathcal{F}, C, p) = C.C, \tag{20}$$

and, since the adjunction formula (18) holds, we have

$$C.C = \frac{{d^o}^2}{w_0 w_1 w_2} > 0 \tag{21}$$

so that $C \cap Sing(\mathcal{F}) \neq \emptyset$. On the other hand, by (8) and (18),

$$\begin{aligned} &\int_C^{orb} c_1\big(TC \otimes \mathcal{O}_{\mathbb{P}_w}(d-1)\big) \\ &= \frac{d^o\left(w_0 + w_1 + w_2 - d^o\right)}{w_0 w_1 w_2} + \frac{(d-1)\, d^o}{w_0 w_1 w_2} \\ &= d^o\ \frac{w_0 + w_1 + w_2 - d^o - 1 + d}{w_0 w_1 w_2}. \end{aligned} \tag{22}$$

Now, $\mathcal{F}_{|C}$ induces a nonzero holomorphic section of $TC \otimes \mathcal{O}_{\mathbb{P}_w}(d-1)$, and the number in (22) is the degree of this line $\mathbb{Q}$-bundle. Since $C \cap Sing(\mathcal{F})$ is nonempty and finite, this degree is positive and it follows that $\mathsf{deg}(C) \leq \mathsf{deg}(\mathcal{F}) + \dfrac{|w| - 2}{w_0 w_1 w_2}$.

□

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