

## A NOTE ON POINCARÉ'S PROBLEM FOR QUASI-HOMOGENEOUS FOLIATIONS

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ABSTRACT. We consider the question of bounding the degree of curves which are invariant by a holomorphic foliation of a given degree on a well-formed weighted projective plane.

### 1. INTRODUCTION AND STATEMENT OF RESULT

In this paper we consider the question of bounding the degree of curves which are invariant by a holomorphic foliation of a given degree on a well-formed weighted projective plane. This question had been considered by several authors ever since H. Poincaré raised it in a more specific setting, that of algebraic integration of polynomial differential equations in  $\mathbb{C}^2$ ; see [1], [2], [5], [6], [10], [11] and [13].

We recall that, in the usual complex projective situation, the presence of dicritical singularities makes the question meaningless, as is shown by simple examples, such as Example 3 below and, in a much more involved and elucidating context, by the examples given by A. Lins Neto in [8].

However, denoting by  $d^\circ$  and  $d$  the degrees of the curve and of the foliation, respectively, in case the invariant curve is nonsingular we get the bound  $d^\circ \leq d + 1$ , which holds also in higher dimensions [13] regardless of the nature of the singularities of the foliation along the invariant variety. Also, as shown in [1], for curves with only nodal singularities, we get the same estimate in case it is irreducible, and  $d^\circ \leq d + 2$  in the reducible case.

We consider this question in  $\mathbb{P}_w := \mathbb{P}(w_0, w_1, w_2)$ ,  $w_i \in \mathbb{N}$  two-by-two coprimes, endowed with a singular holomorphic foliation which admits an invariant quasi-smooth curve. Let  $Sing(\mathbb{P}_w)$  and  $Sing(\mathcal{F})$  denote the singular sets of  $\mathbb{P}_w$  and of  $\mathcal{F}$ , respectively. The result is as follows:

**Theorem.** *Let  $\mathcal{F}$  be a singular holomorphic foliation on  $\mathbb{P}_w$ , of degree  $deg(\mathcal{F})$ , with  $Sing(\mathcal{F}) \cap Sing(\mathbb{P}_w) = \emptyset$  and  $C$  a quasi-smooth curve of degree  $deg(C)$  which is invariant by  $\mathcal{F}$ . Then,*

$$(1) \quad deg(C) \leq deg(\mathcal{F}) + \frac{w_0 + w_1 + w_2 - 2}{w_0 w_1 w_2}.$$

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This bound cannot be improved, as is shown by

**Example 1.** Let  $f(x, y, z) = x^{mk} + y^{mk} - z^k$  and  $g(x, y, z) = ax^m + by^m + cz$ ,  $m, k \in \mathbb{N}$ . These are quasi-homogeneous polynomials of type  $(1, 1, m)$  and degrees  $km$  and  $m$ , both defining quasi-smooth curves of degrees  $k$  and  $1$ , respectively, which avoid the singularity of  $\mathbb{P}(1, 1, m)$ . The 1-form  $\eta = k f dg - g df$  defines a singular foliation  $\mathcal{F}$  on  $\mathbb{P}(1, 1, m)$  of degree  $\text{deg}(\mathcal{F}) = k - 1/m$ . The orbifold  $C = (f = 0)$  is  $\mathcal{F}$ -invariant and

$$(2) \quad \text{deg}(C) = k \leq k - \frac{1}{m} + 1 = \text{deg}(\mathcal{F}) + 1.$$

**Example 2.** Let  $f(x, y, z) = x^{m+1} + y^{m+1} - (x+y)z$  and  $g(x, y, z) = ax^m + by^m + cz$ ,  $2 \leq m \in \mathbb{N}$ . These are quasi-homogeneous polynomials of type  $(1, 1, m)$  and degrees  $m + 1$  and  $m$ , both defining quasi-smooth curves of degrees  $1 + 1/m$  and  $1$ . Note that the curve  $C = (f = 0)$  passes through the singular point  $[0 : 0 : 1]_{(1,1,m)}$  of  $\mathbb{P}(1, 1, m)$  and is smooth there. The 1-form  $\eta = (1 + 1/m) f dg - g df$  defines a singular foliation  $\mathcal{F}$  on  $\mathbb{P}(1, 1, m)$ , of degree  $\text{deg}(\mathcal{F}) = 1$ , which is nonsingular at  $[0 : 0 : 1]_{(1,1,m)}$ .  $C$  is  $\mathcal{F}$ -invariant and

$$(3) \quad \text{deg}(C) = 1 + 1/m \leq 1 + 1 = \text{deg}(\mathcal{F}) + 1.$$

**Example 3.** Let  $k$  be a positive integer and

$$(4) \quad \eta = \frac{yz}{k} dx + \left(m - \frac{1}{k}\right) xz dy - xy dz.$$

$\eta$  induces a singular foliation  $\mathcal{F}$  on  $\mathbb{P}(1, 1, m)$  of degree  $\text{deg}(\mathcal{F}) = 1/m$ . The curve  $C = \{z^k - xy^{km-1} = 0\}$  has degree  $k$  and is  $\mathcal{F}$ -invariant. Since  $k$  is arbitrary, the theorem does not hold. Here we have  $[0 : 0 : 1]_{(1,1,m)}$  as a singular point of the foliation and also,  $[1 : 0 : 0]_{(1,1,m)}$  is a singularity of  $C$  and a dicritical singularity of  $\mathcal{F}$ .

## 2. FOLIATIONS ON WEIGHTED PROJECTIVE PLANES

We start by recalling weighted projective planes. Let  $w := w_0, w_1, w_2$  be positive integers which are two-by-two coprimes.  $\mathbb{C}^*$  acts on  $\mathbb{C}^3 \setminus \{0\}$  by  $\lambda.(z_0, z_1, z_2) = (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2)$ . The quotient space by this action is the weighted projective plane of type  $w$ ,  $\mathbb{P}(w_0, w_1, w_2) := \mathbb{P}_w$ . In case  $w_i > 1$  for some  $i$ ,  $\mathbb{P}(w)$  is a compact algebraic surface with cyclic quotient singularities. Following [9], we have a natural orbifold map

$$(5) \quad \begin{aligned} f_w : \mathbb{P}^2 &\longrightarrow \mathbb{P}_w \\ [z_0 : z_1 : z_2] &\longmapsto [z_0^{w_0} : z_1^{w_1} : z_2^{w_2}]_w, \end{aligned}$$

which allows us to show that there exists a unique (up to isomorphism) rank 1 complex  $\mathbb{Q}$ -bundle  $\mathcal{O}_{\mathbb{P}_w}(1)$  over  $\mathbb{P}_w$ , such that

$$(6) \quad f_w^* \mathcal{O}_{\mathbb{P}_w}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1).$$

The Euler sequence carries over to the weighted case, and we have the exact sequence of  $\mathbb{Q}$ -bundles over  $\mathbb{P}_w$ :

$$(7) \quad 0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_{\mathbb{P}_w}(w_0) \oplus \mathcal{O}_{\mathbb{P}_w}(w_1) \oplus \mathcal{O}_{\mathbb{P}_w}(w_2) \longrightarrow T\mathbb{P}_w \longrightarrow 0,$$

where  $\underline{\mathbb{C}}$  is the trivial orbifold bundle and  $T\mathbb{P}_w$  is the orbifold tangent bundle of  $\mathbb{P}_w$ . Also, see [9], the Chern-Weil theory of Chern classes holds as well in  $\mathbb{P}_w$  as in projective spaces and, denoting by  $\zeta = c_1(\mathcal{O}_{\mathbb{P}_w}(1))$  we have, from (7),

$$(8) \quad c(T\mathbb{P}_w) = (1 + w_0\zeta)(1 + w_1\zeta)(1 + w_2\zeta)$$

and hence

$$(9) \quad c_i(T\mathbb{P}_w) = \sigma_i(w_0, w_1, w_2) \zeta^i$$

where  $\sigma_i$  is the  $i$ -th elementary symmetric function.

Now, let  $X$  be a quasi-homogeneous vector field of type  $(w_0, w_1, w_2)$  and degree  $d$  in  $\mathbb{C}^3$ ; that is, writing  $X = \sum_{i=0}^2 P_i(z) \frac{\partial}{\partial z_i}$  we have that  $P_i(\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2) = \lambda^{d+w_i-1} P_i(z_0, z_1, z_2)$ . These descend well to  $\mathbb{P}_w$ . In fact, tensorize (1) by  $\mathcal{O}_{\mathbb{P}_w}(d-1)$  to get

$$(10) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}_w}(d-1) \longrightarrow \bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}_w}(d+w_i-1) \longrightarrow T\mathbb{P}_w \otimes \mathcal{O}_{\mathbb{P}_w}(d-1) \longrightarrow 0.$$

It follows that a quasi-homogeneous vector field  $X$  induces a foliation  $\mathcal{F}$  of  $\mathbb{P}_w$  and that  $gR_w + X$  define the same foliation as  $X$ , where  $R_w$  is the adapted radial vector field  $R_w = w_0 z_0 \frac{\partial}{\partial z_0} + w_1 z_1 \frac{\partial}{\partial z_1} + w_2 z_2 \frac{\partial}{\partial z_2}$ , with  $g$  a quasi-homogeneous polynomial of type  $(w_0, w_1, w_2)$  and degree  $d-1$ .

Dually, noting that  $|w| = w_0 + w_1 + w_2$ , we have the exact sequence

$$(11) \quad 0 \rightarrow \Omega_{\mathbb{P}_w}^1 \otimes \mathcal{O}_{\mathbb{P}_w}(d+|w|-1) \rightarrow \bigoplus_{i=0}^2 \mathcal{O}_{\mathbb{P}_w}(d+|w|-w_i-1) \rightarrow \mathcal{O}_{\mathbb{P}_w}(d+|w|-1) \rightarrow 0.$$

Hence, a foliation  $\mathcal{F}$  of  $\mathbb{P}_w$  is also induced by a 1-form  $\eta = A_0 dz_0 + A_1 dz_1 + A_2 dz_2$ , with  $A_i$  a quasi-homogeneous polynomial of type  $(w_0, w_1, w_2)$ , degree  $d+|w|-w_i-1$  and  $\iota_{R_w} \eta = w_0 z_0 A_0 + w_1 z_1 A_1 + w_2 z_2 A_2 \equiv 0$ .

We shall assume that

$$(12) \quad \text{Sing}(\mathcal{F}) \cap \text{Sing}(\mathbb{P}_w) = \emptyset.$$

The reason for this assumption is the following argument, given in [4]: around an orbifold point  $p_i \in \text{Sing}(\mathbb{P}_w)$  the surface is of the type  $B^2/\Gamma_i$ , where  $B^2$  is the unit ball in  $\mathbb{C}^2$  and  $\Gamma_i$  is a  $w_i$ -cyclic group. Suppose we have a foliation  $\mathcal{F}$  on  $\mathbb{P}_w \setminus \text{Sing}(\mathbb{P}_w)$  and consider the neighbourhood  $U = B^2/\Gamma_i$  of  $p_i$ . Then we can lift  $\mathcal{F}|_{U \setminus p_i}$  to  $B^2 \setminus \{0\}$ , and on this covering the foliation can be defined by a holomorphic vector field which admits a holomorphic extension to 0.  $\mathcal{F}$  is nonsingular at  $p_i$  if this extension is nonvanishing at 0. In this case the foliation is, after a  $\Gamma_i$ -equivariant holomorphic change of coordinates, the quotient of, say, the vertical foliation on  $B^2$ . Hence, the leaves are defined as in the smooth situation, by gluing the local leaves of  $\mathcal{F}$  at regular points of the foliation and are therefore orbifolds.

We proceed now to define the “degree”,  $\text{deg } \mathcal{F}$ , of such a foliation. Recall that, in the usual projective situation,  $\text{deg } \mathcal{F}$  is the degree of the variety of tangencies of  $\mathcal{F}$  with a generic hyperplane.

An analogous geometric interpretation holds in the weighted situation, and we similarly have the corresponding canonical  $\mathbb{Q}$ -bundles  $K_{\mathbb{P}_w}, K_{\mathcal{F}}$  and the  $\mathbb{Q}$ -bundles  $T_{\mathcal{F}}, N_{\mathcal{F}}, N_{\mathcal{F}}^*$ , all lying in  $\text{Pic}(\mathbb{P}_w) \otimes \mathbb{Q}$ . The adjunction formula

$$(13) \quad K_{\mathbb{P}_w} = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^*$$

still holds, and we point out that  $K_{\mathbb{P}_w} = \mathcal{O}_{\mathbb{P}_w}(-|w|)$ ,  $K_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}_w}(d - 1)$  and  $N_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}_w}(d + |w| - 1)$ .

Let  $C$  be a compact connected curve (possibly singular) whose irreducible components are not  $\mathcal{F}$ -invariant. Then, for  $p \in C$ , the index  $\text{tang}(\mathcal{F}, C, p)$  is defined as in [3] and, writing  $\text{tang}(\mathcal{F}, C) = \sum_{p \in C} \text{tang}(\mathcal{F}, C, p)$ , we have that

$$(14) \quad \text{tang}(\mathcal{F}, C) = K_{\mathcal{F}}.C + C.C \geq 0.$$

We define the degree of  $\mathcal{F}$  just as in the usual projective situation, that is,

$$(15) \quad \text{deg}(\mathcal{F}) := \text{tang}(\mathcal{F}, H),$$

where  $H$  is a curve defined by a generic irreducible quasi-homogeneous polynomial of degree  $w_0 w_1 w_2$ .

After the work of I. Satake (see [9] and [12]), Poincaré’s duality holds, so that  $H$  can be seen as the Poincaré dual of  $c_1(\mathcal{O}_{\mathbb{P}_w}(1))$ .

Before calculating (15) we recall that

$$(16) \quad \int_{\mathbb{P}_w}^{orb} c_1^2(\mathcal{O}_{\mathbb{P}_w}(1)) = \frac{1}{w_0 w_1 w_2}.$$

Hence, (15) reads

$$(17) \quad \begin{aligned} \text{deg}(\mathcal{F}) &= K_{\mathcal{F}}.H + H.H = \int_H^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(d - 1)) + \int_H^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(1)) \\ &= \int_{\mathbb{P}_w}^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(d - 1)) \wedge c_1(\mathcal{O}_{\mathbb{P}_w}(1)) + \int_{\mathbb{P}_w}^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(1)) \wedge c_1(\mathcal{O}_{\mathbb{P}_w}(1)) \\ &= \frac{d - 1}{w_0 w_1 w_2} + \frac{1}{w_0 w_1 w_2} = \frac{d}{w_0 w_1 w_2}. \end{aligned}$$

Now suppose  $C$  is a quasi-smooth curve in  $\mathbb{P}_w$ , that is, is defined by a quasi-homogeneous polynomial  $P(z_0, z_1, z_2)$ , of degree  $d^o$ , whose only singularity is at  $0 \in \mathbb{C}^3$ . In particular,  $C$  is smooth at the orbifold points it goes through. In this case we have Kollár-Mori’s adjunction formula since  $C$  is a Cartier divisor in  $\mathbb{P}_w$  (see [7], Proposition 5.73) and hence

$$(18) \quad K_C = K_{\mathbb{P}_w|_C} \otimes N_C.$$

With this at hand we have, using Poincaré’s duality,

$$(19) \quad \text{deg}(C) = \int_C^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(1))^{\dim C} = \int_{\mathbb{P}_w}^{orb} c_1(\mathcal{O}_{\mathbb{P}_w}(d^o)) \wedge c_1(\mathcal{O}_{\mathbb{P}_w}(1)) = \frac{d^o}{w_0 w_1 w_2}.$$

## 3. PROOF OF THE THEOREM

Suppose  $C$  is quasi-smooth, avoids the singularities of  $\mathbb{P}_w$  and is  $\mathcal{F}$ -invariant.

The sum of the Camacho-Sad indices,  $CS(\mathcal{F}, C)$ , over  $C \cap \text{Sing}(\mathcal{F})$  satisfies (see [3])

$$(20) \quad CS(\mathcal{F}, C) = \sum_{p \in C \cap \text{Sing}(\mathcal{F})} CS(\mathcal{F}, C, p) = C.C,$$

and, since the adjunction formula (18) holds, we have

$$(21) \quad C.C = \frac{d^{o^2}}{w_0 w_1 w_2} > 0$$

so that  $C \cap \text{Sing}(\mathcal{F}) \neq \emptyset$ . On the other hand, by (8) and (18),

$$(22) \quad \begin{aligned} & \int_C^{orb} c_1(TC \otimes \mathcal{O}_{\mathbb{P}_w}(d-1)) \\ &= \frac{d^o (w_0 + w_1 + w_2 - d^o)}{w_0 w_1 w_2} + \frac{(d-1) d^o}{w_0 w_1 w_2} \\ &= d^o \frac{w_0 + w_1 + w_2 - d^o - 1 + d}{w_0 w_1 w_2}. \end{aligned}$$

Now,  $\mathcal{F}|_C$  induces a nonzero holomorphic section of  $TC \otimes \mathcal{O}_{\mathbb{P}_w}(d-1)$ , and the number in (22) is the degree of this line  $\mathbb{Q}$ -bundle. Since  $C \cap \text{Sing}(\mathcal{F})$  is nonempty and finite, this degree is positive and it follows that  $\deg(C) \leq \deg(\mathcal{F}) + \frac{|w| - 2}{w_0 w_1 w_2}$ .

□

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