PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 9, September 2012, Pages 3151–3160
S 0002-9939(2012)11260-2
Article electronically published on January 6, 2012

# LINEAR ORTHOGONALITY PRESERVERS OF HILBERT $C^*$-MODULES OVER $C^*$-ALGEBRAS WITH REAL RANK ZERO

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(Communicated by Marius Junge)

Abstract. Let $A$ be a $C^*$-algebra with real rank zero. Let $E$ and $F$ be Hilbert $A$-modules with $E$ being full. Suppose that $\theta : E \to F$ is a linear map preserving orthogonality, i.e.,

$$\langle \theta(x), \theta(y) \rangle \;=\; 0 \quad \text{whenever} \quad \langle x, y \rangle \;=\; 0.$$

We show in this article that if $\theta$ is an $A$-module map (not assumed to be bounded), then there exists a central positive multiplier $u \in M(A)$ such that

$$\langle \theta(x), \theta(y) \rangle \;=\; u\langle x, y \rangle \qquad (x, y \in E).$$

In the case when $A$ is a standard $C^*$-algebra, when $A$ is a real rank zero properly infinite unital $C^*$-algebra, or when $A$ is a $W^*$-algebra, we also get the same conclusion with the assumption of $\theta$ being an $A$-module map weakened to being a local map.

## 1. Introduction and notation

It is common knowledge that, together with linearity, the inner product and the norm structures of a Hilbert space $H$ determine each other. It might be a bit less well known that the orthogonality structure also suffices to determine the inner product up to a scalar. This fact follows from the following easy observation: for any $x, y \in H$, $\|x\| = \|y\|$ if and only if $x + \lambda y$ is orthogonal to $x - \lambda y$ for some scalar $\lambda$ with $|\lambda| = 1$ (see also [3, 6]).

It is natural and interesting to ask whether the linear structure and orthogonality structure of a (complex) Hilbert $C^*$-module determines its $C^*$-algebra-valued inner product. More precisely, let $A$ be a (complex) $C^*$-algebra, and $\theta : E \to F$ be a $\mathbb{C}$-linear map between Hilbert $A$-modules that preserves orthogonality (i.e. preserves zero $A$-valued inner products). We want to study to what extent $\theta$ will respect the $A$-valued inner products. When the underlying $C^*$-algebra is $\mathbb{C}$, it reduces to the case of Hilbert spaces.

We first note that without any further assumption on $\theta$, the above question might have a negative answer.

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Received by the editors May 25, 2010 and, in revised form, March 22, 2011.
2010 *Mathematics Subject Classification.* Primary 46L08, 46H40.
*Key words and phrases.* Hilbert $C^*$-modules, orthogonality preservers, module maps, local maps, real rank zero $C^*$-algebras.
The authors are supported by The Chinese University of Hong Kong Direct Grant (2060389), National Natural Science Foundation of China (10771106), and Taiwan NSC grant (NSC96-2115-M-110-004-MY3).

**Example 1.1.** Let $H$ be an infinite dimensional (complex) Hilbert space and $A = \mathcal{K}(H)$ be the $C^*$-algebra of all compact operators on $H$. Suppose that $\bar{H}$ is a vector space that is conjugate-linear isomorphic to $H$. When equipped with the operations: $\langle \overline{\eta_1}, \overline{\eta_2} \rangle := \eta_1 \otimes \eta_2$ and $\overline{\eta_1} T := \overline{T^* \eta_1}$ ($\overline{\eta_1}, \overline{\eta_2} \in \bar{H}; T \in A$), we see that $\bar{H}$ is a Hilbert $A$-module. Suppose that $\theta$ is any unbounded bijective $\mathbb{C}$-linear map from $\bar{H}$ onto $\bar{H}$. Since $\langle x, y \rangle = 0$ if and only if $x = 0$ or $y = 0$, we see that both $\theta$ and $\theta^{-1}$ preserve orthogonality.

In fact, assuming norm continuity, Schweizer obtains his pioneering work in his thesis (see [20, 9.6]) on a characterization of *bounded* orthogonality-preserving $\mathbb{C}$-linear maps. His result ensures that all *bounded* orthogonality-preserving $\mathbb{C}$-linear maps from a full Hilbert $C$-module $X$ into a full Hilbert $D$-module $Y$ are exactly those maps $T$ which induce a sort of "correspondence" between the $C^*$-algebras $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, where $\mathcal{K}(X)$ (respectively, $\mathcal{K}(Y)$) is generated by "rank one operators" $\theta_{\zeta,\eta}$ defined as $\theta_{\zeta,\eta}(\xi) := \zeta \langle \eta, \xi \rangle$ for $\zeta, \eta, \xi \in X$ (respectively, $Y$). More precisely, there is a "local morphism" $\pi : \mathcal{K}(X) \to \mathcal{K}(Y)$ such that

$$T(ax) = \pi(a)T(x) \quad (a \in \mathcal{K}(X)),$$

or equivalently,

$$\theta_{T\zeta,T\zeta} \leq \|T\|^2 \, \pi(\theta_{\zeta,\zeta}) \quad (\zeta \in X).$$

As a result, in the case $C = D$, we can see that even a *bounded* $\mathbb{C}$-linear orthogonality preserver does not necessarily preserve the $C^*$-algebra inner products. Hence, we still need more assumptions to ensure our question to have a positive answer.

As we are dealing with Hilbert $A$-modules, a natural additional assumption is that $\theta$ is an $A$-module map, i.e., $\theta(xa) = \theta(x)a$ ($x \in E, a \in A$). In [9], Ilišević and Turnšek showed that if $A$ is a standard $C^*$-algebra, then for every orthogonality-preserving $A$-module map $\theta : E \to F$, there is a scalar $\lambda \geq 0$ such that

$$\langle \theta(x), \theta(y) \rangle = \lambda \langle x, y \rangle \quad (x, y \in E).$$

In particular, all such $\theta$ are scalar multiples of isometries.

In [13], under a weaker assumption on $\theta$, namely $\theta$ being "local", we get a similar conclusion in the case when $A$ is a commutative $C^*$-algebra (in fact, the main difficulties in [13] come from the fact that $\theta$ is not assumed to be an $A$-module map). Recall that a $\mathbb{C}$-linear map $\theta : E \to F$ is *local* if

$$\theta(x)a = 0 \quad \text{whenever} \quad xa = 0 \qquad (x \in E; a \in A).$$

Readers should find the idea of local linear maps familiar. For example, local linear maps on the space of smooth functions defined on a manifold modeled on $\mathbb{R}^n$ are exactly linear differential operators (see, e.g., [19, 16]). See [11, 2] for the vector-valued case, and [1] for the Banach $C^1[0,1]$-module setting. We also mention that there is a bimodule version of local maps as studied by Schweizer in [20, 21] (which is different from ours). Notice that every module map is local, but local linear maps, e.g., linear differential operators, might not be a module map. Nevertheless, it has been shown in [12, Proposition A.1] that every *bounded* local map between Hilbert $C^*$-modules is a module map.

The results in [9] and [13] lead to the following conjecture. We remark that the fullness assumption of $E$ in this conjecture is a necessity. Without this, the conclusion does not hold even in the case when $A$ is commutative (see [13, 3.6]). Here, a Hilbert $A$-module $E$ is said to be *full* if the linear span of $\{\langle x, y \rangle : x, y \in E\}$ is dense in $A$.

**Conjecture 1.2.** *Let $A$ be a $C^*$-algebra. Let $E$ and $F$ be Hilbert $A$-modules with $E$ being full. Assume $\theta : E \to F$ is a ($\mathbb{C}$-linear) local map preserving orthogonality; i.e. for any $x, y \in E$,*

$$\langle x, y\rangle \ = \ 0 \quad \textit{implies} \quad \langle \theta(x), \theta(y)\rangle \ = \ 0.$$

*Then there is a central positive element $u \in M(A)$ such that*

$$\langle \theta(x), \theta(y)\rangle \ = \ u\langle x, y\rangle \qquad (x, y \in E).$$

In this article, positive answers of this conjecture will be given in the following four cases:

(1) $A$ is a $C^*$-algebra of real rank zero and $\theta$ is an $A$-module map (Theorem 2.3).
(2) $A$ is a standard $C^*$-algebra (Corollary 3.2).
(3) $A$ is a properly infinite unital $C^*$-algebra of real rank zero (Corollary 3.3).
(4) $A$ is a $W^*$-algebra (Corollary 3.4).

We note that although the main result, Theorem 9.6, in [20] appears to be more general and more complete (as they deal with $\mathbb{C}$-linear orthogonality preservers), they do not give too much hint to the answer of our conjecture. For example, in the situation of Example 1.1, what one can get from [20, 9.6] is the tautology that a bounded orthogonality-preserving $\mathbb{C}$-linear map $\theta : \bar{H} \to \bar{H}$ is $\mathbb{C}$-linear. This does not give any useful insight into the study of orthogonality-preserving $\mathcal{K}(H)$-module maps in even such a simple case.

As a final remark for the introduction, we note that, unlike the situation in some of the other literature (e.g. [20, 7]), $\theta$ is *not* assumed to be bounded. It is because of the conceptual reason that $A$-linearity (or locality) and orthogonality structures should completely determine the $A$-inner products, as stated in the beginning of this section. However, we shall see that the boundedness of an $A$-linear orthogonality preserver will be an automatic consequence of our results.

## 2. Orthogonality-preserving $A$-module maps

First, we give some notation that will be used throughout this article. In the following, $A$ is a $C^*$-algebra, $E$ and $F$ are Hilbert $A$-modules, and $\Psi, \theta : E \to F$ are orthogonality-preserving $\mathbb{C}$-linear maps, which are not assumed to be bounded.

Let $a \in A_+$. We set $C^*(a)$ to be the $C^*$-subalgebra generated by $a$, and $\mathbf{c}(a)$ to be the *central cover* of $a$, i.e., the smallest central element in $A^{**}_+$ dominating $a$ (see, e.g., [18, 2.6.2]). If, in addition, $\alpha, \beta \in \mathbb{R}_+$, we put $e_a(\alpha, \beta]$ to be the spectral projections of $a$ in $A^{**}$ corresponding to the set $(\alpha, \beta] \cap \sigma(a)$.

We denote by $Z(A)$ the center and by $M(A)$ the space of all multipliers of $A$. On the other hand, $\mathrm{Proj}_1(A)$ is the set of all norm-one (i.e., nonzero) projections in $A$. For any *open projection* $p \in \mathrm{Proj}_1(A^{**})$ (i.e., there exists an increasing net $\{a_i\}$ in $A_+$ such that $a_i \uparrow p$ in the weak-*-topology), we denote by $\mathrm{her}(p) := pA^{**}p \cap A$ the hereditary $C^*$-subalgebra associated to $p$. See, e.g., [4] for more information about open projections.

We recall that $A$ has *real rank zero* if every selfadjoint element in $A$ can be approximated in norm by invertible selfadjoint elements. If $A$ has real rank zero, then the real linear span of $\mathrm{Proj}_1(A)$ is norm dense in $A_{sa}$ (see, e.g., [5]).

Let us start with the following easy lemma, part (a) of which might be well known, but we give an argument here for completeness.

**Lemma 2.1.** *(a) If $p \in \mathrm{Proj}_1(A^{**})$ and $b \in Z(pA^{**}p)_+$, then $\|\mathbf{c}(b)\| = \|b\|$, $\mathbf{c}(b)p = b$ and $\mathbf{c}(b)\mathbf{c}(p) = \mathbf{c}(b)$.*

*(b) Suppose that $A$ has real rank zero and $E$ is full. If $q \in A^{**} \setminus \{0\}$ is an open projection, there are $r \in \mathrm{Proj}_1(A)$ and $y \in Er$ such that $r = \langle y, y\rangle \leq q$.*

*Proof.* (a) Since $b \leq \|b\|1$, we see that $0 \leq b \leq \mathbf{c}(b) \leq \|b\|1$ and $\|\mathbf{c}(b)\| = \|b\|$. Clearly, $\mathbf{c}(b)p = p\mathbf{c}(b)p \geq pbp = b$. Conversely, as $Z(pA^{**}p) = Z(A^{**})p$ (see e.g. [10, 5.5.6]), there is $a \in Z(A^{**})_+$ with $b = ap$ (note that $b^{1/2} \in Z(A^{**})p$). Thus, we have $b = a^{1/2}pa^{1/2} \leq a^{1/2}\mathbf{c}(p)a^{1/2} = a\mathbf{c}(p)$. As $a\mathbf{c}(p)$ is central, $\mathbf{c}(b) \leq a\mathbf{c}(p)$ and $\mathbf{c}(b)p = p\mathbf{c}(b)p \leq ap = b$. The last equality follows from [18, 2.6.4] and the fact that $b\mathbf{c}(p) = b$.

(b) Note that $\mathrm{her}(q) \neq (0)$ and also has real rank zero (see e.g. [5]). Moreover, $E_0 := E \cdot \mathrm{her}(q)$ is a full (and, hence nonzero) Hilbert $\mathrm{her}(q)$-module. Pick any $x \in E_0$ such that $a := \langle x, x\rangle$ is a norm-one element in $\mathrm{her}(q)$. Let $t \in (0, 1/3)$ and $b \in \mathrm{her}(q)_+$ such that $\|a - b\| < t$ and $\sigma(b) = \{\lambda_1, ..., \lambda_n\}$ with $\lambda_1 \leq \cdots \leq \lambda_n = \|b\|$ (see e.g. [5]). Since $\|b\| > 2/3$, we can choose $s \in [t, \|b\|] \setminus \sigma(b)$. If we set $r := e_b(s, 2] \in \mathrm{Proj}_1(A)$, then $\|r - rar\| \leq \|r - rbr\| + \|b - a\| < 1$. If $c := r + \sum_{n=1}^{\infty}(r - rar)^n \in A_+$, then $(rar)c = c(rar) = r$ and so, $\langle xc^{1/2}, xc^{1/2}\rangle = c^{1/2}rarc^{1/2} = r$. Finally, $xc^{1/2} \in Er$ as $c^{1/2}$ is in the $C^*$-subalgebra $rAr + \mathbb{C}r$. □

**Proposition 2.2.** *Let $A$ be a unital $C^*$-algebra of real rank zero. Suppose that $\theta : E \to F$ is an $A$-module map preserving orthogonality, and there is an element $x_0 \in E$ such that $\langle x_0, x_0\rangle = 1$. Then one can find $u \in Z(A)_+$ satisfying*

$$\langle \theta(x), \theta(y)\rangle \; = \; u\langle x, y\rangle \quad (x, y \in E).$$

*Proof.* Let $u := \langle \theta(x_0), \theta(x_0)\rangle \in A_+$. For any symmetry $w \in A$, as $x_0 + x_0w$ and $x_0 - x_0w$ are orthogonal to each other, so are $\theta(x_0) + \theta(x_0)w$ and $\theta(x_0) - \theta(x_0)w$. Consequently, $u + wu - uw - wuw = 0$ and $u + uw - wu - wuw = 0$ (by taking the adjoint). This tells us that $u = wuw$, and so $u \in Z(A)_+$ (as $A$ is generated by projections, and thus also by symmetries). Pick any $z \in E$ with $\langle x_0, z\rangle = 0$. Then $z + x_0\langle z, z\rangle^{1/2}$ is also orthogonal to $z - x_0\langle z, z\rangle^{1/2}$. It follows from the orthogonality-preserving property that

$$\langle \theta(z), \theta(z)\rangle \; = \; \langle z, z\rangle^{1/2}\langle \theta(x_0), \theta(x_0)\rangle\langle z, z\rangle^{1/2} \; = \; u\langle z, z\rangle.$$

For any $y \in E$, the element $z = y - x_0\langle x_0, y\rangle$ is orthogonal to $x_0$. Hence,

$$\langle \theta(y), \theta(y)\rangle \;\; = \;\; \langle y, x_0\rangle\langle \theta(x_0), \theta(x_0)\rangle\langle x_0, y\rangle + \langle \theta(z), \theta(z)\rangle \; = \; u\langle y, y\rangle.$$

A polarization type argument implies that $\langle \theta(x), \theta(y)\rangle = u\langle x, y\rangle$ $(x, y \in E)$. □

**Theorem 2.3.** *Let $A$ be a $C^*$-algebra of real rank zero. Suppose that $E$ is full, and $\theta : E \to F$ is an orthogonality-preserving $A$-module map (not assumed to be bounded). There is $u \in Z(M(A))_+$ such that*

$$\langle \theta(x), \theta(y)\rangle \; = \; u\langle x, y\rangle \quad (x, y \in E).$$

*In particular, $\theta$ is automatically bounded.*

*Proof.* Set

$$P \; := \; \{(x, p) \in E \times \mathrm{Proj}_1(A) : \langle x, x\rangle = p\}.$$

Lemma 2.1(b) tells us that $P \neq \emptyset$. Suppose that $(x,p) \in P$. Then $xp = x$ since we have $\langle xp - x, xp - x\rangle = p\langle x,x\rangle p - \langle x,x\rangle p - p\langle x,x\rangle + \langle x,x\rangle = 0$. Also, $Ep$ is a full Hilbert $pAp$-module and the restriction of $\theta$ on $Ep$ is an orthogonality-preserving $pAp$-module map. Since $p$ is the identity of the $C^*$-algebra $pAp$ and $\theta(Ep) \subseteq Fp$, one can apply Proposition 2.2 to obtain $b_p \in Z(pAp)_+$ which satisfies

$$\langle \theta(x)p, \theta(y)p\rangle \;=\; b_p\langle xp, yp\rangle \qquad (x, y \in E).$$

By Lemma 2.1(a), we have

$$p\left(\langle \theta(x), \theta(y)\rangle - \mathbf{c}(b_p)\langle x, y\rangle\right)p \;=\; 0 \qquad (x, y \in E). \tag{2.1}$$

As the weak-*-closed linear span, $I$, of $\{\langle \theta(x), \theta(y)\rangle - \mathbf{c}(b_p)\langle x, y\rangle : x, y \in E\}$ is an ideal of $A^{**}$, there is a central projection $q_I \in A^{**}$ with $I = q_I A^{**}$. Since $pq_I = pq_I p = 0$, we have $\mathbf{c}(p) \le 1 - q_I$. Consequently,

$$\mathbf{c}(p)\langle \theta(x), \theta(y)\rangle \;=\; \mathbf{c}(p)\mathbf{c}(b_p)\langle x, y\rangle \qquad (x, y \in E). \tag{2.2}$$

Now, let $\mathcal{D} := \{D \subseteq P : \mathbf{c}(p)\mathbf{c}(q) = 0 \text{ whenever } (x,p), (y,q) \in D\}$. If we equip $\mathcal{D}$ with the usual inclusion, then Zorn's Lemma gives a maximal element $D_0 = \{(x_\gamma, p_\gamma)\}_{\gamma\in\Gamma} \in \mathcal{D}$. Set $q_0 := \bigvee_{\gamma\in\Gamma} \mathbf{c}(p_\gamma)$ in $\mathrm{Proj}_1(A^{**})$, which is a central element. Observe that $1 - q_0$ will not dominate a nontrivial open projection. Indeed, if $0 \neq q \le 1 - q_0$ is an open projection, then Lemma 2.1(b) produces an element $(y, r) \in P$ such that $r \le q$. Therefore, $D_0 \cup \{(y,r)\} \in \mathcal{D}$, which contradicts the maximality of $D_0$. We now claim that the $*$-homomorphism $\Phi : A \to q_0 A \subseteq A^{**}$ defined by $\Phi(a) = q_0 a$ is injective. Suppose on the contrary that there exists $a \in A_+$ with $\|a\| = 1$ and $\Phi(a) = 0$. Take any $\epsilon \in (0,1)$, and put $q_\epsilon$ to be the nonzero open projection $e_a(\epsilon, 1]$. As $a - \epsilon q_\epsilon \ge 0$, we have $q_\epsilon q_0 = q_0 q_\epsilon q_0 \le q_0 a q_0/\epsilon = 0$. So, $q_\epsilon \le 1 - q_0$, which implies the contradiction that $q_\epsilon = 0$.

As $x_\gamma \mathbf{c}(p_\gamma) = x_\gamma p_\gamma \mathbf{c}(p_\gamma) = x_\gamma$ $(\gamma \in \Gamma)$, we see that $x_\gamma$ and $x_{\gamma'}$ are orthogonal if $\gamma \neq \gamma'$. We now claim that the $\mathbf{c}(b_\gamma)$'s are uniformly bounded (where $b_\gamma \in Z(p_\gamma A p_\gamma)_+$ is the element associated with $(x_\gamma, p_\gamma) \in P$ that satisfies (2.2)). Suppose on the contrary that there are $\mathbf{c}(b_{\gamma_n})$ with $\|\mathbf{c}(b_{\gamma_n})\| = \|b_{\gamma_n}\| \ge n^3$ $(n \in \mathbb{N})$. Note that the orthogonal sum $x := \sum_n \frac{x_{\gamma_n}}{n}$ is convergent in norm in $E$. By the orthogonality-preserving property of $\theta$, Lemma 2.1(a) as well as equality (2.1), for any $m \in \mathbb{N}$,

$$\begin{aligned}\langle \theta(x), \theta(x)\rangle = \;& \langle \theta\left(x_{\gamma_m}/m\right), \theta\left(x_{\gamma_m}/m\right)\rangle + \langle \theta\left(x - x_{\gamma_m}/m\right), \theta\left(x - x_{\gamma_m}/m\right)\rangle \\ \ge \;& \langle \theta\left(x_{\gamma_m}/m\right), \theta\left(x_{\gamma_m}/m\right)\rangle \;=\; \frac{\mathbf{c}(b_{\gamma_m})\left\langle x_{\gamma_m}, x_{\gamma_m}\right\rangle}{m^2} \;=\; \frac{b_{\gamma_m}}{m^2}.\end{aligned}$$

As the norm of the last term goes to infinity as $n \to \infty$, we reach a contradiction.

Finally, let $d$ be the weak-*-limit in $A^{**}$ of finite sums of the uniformly bounded mutually orthogonal elements $\mathbf{c}(b_\gamma)$ (see Lemma 2.1(a)). By relation (2.2) and the fact that $q_0$ is the weak-*-limit of finite sums of $\mathbf{c}(p_\gamma)$'s, we have

$$dq_0\langle x, y\rangle \;=\; q_0\langle \theta(x), \theta(y)\rangle \;\in\; q_0 A \qquad (x, y \in E).$$

Since $E$ is full, we see that $d$ induces an element $m \in Z(M(q_0A))_+$ such that $mq_0\langle x, y\rangle = q_0\langle\theta(x), \theta(y)\rangle$ $(x, y \in E)$. Since $\Phi : A \to q_0A$ extends to a $*$-isomorphism $\tilde{\Phi} : M(A) \to M(q_0A)$, there is $u \in Z(M(A))_+$ such that $\tilde{\Phi}(u) = m$. This means that

$$\Phi(u\langle x, y\rangle - \langle\theta(x), \theta(y)\rangle) = 0 \qquad (x, y \in E),$$

which gives the required conclusion. □

*Remark* 2.4. Let $A$ be a general $C^*$-algebra. Suppose that there exist Hilbert $A^{**}$-modules $\tilde{E}$ and $\tilde{F}$ containing $E$ and $F$ respectively such that the Hilbert $A^{**}$-module structures extend the corresponding Hilbert $A$-module structures and $\theta$ extends to an orthogonality-preserving $A^{**}$-module map $\tilde{\theta} : \tilde{E} \to \tilde{F}$. Then one can use Theorem 2.3 to show that $\theta$ satisfies the conclusion of Conjecture 1.2 (since $A^{**}$ has real rank zero). In the situation when $\theta$ is a bounded orthogonality-preserving $A$-module map, we have tried $\tilde{E} = E^{**}$ and $\tilde{F} = F^{**}$ but encountered some difficulties in showing that $\theta^{**}$ is orthogonality preserving. It was claimed in [7] that, when $\theta$ is bounded, such $\tilde{E}$, $\tilde{F}$ and $\tilde{\theta}$ could be found. However, instead of manipulating the difficulties in the arguments in [7], we are working on a proof for the case of general $C^*$-algebras, *without* the boundedness assumption on $\theta$ (but $\theta$ is assumed to be an $A$-module map), using completely different ideas from those in this article and in [7], [9], and [13].

## 3. Orthogonality-preserving $\mathbb{C}$-linear local maps

In this section, we consider ($\mathbb{C}$-linear) local maps (see the Introduction) that preserve orthogonality. Let us first give the following useful observation.

**Lemma 3.1.** *Let $A_0$ be the $*$-algebra generated by all the idempotents in $A$. If $\Psi : E \to F$ is a local map, then $\Psi$ is an $A_0$-module map.*

*Proof.* Let $p \in A$ be an idempotent and $x \in E$. As $\Psi$ is local, one has $\Psi(x - xp)p = 0$. If $\{u_i\}$ is an approximate unit for $A$, then $(1-p)u_i \in A$ will strictly converge to $(1-p)$. Since $\Psi(xp)(1-p)u_i = 0$, we have $\langle y, \Psi(xp)\rangle(1-p) = \lim\ \langle y, \Psi(xp)\rangle(1-p)u_i = 0$ $(y \in F)$. This implies that $\Psi(xp) - \Psi(xp)p = \Psi(xp)(1-p) = 0$. Thus, $\Psi(x)p = \Psi(xp)$, and so $\Psi(xv) = \Psi(x)v$ for any $v \in A_0$. □

Note that if $A$ has real rank zero, then $A_0$ is dense in $A$. We remark however that $A_0$ can be $\{0\}$ (e.g. if $A = C_0(0, 1)$).

Recall that $A$ is a *standard $C^*$-algebra* on a Hilbert space $H$ if $\mathcal{K}(H) \subseteq A \subseteq \mathcal{L}(H)$. In this case, $A_0$ contains a "big enough" ideal $\mathcal{F}(H)$ of $A$, in the sense that $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$ is an essential ideal. As a consequence, we can use Lemma 3.1 and Theorem 2.3 to give a self-contained proof of the following slight extension of [9, 2.3] (note that the $A$-linearity is replaced by the local property).

**Corollary 3.2** (cf. [9, 2.3])**.** *Suppose that $A$ is a standard $C^*$-algebra on a Hilbert space $H$. If $\Psi : E \to F$ is local and orthogonality preserving, then there is $\lambda \in \mathbb{R}_+$ such that*

$$\langle\Psi(x), \Psi(y)\rangle \ = \ \lambda\langle x, y\rangle \qquad (x, y \in E).$$

*Proof.* Consider $E_0 := E \cdot \mathcal{K}(H)$ and $F_0 := F \cdot \mathcal{K}(H)$ (both of them being Hilbert $\mathcal{K}(H)$-modules). Let $\{v_\gamma\}_{\gamma\in\Gamma}$ be an approximate unit in $\mathcal{K}(H)$ consisting of finite rank positive operators. By Lemma 3.1, $\Psi(xv) = \Psi(x)v$ for every finite rank operator $v$ and every $x \in E$. On the other hand, for any $y \in E_0$, there exist $a \in \mathcal{K}(H)$ and $x \in E_0$ with $y = xa$ (by the Cohen factorization theorem), and so

$$\Psi(y)v_\gamma \ = \ \Psi(xav_\gamma) \ = \ \Psi(x)av_\gamma \quad (\gamma \in \Gamma),$$

which shows that $\|\Psi(y)v_\gamma - \Psi(x)a\| \to 0$ (along $\gamma$). Define $\Phi : E_0 \to F_0$ by setting $\Phi(y)$ to be the norm limit of $\Psi(y)v_\gamma$. As $\Phi(y) = \Psi(x)a$ as well, we see that $\Phi(y)$ does not depend on the choice of $\{v_\gamma\}_{\gamma\in\Gamma}$, nor on the decomposition $y = xa$. If $b \in \mathcal{K}(H)$, then

$$\Phi(yb) \ = \ \Phi(xab) \ = \ \Psi(x)ab \ = \ \Phi(y)b.$$

Moreover, if $x, y \in E_0$ with $\langle x, y\rangle = 0$, then $\langle \Psi(x), \Psi(y)\rangle = 0$, which implies that $\langle \Psi(x)v_\gamma, \Psi(y)v_{\gamma'}\rangle = 0$ $(\gamma, \gamma' \in \Gamma)$, and so, $\langle \Phi(x), \Phi(y)\rangle = 0$. On the other hand, since $\mathcal{K}(H)$ is simple, we see that either $E_0$ is a full $\mathcal{K}(H)$-module or $E_0 = \{0\}$. By Theorem 2.3, there exists $\lambda \in Z(M(\mathcal{K}(H)))_+ = \mathbb{R}_+$ such that for every $x, y \in E_0$, one has $\langle \Phi(x), \Phi(y)\rangle = \lambda\langle x, y\rangle$ (note that one can take any $\lambda$ if $E_0 = \{0\}$). Thus, for any $x, y \in E$ and $\gamma, \gamma' \in \Gamma$,

$$v_\gamma\langle \Psi(x), \Psi(y)\rangle v_{\gamma'} \ = \ \langle \Phi(xv_\gamma), \Phi(yv_{\gamma'})\rangle \ = \ \lambda\langle xv_\gamma, yv_{\gamma'}\rangle \ = \ \lambda v_\gamma\langle x, y\rangle v_{\gamma'}.$$

Consequently, if $b_{x,y} := \langle \Psi(x), \Psi(y)\rangle - \lambda\langle x, y\rangle \in A$, then $v_\gamma b_{x,y} v_{\gamma'} = 0$ $(\gamma, \gamma' \in \Gamma)$, which shows that $b_{x,y} = 0$ (as $v_\gamma \to 1$ in the strong operator topology). □

We recall that a unital $C^*$-algebra $A$ is said to be *properly infinite* if there exists $p \in \mathrm{Proj}_1(A)$ such that $p \sim 1 \sim 1 - p$.

**Corollary 3.3.** *Let $A$ be a properly infinite real rank zero unital $C^*$-algebra. If $E$ is full and $\Psi : E \to F$ is an orthogonality-preserving local map, then there is $u \in Z(A)_+$ such that*

$$\langle \Psi(x), \Psi(y)\rangle \ = \ u\langle x, y\rangle \qquad (x, y \in E).$$

*Proof.* Let $A_1$ be the linear span of projections in $A$. By [15, Corollary 2.2], we see that $A = A_1$. Therefore, Lemma 3.1 tells us that $\Psi$ is an $A$-module map. Now, we can apply Theorem 2.3 to obtain the conclusion. □

**Corollary 3.4.** *Let $A$ be a $W^*$-algebra. If $E$ is full and $\Psi : E \to F$ is an orthogonality-preserving local map, then there is $u \in Z(A)_+$ such that*

$$\langle \Psi(x), \Psi(y)\rangle \ = \ u\langle x, y\rangle \qquad (x, y \in E).$$

*Proof.* By Theorem 2.3, it suffices to show that $\Psi$ is an $A$-module map. Recall that there are mutually orthogonal central projections $q_{11}$, $q_{21}$ and $q_\infty$ in $A$ summing up to 1 such that $q_{11}A$ is a finite $W^*$-algebra of type $I$, $q_{21}A$ is a finite $W^*$-algebra of type II, and $q_\infty A$ is a properly infinite $W^*$-algebra (see, e.g., [14, 6.1.9]). Thus, $E = Eq_{11} \oplus Eq_{21} \oplus Eq_\infty$. The restriction $\Psi|_{Eq_\infty}$ is a $(q_\infty A)$-module map because of Lemma 3.1 and the fact that every element in $q_\infty A$ is a sum of at most five idempotents (see [17, Theorem 4]). Similarly, the restriction $\Psi|_{Eq_{21}}$ is a $(q_{21}A)$-module map since every element in $q_{21}A$ is a complex linear combination of at most twenty-four projections [8, Theorem 2]. Thus, it remains to verify the case when $A$ is a finite $W^*$-algebra of type $I$.

In this case, for each $n \in \mathbb{N}$, there exist a hyperstonean space $\Omega_n$ (could be empty) and a projection $q_n \in Z(A)$ such that $\{q_n\}$ are orthogonal to one another, $\sum_n q_n$ weak-*-converges to 1 and $q_n A \cong C(\Omega_n) \otimes M_n$ (see e.g. [14, 6.7.7]). Here we use the convention that $C(\Omega_n) = \{0\}$ if $\Omega_n = \emptyset$. Let $n \in \mathbb{N}$ such that $\Omega_n \neq \emptyset$ and $e \in C(\Omega_n)$ be the identity. Pick any rank one projection $p \in M_n$. If $r := e \otimes p \in \mathrm{Proj}_1(q_n A)$, then $rAr$ is isomorphic to $C(\Omega_n)$. By Lemma 3.1, the induced map $\Psi_r : Er \to Fr$ is an orthogonality-preserving local map between Hilbert $rAr$-modules. Using [13, 3.5], we see that $\Psi_r$ is an $rAr$-module map. In particular, for any $a \in C(\Omega_n)$ and $x \in E$, one has

$$\Psi(x(a \otimes p)) \;=\; \Psi_r(xr(a \otimes I)r) \;=\; \Psi_r(xr)r(a \otimes I)r \;=\; \Psi(x)(a \otimes p),$$

where $I \in M_n$ is the identity. Now, let $\{e_{kl}\}_{k,l=1}^n$ be the matrix unit of $M_n$. As $\frac{e_{kl}+e_{kl}^*+e_{kk}+e_{ll}}{2}$ and $\frac{\mathrm{i}(e_{kl}-e_{kl}^*)+e_{kk}+e_{ll}}{2}$ are rank one projections, we see that $e_{kl}$ is a linear combination of rank one projections $(k, l \in \{1, ..., n\})$. Since any $a \in q_n A$ is of the form $a = \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$ $(a_{ij} \in C(\Omega_n))$, we see that $\Psi(xa) = \Psi(x)a$ $(x \in E; a \in q_n A)$. It follows that for any $x \in E$ and $a \in A$,

$$\Psi(xa)\sum_{k=1}^n q_k \;=\; \Psi\left(xa\sum_{k=1}^n q_k\right) \;=\; \Psi(x)\left(a\sum_{k=1}^n q_k\right) \;=\; (\Psi(x)a)\sum_{k=1}^n q_k \qquad (n \in \mathbb{N}).$$

Consequently, for any $y \in F$, we have $\langle y, \Psi(xa) - \Psi(x)a\rangle \sum_{k=1}^n q_k = 0$, which implies that $\langle y, \Psi(xa) - \Psi(x)a\rangle = 0$, and so $\Psi(xa) = \Psi(x)a$ as required. □

We end this article with a result that could be a first step towards a positive answer for Conjecture 1.2 in the case when $A$ has real rank zero (with $\theta$ not being assumed to be an $A$-module map nor bounded). This result is also interesting by itself and gives us a rough idea what kind of difficulties we will come across without the $A$-linearity.

**Proposition 3.5.** *Let $A$ be a unital $C^*$-algebra of real rank zero, and $A_0$ be the $*$-algebra generated by the idempotents in $A$. Suppose that there is an element $x_0 \in E$ such that $\langle x_0, x_0\rangle = 1$. If $\Psi : E \to F$ is a local map preserving orthogonality, then one can find $u \in Z(A)_+$ as well as an $A_0$-submodule $E_0 \subseteq E$ containing $x_0$ with $E_0^\perp = \{0\}$ such that*

$$\langle \Psi(x), \Psi(y)\rangle \;=\; u\langle x, y\rangle \quad (x, y \in E_0).$$

*Proof.* Define $u := \langle \Psi(x_0), \Psi(x_0)\rangle \in A_+$. Note that by Lemma 3.1, $\Psi$ is an $A_0$-module map. Thus, $\Psi(xw) = \Psi(x)w$ for any symmetry $w \in A$ (as $w \in A_0$). Now, the argument of Proposition 2.2 tells us that $u \in Z(A)_+$. Let $z \in E$ such that $\langle x_0, z\rangle = 0$ and $\langle z, z\rangle \in A_0$. Then $z + x_0$ is also orthogonal to $z - x_0\langle z, z\rangle$. It follows from the orthogonality-preserving property and the $A_0$-linearity of $\Psi$ that

$$\langle \Psi(z), \Psi(z)\rangle \;=\; \langle \Psi(x_0), \Psi(x_0)\rangle\langle z, z\rangle \;=\; u\langle z, z\rangle.$$

Let $\mathcal{D} := \{D \subseteq E : x_0 \in D; \langle x, x\rangle \in A_0 \text{ and } \langle x, y\rangle = 0 \text{ for any } x \neq y \in D\}$. Take any maximal element $M$ in $\mathcal{D}$, and define $E_0$ to be the linear spans of $x \cdot a$ $(x \in M; a \in A_0)$. For any $y \in E_0$, we know that $\langle y, y\rangle, \langle x_0, y\rangle \in A_0$. Thus, $z = y - x_0\langle x_0, y\rangle$ is orthogonal to $x_0$ and $\langle z, z\rangle = \langle y, y\rangle - \langle y, x_0\rangle\langle x_0, y\rangle \in A_0$. Hence, the above implies that

$$\langle \Psi(y), \Psi(y)\rangle \;\;=\;\; \langle y, x_0\rangle\langle \Psi(x_0), \Psi(x_0)\rangle\langle x_0, y\rangle + \langle \Psi(z), \Psi(z)\rangle \;=\; u\langle y, y\rangle.$$

A polarization type argument tells us that $\langle\Psi(x),\Psi(y)\rangle = u\langle x,y\rangle$ $(x,y \in E_0)$. Suppose on the contrary that there exists $z \in E$ with $\|z\| = 1$ and $\langle z,x\rangle = 0$ for any $x \in E_0$. Let $a := \langle z,z\rangle$ and $q_n := e_a(\frac{1}{2^n},1]$ $(n \in \mathbb{N})$. There exist $d,b \in C^*(a)_+$ such that $q_5 \le ab \le 1$, $d \le a$, $dq_4 = aq_4$ and $dq_5 = d$. As $b^{1/2}d^{1/2} \le 1$, $b^{1/2}d^{1/2}q_4 = b^{1/2}a^{1/2}q_4 = q_4$ and $b^{1/2}d^{1/2}q_5 = b^{1/2}d^{1/2}$, we see that

$$\|z - zb^{1/2}d^{1/2}\|^2 \ = \ \|a - 2ab^{1/2}d^{1/2} + abd\| \ \le \ 2\|a(1 - b^{1/2}d^{1/2})\| \ < \ 1/8.$$

Since $d^{1/2} \in \mathrm{her}(q_5)$, there exists $c \in A_0 \cap \mathrm{her}(q_5)_+$ such that $\|d^{1/2} - c\| < 1/8$ (because $\mathrm{her}(q_5)$ also has real rank zero; see e.g. [5]). If $z' := zb^{1/2}c$, then $\langle z',z'\rangle = cq_5abq_5c = c^2 \in A_0$ (as $abq_5 = q_5$) and $\|z - z'\| \le \|z - zb^{1/2}d^{1/2}\| + \|zb^{1/2}d^{1/2} - zb^{1/2}c\| \le 1/2$, which implies that $z' \ne 0$. Moreover, since $\langle x,z'\rangle = \langle x,z\rangle b^{1/2}c = 0$ for any $x \in M$, we see that $M \cup \{z'\} \in \mathcal{D}$, which is a contradiction. □

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