PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 9, September 2012, Pages 3169–3178
S 0002-9939(2012)11356-5
Article electronically published on January 13, 2012

COMPACT OPERATORS IN TRO'S

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(Communicated by Marius Junge)

ABSTRACT. We give a geometric characterization of the elements of a ternary ring of operators (or simply, TRO) that can be represented as compact operators by a faithful representation of the TRO.

1. INTRODUCTION

A *ternary ring of operators* (or simply, TRO) between Hilbert spaces H_2 and H_1 is a norm closed subspace $\mathcal{V}$ of $\mathcal{B}(H_2, H_1)$ which is closed under the triple product

$$\mathcal{V} \times \mathcal{V} \times \mathcal{V} \ni (x, y, z) \mapsto xy^*z \in \mathcal{V}.$$

A TRO $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ is called a w*-TRO if it is w*-closed (equivalently, weak operator closed, or strong operator closed) in $\mathcal{B}(H_2, H_1)$. TRO's were first introduced by Hestenes [9] and since then they have been studied by many authors. In general, a TRO $\mathcal{V}$ can be identified with the off-diagonal corner (at the (1,2) position) of its *linking* C^*-algebra

$$A(\mathcal{V}) = \begin{pmatrix} \mathcal{C} & \mathcal{V} \\ \mathcal{V}^* & \mathcal{D} \end{pmatrix} \subseteq \mathcal{B}(H_1 \oplus H_2),$$

where $\mathcal{C}$ and $\mathcal{D}$ are the C^*-algebras generated by $\mathcal{V}\mathcal{V}^*$ and $\mathcal{V}^*\mathcal{V}$ respectively.

If $\mathcal{S}$ is a nonempty subset of the unit ball of a normed space $\mathcal{X}$, then the *contractive perturbations* of $\mathcal{S}$ are defined as

$$\mathrm{cp}(\mathcal{S}) = \{x \in \mathcal{X} \quad \|x \pm s\| \leq 1 \ \forall s \in \mathcal{S}\}.$$

It is clear that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ implies $\mathrm{cp}(\mathcal{S}_1) \supseteq \mathrm{cp}(\mathcal{S}_2)$. Also, an element x of the unit ball of $\mathcal{X}$ is an extreme point if and only if $\mathrm{cp}(\{x\}) = \{0\}$. We shall write $\mathrm{cp}(x)$ instead of $\mathrm{cp}(\{x\})$.

One may define contractive perturbations of higher order by using the recursive formula $\mathrm{cp}^{n+1}(\mathcal{S}) = \mathrm{cp}\,(\mathrm{cp}^n(\mathcal{S}))$, $n \in \mathbb{N}$. It is clear that $\mathrm{cp}(\mathcal{S})$ is a norm-closed convex subset of the closed unit ball of $\mathcal{X}$. One can also verify that $\mathcal{S} \subseteq \mathrm{cp}^2(\mathcal{S})$; from this it follows that $\mathrm{cp}^3(\mathcal{S}) = \mathrm{cp}(\mathcal{S})$. The second contractive perturbations were introduced in [2]. In [2] it is proved that the set of the second contractive perturbations of an element a of a C^*-algebra $\mathcal{A}$ is compact in the norm topology if and only if there exists a faithful representation ϕ of $\mathcal{A}$ such that $\phi(a)$ is a compact operator. Further study was conducted in [1], [3], [4] and [11]. We shall see that this characterization is not valid for the elements of a TRO.

Received by the editors November 11, 2010 and, in revised form, March 24, 2011.
2010 *Mathematics Subject Classification.* Primary 47B07; Secondary 47Cxx.
Key words and phrases. Contractive perturbations, compact operators, weakly compact, ternary ring of operators, TRO's.

In this work we characterize the elements of a TRO that are represented as compact operators by a faithful representation of the TRO, in terms of the size of their contractive perturbations. We show that there exists a faithful representation ϕ of the TRO $\mathcal{V}$ that maps an element a of the unit ball of $\mathcal{V}$ to a compact operator if and only if the set of its second contractive perturbations is weakly compact. It follows from [2] and our result that for an element a of a C^*-algebra $\mathcal{A}$ the set $\mathrm{cp}^2(a)$ is compact if and only if it is weakly compact or, equivalently, there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator.

Ylinen proved in [16] and [17] that for an element a of a C^*-algebra $\mathcal{A}$ the operator $x \to axa$ on $\mathcal{A}$ is compact if and only if it is weakly compact or, equivalently, there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator. We obtain an analogous result for the operator $x \to ax^*a$ on a TRO.

Notation. Throughout, we adopt the following notation: H_1 and H_2 are Hilbert spaces, $\mathcal{B}(H_2, H_1)$ the space of all bounded linear operators $H_2 \to H_1$ and $\mathcal{K}(H_2, H_1)$ the space of all compact operators $H_2 \to H_1$. In particular, $\mathcal{B}(H_1) = \mathcal{B}(H_1, H_1)$ and $\mathcal{K}(H_1, H_1) = \mathcal{K}(H_1)$. $\mathcal{V}$ is a TRO that is a subspace of $\mathcal{B}(H_2, H_1)$. Let $\mathcal{X}$ be a Banach space, $\mathcal{Y} \subseteq \mathcal{X}$ a subspace and $a \in \mathcal{Y}$. Then by $\mathrm{cp}^n_{\mathcal{Y}}(a)$ we denote the set of the n-th contractive perturbations of a computed with respect to $\mathcal{Y}$. If r is a positive number, then by $\mathcal{X}_r$ we denote the closed ball of center 0 and radius r. Let x, y be elements of a Hilbert space H. We denote by $x \otimes y$ the rank one operator on H defined by

$$(x \otimes y)(z) = \langle z, x \rangle y.$$

2. Preliminaries

Let $\mathcal{V}$ and $\mathcal{W}$ be two TRO's. A linear map $\phi : \mathcal{V} \to \mathcal{W}$ is called a *TRO-homomorphism* if it preserves the ternary product

$$\phi(xy^*z) = \phi(x)\phi(y)^*\phi(z)$$

for all $x, y, z \in \mathcal{V}$. If, in addition, ϕ is an injection from $\mathcal{V}$ onto $\mathcal{W}$, we call ϕ a *TRO-isomorphism* from $\mathcal{V}$ onto $\mathcal{W}$. A TRO-homomorphism ϕ from a TRO $\mathcal{V}$ into the set of all bounded operators from one Hilbert space to another is called a *representation* of $\mathcal{V}$. We will say that a representation $\phi : \mathcal{V} \to \mathcal{B}(H_2, H_1)$ is a *faithful* representation of $\mathcal{V}$ if ϕ is injective. It was shown in [8, Proposition 3.4] that every faithful TRO-representation is an isometry.

Proposition 2.1. *Let H_1 and H_2 be Hilbert spaces, $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ a TRO and $A(\mathcal{V})$ its linking algebra. If a is in the unit ball of $\mathcal{V}$, then $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$.*

Proof. First we note that if $\mathcal{E}$ is a Banach space and $b \in \mathcal{E}$ has the property $\|x + b\| \leq 1$ for all $x \in \mathcal{E}$ with $\|x\| \leq 1$, then $b = 0$. Indeed, if the above property holds for some $b \neq 0$, then taking $x = b/\|b\|$ we have $\|b/\|b\| + b\| \leq 1$, which implies that $\|b\| = 0$. This yields a contradiction.

Let

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathrm{cp}^2_{A(\mathcal{V})}(a).$$

We can easily see that

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in \mathrm{cp}_{A(\mathcal{V})}(a)$$

for every $x \in \mathrm{cp}_{\mathcal{V}}(a)$ and $y \in \mathcal{V}^*$ with $\|y\| \leq 1$. So, it follows directly that $b_2 \in \mathrm{cp}^2_{\mathcal{V}}(a)$, while from the remark at the beginning of the proof it follows that $b_3 = 0$. We have that $\begin{pmatrix} b_1 & b_2 \\ y & b_4 \end{pmatrix}$ is a contraction for every $y \in \mathcal{V}^*$ with $\|y\| \leq 1$. Thus, if $\eta \in H_1$, we have that

$$\|b_1\eta\|^2 + \|y\eta\|^2 \leq \|\eta\|^2 \tag{1}$$

for all $y \in \mathcal{V}^*$ with $\|y\| \leq 1$. Since the strong*-topology of $\mathcal{V}^*$ is finer than its strong topology, it follows from [8, Theorem 3.6 (Kaplansky density theorem)] that the inequality (1) holds for all y in the closed unit ball of $\overline{\mathcal{V}^*}^{w^*}$. Therefore, for all partial isometries $y \in \overline{\mathcal{V}^*}^{w^*}$, we have $0 \leq b_1^*b_1 + y^*y \leq 1$. Denoting by p_y the domain projection y^*y of a partial isometry $y \in \overline{\mathcal{V}^*}^{w^*}$, it follows that $0 \leq b_1^*b_1 \leq 1 - p_y$. Multiplying by p_y, we deduce that $p_y b_1^* b_1 p_y = 0$ or $b_1 p_y = 0$. Let Π be the set of all partial isometries of $\overline{\mathcal{V}^*}^{w^*}$ and p the orthogonal projection onto the closed linear span of the subspaces $\{p_y(H_2)\}_{y\in\Pi}$. Then we have proved that $b_1 p = 0$. On the other hand, we can see that $b_1 p^\perp = 0$, since b_1 is in the C^*-algebra generated by $\mathcal{V}\mathcal{V}^*$ and $\overline{\mathcal{V}^*}^{w^*}$ is generated by its partial isometries [8, Theorem 3.2]. Hence, we have proved that $b_1 = 0$. By symmetry, we obtain $b_4 = 0$. Thus, we showed that each element of $\mathrm{cp}^2_{A(\mathcal{V})}(a)$ is in $\mathcal{V}$. The fact that $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$ is immediate. □

Remark 2.2. The containment in the last proposition may be strict. We shall give an example. Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = \infty$ and $\dim H_2 < \infty$ and $u : H_2 \to H_1$ an isometry. Let $\mathcal{V} = \mathcal{B}(H_2, H_1)$. Since u is an extreme point of $\mathcal{V}$ [18], the set $\mathrm{cp}^2_{\mathcal{V}}(u)$ is equal to the unit ball of $\mathcal{V}$. Now, $A(\mathcal{V}) = \mathcal{B}(H_1 \oplus H_2)$ and it follows from [2, Corollary 2.4] that $\mathrm{cp}^2_{A(\mathcal{V})}(u)$ is compact. Hence, the inclusion $\mathrm{cp}^2_{A(\mathcal{V})}(u) \subset \mathrm{cp}^2_{\mathcal{V}}(u)$ is strict. Considering the identity representation of $\mathcal{V}$ in this example, one can see that the implication $(i) \Rightarrow (ii)$ of [2, Theorem 2.2] does not hold for TRO's.

Remark 2.3. It is known that the linking algebra $A(\mathcal{V})$ is just the C^*-envelope $C^*_e(\mathcal{V})$ of the TRO $\mathcal{V}$. Therefore, the inclusion in Proposition 2.1 in the case of an operator space $\mathcal{O}$ would be $\mathrm{cp}^2_{C^*_e(\mathcal{O})}(a) \subseteq \mathrm{cp}^2_{\mathcal{O}}(a)$. Now, we shall see that this inclusion does not hold in operator spaces in general.

Let H be an infinite dimensional Hilbert space,

$$\mathcal{O} = \left\{ \begin{bmatrix} \lambda\, Id & a \\ 0 & \mu\, Id \end{bmatrix} \;:\; a \in \mathcal{K}(H), \lambda, \mu \in \mathbb{C} \right\}.$$

The C^*-algebra generated by $\mathcal{O}$ in $\mathcal{B}(H \oplus H)$ is

$$C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O}) = \left\{ \begin{bmatrix} \lambda\, Id + a & b \\ c & \mu\, Id + d \end{bmatrix} :\; a, b, c, d \in \mathcal{K}(H), \lambda, \mu \in \mathbb{C} \right\}.$$

If $\mathcal{I}$ is a proper ideal of $C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O})$, then $\mathcal{I}$ contains $\mathcal{K}(H \oplus H)$, the compact operators on $H \oplus H$ and, consequently, the quotient space $C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O})/\mathcal{I}$ is finite dimensional. Therefore, $C^*_e(\mathcal{O}) = C^*_{\mathcal{B}(H\oplus H)}(\mathcal{O})$. Then if we consider the operator

$$s = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix},$$

we see that

$$\mathrm{cp}^2_{C_e^*(\mathcal{O})}(s) = \left\{ \left[\begin{array}{cc} \lambda\, Id + x & 0 \\ 0 & 0 \end{array} \right] : \ \lambda \in \mathbb{C},\, x \in \mathcal{K}(H),\, \|\lambda\, Id + x\| \leq 1 \right\}$$

and that

$$\mathrm{cp}^2_{\mathcal{O}}(s) = \left\{ \left[\begin{array}{cc} \lambda & 0 \\ 0 & 0 \end{array} \right] : \ \lambda \in \mathbb{C},\, |\lambda| \leq 1 \right\}.$$

Proposition 2.4. *Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = \infty$ and $\dim H_2 = \infty$ and $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ a TRO. Let $a = \sum_{i=1}^{\infty} \lambda_i u_i \in \mathcal{V}$ be a norm one compact operator, where $\{u_i\}_{i=1}^{\infty}$ are finite rank partial isometries such that $u_i u_j^* = 0$, $u_i^* u_j = 0$ for $i \neq j$ and $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of positive numbers decreasing to 0. Let $e_k = \sum_{i=1}^{k} u_i^* u_i$ and $f_k = \sum_{i=1}^{k} u_i u_i^*$. If x is any contraction in $\mathcal{V}$, then*

(1) $\|a \pm (1 - \lambda_k) f_k^{\perp} x e_k^{\perp}\| \leq 1$,
(2) *e_k and f_k are in the C^*-algebra generated by $\mathcal{V}^*\mathcal{V}$ and $\mathcal{V}\mathcal{V}^*$ respectively,*
(3) $f_k^{\perp} x e_k^{\perp} \in \mathcal{V}$.

Proof. (1) Let y be a contraction in $\mathcal{V}$. From [12] we know that

$$\|a \pm (\mathbf{1} - |a^*|)^{1/2} y (\mathbf{1} - |a|)^{1/2}\| \leq 1.$$

Simple computations show that

$$\left\| a \pm \left(\sum_{i=1}^{\infty} (1 - \lambda_i)^{1/2} u_i u_i^* + f^{\perp} \right) y \left(\sum_{j=1}^{\infty} (1 - \lambda_j)^{1/2} u_j^* u_j + e^{\perp} \right) \right\| \leq 1,$$

where $e = [\{u_i^* u_i\}_{i \in \mathbb{N}}]$ and $f = [\{u_i u_i^*\}_{i \in \mathbb{N}}]$. Now setting

$$y = \left(\sum_{i=k+1}^{\infty} \frac{(1 - \lambda_k)^{1/2}}{(1 - \lambda_i)^{1/2}} u_i u_i^* + (1 - \lambda_k)^{1/2} f^{\perp} \right) x \times \left(\sum_{j=k+1}^{\infty} \frac{(1 - \lambda_k)^{1/2}}{(1 - \lambda_j)^{1/2}} u_j^* u_j + (1 - \lambda_k)^{1/2} e^{\perp} \right),$$

where $x \in \mathcal{V}$ is a contraction, we get the result.

(2) Assume that $\lambda_1 = 1$. We define a sequence $(a_i)_{i \in \mathbb{N}}$ in $\mathcal{V}$, where $a_1 = a$ and $a_n = a_{n-1} a_{n-1}^* a_{n-1}$. Simple computations show that $\lim_{n \to \infty} a_n = u_1$ is in $\mathcal{V}$. That means that $a - u_1 = \sum_{i=2}^{\infty} \lambda_i u_i$ is in $\mathcal{V}$ and using the same argument for $a - u_1$ we deduce $u_2 \in \mathcal{V}$ and continuing in the above fashion, we inductively get $u_n \in \mathcal{V}$ for all $n \in \mathbb{N}$. Hence, $u_n^* \in \mathcal{V}^*$ for all $n \in \mathbb{N}$. It follows that $e_k = \sum_{i=1}^{k} u_i^* u_i = (\sum_{l=1}^{k} u_l^*)(\sum_{m=1}^{k} u_m) \in \mathcal{V}^*\mathcal{V}$ and $f_k = \sum_{i=1}^{k} u_i u_i^* = (\sum_{l=1}^{k} u_l)(\sum_{m=1}^{k} u_m^*) \in \mathcal{V}\mathcal{V}^*$.

(3) Since $x \in \mathcal{V}$, $x e_k \in \mathcal{V}$, $f_k x \in \mathcal{V}$ and $f_k x f_k \in \mathcal{V}$, it follows that $f_k^{\perp} x e_k^{\perp} = (\mathbf{1} - f_k) x (\mathbf{1} - e_k) = x - x e_k - f_k x + f_k x e_k$ is in $\mathcal{V}$. □

Proposition 2.5. *Let H_1 and H_2 be Hilbert spaces with $\dim H_1 = \infty$ and $\dim H_2 = \infty$ and $\mathcal{V} \subseteq \mathcal{B}(H_2, H_1)$ a TRO. If $\mathcal{C}$ is the TRO that consists of all compact operators of $\mathcal{V}$, then $\mathrm{cp}^2_{\mathcal{V}}(a) \subseteq \mathrm{cp}^2_{\mathcal{C}}(a)$ for all $a \in \mathcal{C}$.*

Proof. Let $a \in \mathcal{C}$. It suffices to show that $\mathrm{cp}^2_{\mathcal{V}}(a) \subseteq \mathcal{K}(H_2, H_1)$. We shall show that if $x \in \mathcal{V} \backslash \mathcal{C}$, then $x \notin \mathrm{cp}^2_{\mathcal{V}}(a)$. Since the operator x is not compact, there exists an

$\varepsilon > 0$ such that for all finite rank projections f, e on H_1, and H_2 respectively, the inequality

$$\|f^\perp x e^\perp\| > \varepsilon$$

holds, where $f^\perp = \mathbf{1} - f$ and $e^\perp = \mathbf{1} - e$. Given that the operator a is compact, there exists a unique sequence of positive numbers $(\lambda_i)_{i\in\mathbb{N}}$ decreasing to 0 and a sequence $\{u_i\}_{i=1}^\infty$ of finite rank partial isometries with $u_i u_j^* = 0$, $u_i^* u_j = 0$ for $i \neq j$ such that

$$a = \sum_{i=1}^\infty \lambda_i u_i.$$

Let $e_k = \sum_{i=1}^k u_i^* u_i$ and $f_k = \sum_{i=1}^k u_i u_i^*$ for all $k \in \mathbb{N}$. From Proposition 2.4 we know that if y is any contraction in $\mathcal{V}$, then $(1-\lambda_k) f_k^\perp y e_k^\perp \in \mathrm{cp}_{\mathcal{V}}(a)$. Thus, it suffices to find a $k \in \mathbb{N}$ and a contraction $y \in \mathcal{V}$ such that $\|x \pm (1-\lambda_k) f_k^\perp y e_k^\perp\| > 1$.

We choose k so that $\lambda_k < \varepsilon$, set $x_k = f_k^\perp x e_k^\perp$ and $y = x_k/\|x_k\|$. The following computations complete the proof:

$$\begin{aligned}\|x + (1-\lambda_k) f_k^\perp y e_k^\perp\| &\geq \|x_k + (1-\lambda_k) y\| \\ &= \left\| x_k + (1-\lambda_k)\frac{x_k}{\|x_k\|}\right\| = \|x_k\| \left| 1 + (1-\lambda_k)\frac{1}{\|x_k\|}\right| \\ &= \|x_k\| + (1-\lambda_k) > \varepsilon + 1 - \varepsilon = 1.\end{aligned}$$

□

3. The main results

We have seen in Remark 2.2 that the characterization given in [2, Theorem 2.2] does not hold for TRO's. In this section we shall show that there exists a faithful representation ϕ of the TRO $\mathcal{V}$ that maps an element $a \in \mathcal{V}_1$ to a compact operator if and only if the set $\mathrm{cp}^2_{\mathcal{V}}(a)$ is weakly compact. This is one of the main results of this work.

Note that if π is a faithful representation of a TRO $\mathcal{V}$, we can identify $\mathcal{V}$ with $\pi(\mathcal{V})$.

Lemma 3.1. *Let a be a non-compact selfadjoint operator in $\mathcal{B}(H)_1$. Then there exists $\varepsilon > 0$ and an infinite dimensional projection p on H such that $\mathcal{B}(p(H))_{\varepsilon^2/2} \subseteq a\mathcal{B}(H)_{1/2}a$.*

Proof. Let us assume that a is a non-compact selfadjoint contraction. We shall denote by E the unique spectral measure relative to $(\sigma(a), H)$ such that $a = \int z dE$, where z is the inclusion map of $\sigma(a)$ in $\mathbb{C}$. From [7, Proposition 4.1] there exists an $\varepsilon > 0$ such that the projection $p = E(\{z \in \sigma(a) : |z| > \varepsilon\})$ is infinite dimensional. Denote by a_p the operator in $\mathcal{B}(p(H))$ such that $a_p(h) = ap(h) = pa(h)$ for all $h \in p(H)$. The operator a_p is invertible. Let us assume that the operator T is in $p\mathcal{B}(H)_{\varepsilon^2/2}p = \mathcal{B}(p(H))_{\varepsilon^2/2}$. Then

$$\|(a_p)^{-1} T (a_p)^{-1}\| \leq \|(a_p)^{-1}\|^2 \|T\| \leq \frac{1}{\varepsilon^2}\frac{\varepsilon^2}{2} = \frac{1}{2}.$$

Therefore,

$$T = a_p((a_p)^{-1} T (a_p)^{-1}) a_p \in a_p \mathcal{B}(p(H))_{1/2} a_p \subseteq ap\mathcal{B}(H)_{1/2}pa.$$

So,

$$\mathcal{B}(p(H))_{\varepsilon^2/2} = p\mathcal{B}(H)_{\varepsilon^2/2}p \subseteq ap\mathcal{B}(H)_{1/2}pa \subseteq a\mathcal{B}(H)_{1/2}a.$$

□

Proposition 3.2. *Let a be a contractive operator on a Hilbert space H. Then the operator a is compact if and only if the set* $\mathrm{cp}^2_{\mathcal{B}(H)}(a)$ *is weakly compact.*

Proof. The forward implication is trivial from [2, Corollary 2.4].

Conversely, suppose that the operator a is non-compact. The polar decomposition of a has the following form:

$$a = v|a|,$$

where v is a partial isometry, such that $v^*v|a| = |a|$ and $\mathrm{dom}(v) = \overline{|a|(H)}$. From Lemma 3.1 we know that there exists $\varepsilon > 0$ and an infinite dimensional projection p such that

$$vp\mathcal{B}(H)_{\varepsilon^2/2}p \subseteq v|a|\mathcal{B}(H)_{1/2}|a|.$$

Therefore, the following inclusions hold:

$$\mathcal{B}(p(H), vp(H))_{\varepsilon^2/2} = vp\mathcal{B}(H)_{\varepsilon^2/2}p \subseteq v|a|\mathcal{B}(H)_{1/2}|a|$$
$$= v|a|\mathcal{B}(H)_{1/2}v^*v|a| \subseteq v|a|\mathcal{B}(H)_{1/2}v|a| = a\mathcal{B}(H)_{1/2}a \subseteq \mathrm{cp}^2_{\mathcal{B}(H)}(a).$$

The last inclusion follows from [2, Proposition 1.2]. Since v is a non-compact partial isometry, $\mathcal{B}(p(H), vp(H))_{\varepsilon^2/2}$ is not weakly compact [7, Chapter V, Theorem 4.2]. The proof is complete. ☐

Theorem 3.3. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. Then there exists a faithful representation ϕ of $\mathcal{A}$ such that $\phi(a)$ is a compact operator if and only if* $\mathrm{cp}^2(a)$ *is a weakly compact set.*

Proof. The forward implication follows from [2, Theorem 2.2].

Conversely assume that $\phi(a)$ is a non-compact operator for all faithful representations ϕ of $\mathcal{A}$. Let $\{(\phi_i, H_i)\}$ be a maximal family of pairwise inequivalent irreducible representations of $\mathcal{A}$ and let ϕ be the reduced atomic representation $(\sum_{i\in I} \oplus\phi_i, \sum_{i\in I} \oplus H_i)$. Since all ϕ_i are irreducible representations, the SOT-closure of $\phi(\mathcal{A})$ equals $\sum_{i\in I} \oplus\mathcal{B}(H_i)$. Kaplansky's Density Theorem shows that $\phi(\mathcal{A}_1)$ is SOT-dense in $(\sum_{i\in I} \oplus\mathcal{B}(H_i))_1$ and so $\phi(a)\phi(\mathcal{A}_{1/2})\phi(a)$ is SOT-dense in

$$\sum_{i\in I} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a).$$

However, [2, Proposition 1.2] shows that $\phi(a)\phi(\mathcal{A}_{1/2})\phi(a)$ is contained in the set $\mathrm{cp}^2(\phi(a))$, which is a SOT-closed set. Thus

$$\left(\sum_{i\in I} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a)\right) \subseteq \mathrm{cp}^2(\phi(a)).$$

The operator $\phi(a)$ is not compact, since the reduced atomic representation is faithful. Thus, there are two cases.

Assume first that there exists an $i_o \in I$ such that $\phi_{i_o}(a)$ is a non-compact operator on H_{i_o}. Therefore, from the proof of Proposition 3.2 there exists an infinite dimensional projection $p \in \mathcal{B}(H_{i_o})$, a non-compact partial isometry v and an $\varepsilon > 0$ such that $\mathcal{B}(p(H_{i_o}), vp(H_{i_o}))_{\varepsilon^2/2} \subseteq \phi_{i_o}(a)\mathcal{B}(H_{i_o})_{1/2}\phi_{i_o}(a)$. It follows that $\mathcal{B}(p(H_{i_o}), vp(H_{i_o}))_{\varepsilon^2/2} \oplus \sum_{i\in I-\{i_o\}} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a) \subseteq \mathrm{cp}^2(a)$. Therefore the set $\mathrm{cp}^2(\phi(a))$ is not weakly compact since $\mathcal{B}(p(H_{i_o}), vp(H_{i_o}))_{\varepsilon^2/2}$ is not a weakly compact set.

Assume now that $\phi_i(a)$ is compact for all $i \in I$. Since $\phi(a)$ is not compact there exists an $\varepsilon > 0$ such that the set $\{i \in I : \|\phi_i(a)\| \geq \varepsilon\}$ is infinite. Then the set

$\sum_{i\in I} \oplus\phi_i(a)\mathcal{B}(H_i)_{1/2}\phi_i(a)$ is not compact since it contains a copy of an l^∞ ball [7, Chapter V, Theorem 4.2]. This completes the proof, as the last set is contained in $\mathrm{cp}^2(a)$. $\square$

Remark 3.4. Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. Then by the theorem above and [2, Theorem 2.2], the following assertions are equivalent:

(1) There exists a faithful representation (ϕ, H) of $\mathcal{A}$ so that $\phi(a)$ is a compact operator.
(2) The set $\mathrm{cp}^2(a)$ is norm compact.
(3) The set $\mathrm{cp}^2(a)$ is weakly compact.

Let $\phi : \mathcal{V} \to \mathcal{B}(H_2, H_1)$ be a representation of a TRO $\mathcal{V}$ and $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$ closed subspaces. A pair of subspaces (K_2, K_1) is said to be *ϕ-invariant* if $\phi(\mathcal{V})K_2 \subseteq K_1$ and $\phi(\mathcal{V})^* K_1 \subseteq K_2$. The representation ϕ is said to be *irreducible* if $(0, 0)$ and (H_2, H_1) are the only ϕ-invariant pairs.

Two representations $\phi_i : \mathcal{V} \to \mathcal{B}(H_1^i, H_2^i)$ of $\mathcal{V}$, $i = 1, 2$ are said to be unitarily equivalent if there are unitary operators $U_i : H_i^1 \to H_i^2$, $i = 1, 2$ such that $\phi_1(x) = U_2^* \phi_2(x) U_1$, for all $x \in \mathcal{V}$.

Let $(\phi_i)_{i\in I}$ be a maximal family of pairwise inequivalent irreducible representations of $\mathcal{V}$, $\phi_i : \mathcal{V} \to \mathcal{B}(H_{2,i}, H_{1,i})$. Their direct sum $\phi = \sum_{i\in I} \oplus\phi_i$ is the reduced atomic representation of $\mathcal{V}$. It follows from [5, Lemma 3.5] that an irreducible representation of a TRO is the restriction of an irreducible representation of its linking algebra. Therefore, the reduced atomic representation of a TRO $\mathcal{V}$ is the restriction of the reduced atomic representation of its linking algebra $A(\mathcal{V})$.

Theorem 3.5. *Let $\mathcal{V}$ be a TRO and $a \in \mathcal{V}_1$. The following are equivalent:*

(1) $\mathrm{cp}^2_{\mathcal{V}}(a)$ *is a weakly compact set.*
(2) *There exists a faithful representation π of $\mathcal{V}$ such that $\pi(a)$ is a compact operator.*
(3) *$\phi(a)$ is a compact operator where ϕ is the reduced atomic representation of $\mathcal{V}$.*

Proof. First we show that (1) is equivalent to (2). Suppose that the set $\mathrm{cp}^2_{\mathcal{V}}(a)$ is weakly compact. From Proposition 2.1 we know that $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$ and therefore the set $\mathrm{cp}^2_{A(\mathcal{V})}(a)$ is weakly compact. Now, by Theorem 3.3, there exists a faithful representation π of $\mathcal{V}$ that maps a to a compact operator.

Conversely, suppose that π is a faithful representation of $\mathcal{V}$ such that $\pi(a)$ is a compact operator. We may assume that both H_1 and H_2 are infinite dimensional Hilbert spaces. Identifying $\mathcal{V}$ with $\pi(\mathcal{V})$, Proposition 2.5 states that $\mathrm{cp}^2_{\mathcal{V}}(a) \subseteq \mathcal{V} \cap \mathcal{K}(H_2, H_1) \subseteq \mathcal{K}(H_1 \oplus H_2)$. The set $\mathrm{cp}^2_{\mathcal{V}}(a)$ is WOT-closed. Since the relative w* and WOT-topologies on the closed unit ball of $\mathcal{B}(H_1 \oplus H_2)$ coincide, [13, Theorem 4.2.4.], $\mathrm{cp}^2_{\mathcal{V}}(a)$ is a w*-closed set. From the Banach-Alaoglu theorem we deduce that $\mathrm{cp}^2_{\mathcal{V}}(a)$ is a w*-compact set. By [10, Proposition 10.4.3], the weak topology on $\mathcal{K}(H_1 \oplus H_2)$ coincides with the relative w*-topology on $\mathcal{K}(H_1 \oplus H_2)$ and therefore $\mathrm{cp}^2_{\mathcal{V}}(a)$ is a weakly compact set.

Obviously (3) implies (2) since ϕ is a faithful representation. So, we only need to show that (1) implies (3). From Proposition 2.1 we know that $\mathrm{cp}^2_{A(\mathcal{V})}(a) \subseteq \mathrm{cp}^2_{\mathcal{V}}(a)$. Therefore the set $\mathrm{cp}^2_{A(\mathcal{V})}(a)$ is weakly compact and from Theorem 3.3,

$\rho(a)$ is a compact operator, where ρ is the reduced atomic representation of $A(\mathcal{V})$ [2, Theorem 2.2]. The operator $\phi(a)$ is compact since $\phi = \rho|_{\mathcal{V}}$. □

Statement (3) of Theorem 3.5 ensures that the elements of $\mathcal{V}$ that are mapped to a compact operator by a faithful representation of $\mathcal{V}$ form a subTRO.

Remark 3.6. Let $\mathcal{A}$ be a C^*-algebra which acts on a Hilbert space H and contains $\mathcal{K}(H)$, the set of compact operators on H. If $a \in \mathcal{A}_1$, the following assertions are equivalent:

(1) a is a compact operator.
(2) The set $\mathrm{cp}^2(a)$ is norm compact.
(3) The set $\mathrm{cp}^2(a)$ is SOT-compact.
(4) The set $\mathrm{cp}^2(a)$ is weakly compact.

Proof. From [2, Corollary 2.4.], we know that (1) and (2) are equivalent.

That (2) implies (3) is obvious.

Now we show that (3) implies (1). The following arguments are similar to those of [2, Lemma 2.1]. Since $\mathrm{cp}^2(a)$ is SOT-compact and $a(\mathcal{K}(H))_{1/2}a \subseteq \mathrm{cp}^2(a)$, the set $a(\mathcal{K}(H))_{1/2}a$ is SOT-precompact. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in H. Without loss of generality we may assume that $\|f_n\| \leq 1/2$, for all $n \in \mathbb{N}$. Let e be a unit vector in $(\ker a^*)^{\perp}$. For every $n \in \mathbb{N}$, let $x_n = e \otimes f_n$. Then $ax_na = a^*e \otimes af_n$. Since $a(\mathcal{K}(H))_{1/2}a$ is a SOT-precompact set, the sequence $\{(a^*e \otimes af_n)(h)\}_{n\in\mathbb{N}}$ has a convergent subsequence for every $h \in H$. Thus, the sequence $\{\langle h, a^*e\rangle af_n\}_{n\in\mathbb{N}}$ has a convergent subsequence and therefore $\{af_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence, a is a compact operator.

Obviously (1) implies (4). So, it suffices to see that (4) implies (1). Let us assume that a is a non-compact operator in $\mathcal{A}$. Following the arguments of the proof of Proposition 3.2 we can easily see that there exists an $\varepsilon > 0$, an infinite dimensional projection p and a non-compact partial isometry v on $p(H)$ such that $\mathcal{K}(p(H), vp(H))_{\varepsilon^2/2} \subseteq \mathrm{cp}^2_{\mathcal{A}}(a)$. Since the ball $\mathcal{K}(p(H), vp(H))_{\varepsilon^2/2}$ is not weakly compact, the set $\mathrm{cp}^2_{\mathcal{A}}(a)$ is not weakly compact. □

The following example shows that the compactness of an element u of a TRO does not imply the SOT-compactness of $\mathrm{cp}^2(u)$.

Example 3.7. Let $\mathcal{V}$ be the TRO $\mathcal{B}(H_2, H_1)$, where H_1 is an infinite dimensional Hilbert space and H_2 a one dimensional Hilbert space. The unit ball of $\mathcal{B}(H_2, H_1)$ is not SOT-compact. Indeed, if $\{e \otimes f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(H_2, H_1)$, where e is a unit vector of H_2 and $\{f_n\}$ an orthonormal sequence of H_1, then the sequence $\{(e \otimes f_n)(e)\} = \{\langle e, e\rangle f_n\} = \{f_n\}$ does not have a convergent subsequence. Consider an isometry $u \in \mathcal{V}$. Then u is compact and $\mathrm{cp}^2(u) = \mathcal{V}_1$ is not a SOT-compact set.

The set $\mathrm{cp}^2(a)$ of the remark above is always WOT-compact since the WOT-topology of $\mathcal{B}(H)$ coincides with its w*-topology on its closed unit ball. Therefore, the WOT-compactness of $\mathrm{cp}^2(a)$ cannot be equivalent with the statements of Remark 3.6.

Vala introduced the notion of compactness in a normed algebra in [15]. He defined an element a of a normed algebra to be compact if the mapping $x \to axa$ is compact.

Definition 3.8. A linear mapping $u : \mathcal{V} \to \mathcal{V}$ is called a *weakly compact operator* on $\mathcal{V}$ if $\{u(x) : \|x\| \leq 1\}$ is relatively weakly compact in $\mathcal{V}$.

We shall use the following theorem. It was proved by K. Ylinen in [16] and [17].

Theorem 3.9. *Let a be an element of the C^*-algebra $\mathcal{A}$. The following conditions are equivalent:*

(1) *There exists a faithful representation ϕ that maps a to a compact operator.*
(2) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = axa$ is compact.*
(3) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = axa$ is weakly compact.*

Bunce and Chu in [6] establish several theorems classifying compact and weakly compact JB^*-triples. A JB^*-triple $\mathcal{A}$ is said to be (weakly) compact if the antilinear operator $x \to \{axa\}$ is (weakly) compact for each $a \in \mathcal{A}$, where $\{\ \}$ denotes the ternary product. It follows from [6, Theorem 3.6] that a TRO $\mathcal{V}$ is isomorphic to a subTRO of $\mathcal{K}(H)$ for some Hilbert space H if and only if the mapping $a \to ax^*a$ is compact or equivalently weakly compact, for all $a \in \mathcal{V}$. The next theorem characterizes the compact elements of a TRO $\mathcal{V}$.

Theorem 3.10. *Let a be an element of a TRO $\mathcal{V}$. The following conditions are equivalent:*

(1) *There exists a faithful representation π that maps a to a compact operator.*
(2) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ is compact.*
(3) *The operator $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ is weakly compact.*

Proof. First we show that (2) implies (1). Let $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ be a compact operator. Then the extension of u to the linking algebra $A(\mathcal{V})$ of $\mathcal{V}$ is compact as well, since

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax_3a \\ 0 & 0 \end{pmatrix} \in \mathcal{V},$$

where $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in A(\mathcal{V})$. It follows that the operator $\tilde{u} : A(\mathcal{V}) \to A(\mathcal{V})$, $\tilde{u}(x) = axa$ is compact. From [16] there exists a faithful representation π of $A(\mathcal{V})$ such that $\pi(a)$ is a compact operator.

Now we show the implication (1)$\Rightarrow$(2). Suppose there exists an isometric representation π of $\mathcal{V}$ on a Hilbert space H so that $\pi(a)$ is a compact operator on H. Then (see [14]) the map $u_1 : \mathcal{B}(H) \to \mathcal{B}(H)$, $u_1(x) = \pi(a)x\pi(a)$ is compact. Obviously, the map $u_2 : \mathcal{B}(H) \to \mathcal{B}(H)$, $u_2(x) = \pi(a)x^*\pi(a)$ is compact as well. Therefore, the restriction of u_2 to $\pi(\mathcal{V})$ is a compact operator. Since π is an isometry the result follows.

That (1) implies (3) can be readily verified.

Applying the arguments at the beginning of this proof and Theorem 3.9 we deduce that (3) implies (1). □

Remark 3.11. Let $\mathcal{V}$ be a TRO. It follows from Remark 2.2 and Theorem 3.5 that the weak compactness of $\mathrm{cp}^2_{\mathcal{V}}(a)$ does not imply its norm compactness. On the other hand, we would like to note that the norm compactness and weak compactness of the mapping $u : \mathcal{V} \to \mathcal{V}$, $u(x) = ax^*a$ are equivalent.

Acknowledgements

The author would like to thank Prof. M. Anoussis, whose help, support and suggestions have been instrumental in completing this paper. He would also like to thank the referee for useful suggestions.

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