

ON ISOMORPHISM PROBLEMS
FOR VERTEX OPERATOR ALGEBRAS
ASSOCIATED WITH EVEN LATTICES

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ABSTRACT. In this article, we completely determine the isomorphism classes of lattice vertex operator algebras and the vertex operator subalgebras fixed by a lift of the -1 -isometry of the lattice. We also provide similar results for certain even lattices associated with doubly-even binary codes.

INTRODUCTION

The lattice vertex operator algebra (VOA) V_L associated with an even lattice L is a fundamental example in VOA theory ([Bo86, FLM88]). The VOA V_L has an automorphism of order 2 lifted from the -1 -isometry of L , and the fixed-point subspace V_L^+ is a subVOA of V_L . The famous moonshine VOA V^\natural was constructed in [FLM88] as an extension of V_Λ^+ associated with the Leech lattice Λ . Therefore, if L is a sublattice of Λ , then the VOA V_L^+ is contained in V^\natural . Many group-theoretical properties of the automorphism group of V^\natural , the Monster simple group, can be studied using the subVOA V_L^+ . For instance, certain maximal 2-local subgroups of the Monster simple group were described using V_L^+ ([Sh07, Sh11]).

It is well known that the VOAs V_L and V_N are isomorphic if and only if L and N are isomorphic. However it was shown in [Sh06] that the VOAs $V_{D_{16}^+}$ and $V_{E_8^+}$ are isomorphic, where D_{16}^+ and E_8^+ are (nonisomorphic) even unimodular lattices of rank 16. Hence it is natural to ask if this case is the only exceptional case.

Given a doubly-even binary code C , one can obtain an even lattice $\mathcal{L}(C) = \frac{1}{\sqrt{2}}\rho^{-1}(C)$ and a certain sublattice $\mathcal{L}^+(C)$ of index 2, where ρ is the canonical map from \mathbb{Z}^n to \mathbb{Z}_2^n . The Leech lattice Λ can be constructed as an overlattice of $\mathcal{L}^+(G_{24})$ ([CS99]), where G_{24} is the extended binary Golay code. The construction of V^\natural in [FLM88] is, in some sense, analogous to this construction. Since there are many analogies among binary codes, lattices and VOAs ([Hö95, Hö03a, Hö08, Sh11]), it is natural to ask if the VOAs V_L and V_L^+ would behave like $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$, respectively.

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TABLE 1. Isomorphism problems and the answers

Problem	Answer
$\mathcal{L}(C) \cong \mathcal{L}(D)$	$C \cong D$
$V_L \cong V_N$	$L \cong N$
$\mathcal{L}^+(C) \cong \mathcal{L}^+(D)$	$C \cong D$ or $\{C, D\} = \{e_8^2, d_{16}^+\}$
$V_L^+ \cong V_N^+$	$L \cong N$ or $\{L, N\} = \{E_8^2, D_{16}^+\}$
$\mathcal{L}^+(C) \cong \mathcal{L}(D)$	$C \cong \mathcal{C}(K)$ and $D \cong \mathcal{C}^+(K)$
$V_L^+ \cong V_N$	$L \cong \mathcal{L}(C)$ and $N \cong \mathcal{L}^+(C)$

In this article, we completely determine when V_L^+ and V_N^+ are isomorphic as well as when V_L^+ and V_N are isomorphic. We also obtain similar results for the even lattices $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$ associated with doubly-even binary codes C . As an observation, we notice that the isomorphism types of the VOAs V_L and V_L^+ are closely related to those of the lattices $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$ (see Table 1). Note that $\mathcal{C}(K)$ and $\mathcal{C}^+(K)$ in the table are binary codes obtained from Kleinian codes K ([Hö03a]) and that some results for the even lattices were already given in [KKM91].

Next, let us explain our method for the case $V_L^+ \cong V_N^+$ (Theorem 3.9). We note that $V_L \cong V_N$ if and only if $L \cong N$ (cf. Proposition 3.1), and the case $V_L^+ \cong V_N$ (Theorem 3.10) can be solved easily by Theorem 3.9. Clearly, $L \cong N$ implies $V_L^+ \cong V_N^+$, and it was shown in [Sh06] that $V_{E_8^2}^+ \cong V_{D_{16}^+}$. We assume that $V_L^+ \cong V_N^+$ as VOAs. Let us consider the irreducible simple current module V_N^- . By the assumption, we regard the irreducible V_N^+ -module V_N^- as an irreducible V_L^+ -module M . The key is that the properties of M are similar to those of V_N^- . By the classification of the irreducible modules for V_L^+ ([DN99, AD04]), we have $M \cong V_L^-, V_{\lambda+L}^\varepsilon$ ($\lambda \notin L$, $\varepsilon \in \{\pm\}$) or $V_L^{T_{x,\pm}}$ as V_L^+ -modules.

(i) If $M \cong V_L^-$, then $V_N \cong V_N^+ \oplus V_N^- \cong V_L^+ \oplus V_L^- \cong V_L$ as VOAs and $L \cong N$.

(ii) Assume $M \cong V_{\lambda+L}^\varepsilon$. We may assume that $\varepsilon = +$ up to conjugation ([Sh04]).

Since the characters of V_L^- and $V_{\lambda+L}^+$ are the same, we obtain an equation about the numbers of norm 2 vectors in L and $\lambda + L$. By the characterization of $\mathcal{L}^+(C)$ in [Sh06], $L \cong \mathcal{L}^+(C)$ and $L + \mathbb{Z}\lambda \cong \mathcal{L}(C)$ for some doubly-even binary code C . Moreover, since $V_{\mathcal{L}(C)}^+ \cong V_{\mathcal{L}^+(C)}$ ([FLM88]), we obtain $V_L \cong V_L^+ \oplus V_L^- \cong V_L^+ \oplus V_{\lambda+L}^+ \cong V_N$ as VOAs, and hence $L \cong N$.

(iii) Assume $M \cong V_L^{T_{x,+}}$. Since the lowest weights are the same, the rank of L is 8. Since M is a self-dual simple current, $\sqrt{2}L^*$ is even. Hence L contains a sublattice isomorphic to $\sqrt{2}E_8$. By the same arguments for N , N has the same property. Since the discriminant groups of L and N are isomorphic, we have $L \cong N$.

(iv) Assume $M \cong V_L^{T_{x,-}}$. Then the rank of L is 16. Moreover, both L and N are unimodular, or $L \cong \mathcal{L}^+(C)$, $N \cong \mathcal{L}^+(D)$ for some doubly-even binary codes C and D with the same weight enumerator. The first case is an exceptional case. In the latter case, either both C and D are self-dual, or $C \cong \mathcal{C}^+(K)$, $D \cong \mathcal{C}^+(J)$ for some Kleinian codes K and J with the same weight enumerator. If both C and D are self-dual, then $L \cong \mathcal{L}^+(C) \cong \mathcal{L}^+(D) \cong N$. By the classification of even Kleinian codes of small length ([Hö03a]), we have $C \cong \mathcal{C}^+(K) \cong \mathcal{C}^+(J) \cong D$. Hence $L \cong N$.

This article also contains proofs for the corresponding results in lattices. These should be useful for understanding the arguments in VOAs, since the approach in lattices is quite similar to that in VOAs.

1. PRELIMINARIES

In this section, we recall or give some definitions and facts required in this article.

1.1. Kleinian codes. In this subsection, we recall the basic definitions for Kleinian codes (cf. [Hö03a]).

Denote the elements of the Kleinian four group $\mathbb{K} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by $0, a, b, c$, where 0 is the identity. A (linear) *Kleinian code* K of length n is a subgroup of $\mathbb{K}^n \cong \mathbb{Z}_2^{2n}$. The dimension of K is t if K has 4^t elements, where $t \in \frac{1}{2}\mathbb{Z}$. The *weight* $\text{wt}(x)$ of $x = (x_i) \in \mathbb{K}^n$ is the number of nonzero x_i . For $m \in \mathbb{Z}$, let $K(m)$ denote the set of all elements of weight m in K . A Kleinian code K is *even* if $\text{wt}(k) \in 2\mathbb{Z}$ for all $k \in K$. The symmetric bilinear product $\mathbb{K}^n \times \mathbb{K}^n \rightarrow \mathbb{F}_2$ is defined by $x \cdot y = \sum_{i=1}^n x_i \cdot y_i$, where $a \cdot b = b \cdot a = 1$, $a \cdot c = c \cdot a = 1$, $c \cdot b = b \cdot c = 1$, and zero otherwise. A Kleinian code is *self-dual* if it is equal to its orthogonal complement. Two Kleinian codes are *equivalent* if one of them is obtained from the other by a permutation of the coordinates together with a permutation of the symbols a, b and c at each coordinate. The *weight enumerator* of K is defined by

$$W_K(X, Y) = \sum_{k \in K} X^{\text{wt}(k)} Y^{n - \text{wt}(k)}.$$

Up to length 8, self-dual Kleinian codes were classified in [Hö03a].

Lemma 1.1 ([Hö03a]).

- (1) *There is exactly one even self-dual Kleinian code of length 2, up to equivalence. It is equivalent to ϵ_2 generated by (aa) and (bb).*
- (2) *There are exactly two even self-dual Kleinian codes of length 4, up to equivalence. They are equivalent to ϵ_2^2 and δ_4^+ , where δ_4^+ is generated by (aa00), (a0a0), (a00a), and (bbbb).*

1.2. Binary codes. In this subsection, we recall the basic definitions for binary codes (cf. [CS99]).

A *binary (linear) code* of length n is a subspace of \mathbb{F}_2^n . The *weight* $\text{wt}(x)$ of $x = (x_i) \in \mathbb{F}_2^n$ is the number of nonzero x_i . For a subset D of \mathbb{F}_2^n , we denote the set of all elements in D of weight m by $D(m)$. A binary code C is *doubly-even* if $\text{wt}(x) \in 4\mathbb{Z}$ for all $x \in C$. The *dual code* C^\perp of C is defined by $C^\perp = \{x \in \mathbb{F}_2^n \mid \langle x, C \rangle = 0\}$, where $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. A binary code C is *self-dual* if $C = C^\perp$. Two binary codes are *equivalent* if one of them is obtained from the other by a permutation of the coordinates. The *weight enumerator* of a coset $x + C \in C^\perp/C$ is defined by

$$W_{x+C}(X, Y) = \sum_{y \in x+C} X^{\text{wt}(y)} Y^{n - \text{wt}(y)} = \sum_{m=0}^n |(x+C)(m)| X^m Y^{n-m}.$$

We refer to [CS99] for the details about the binary codes e_8 and d_{16}^+ .

Lemma 1.2 (cf. [CS99]).

- (1) *The extended Hamming code e_8 is the unique doubly-even self-dual code of length 8, up to equivalence.*

- (2) *There are exactly two doubly-even self-dual codes of length 16, up to equivalence. They are equivalent to e_8^2 and d_{16}^+ .*

1.3. Lattices. In this subsection, we recall the basic definitions for lattices (cf. [CS99]).

Let (\cdot, \cdot) be a positive-definite symmetric bilinear form on \mathbb{R}^n . The *norm* of $v \in \mathbb{R}^n$ is (v, v) . For a subset $U \subset \mathbb{R}^n$, let us denote by $U(m)$ the set of all vectors of norm m in U . A subset L of \mathbb{R}^n is a (positive-definite) *lattice* of rank n if there is a basis e_1, e_2, \dots, e_n of \mathbb{R}^n satisfying $L = \bigoplus_{i=1}^n \mathbb{Z}e_i$. The *dual lattice* L^* of L is defined by $L^* = \{u \in \mathbb{R}^n \mid (u, L) \subset \mathbb{Z}\}$. The *discriminant group* of L is the quotient group L^*/L . A lattice L is *even* if $(v, v) \in 2\mathbb{Z}$ for all $v \in L$, and L is *unimodular* if $L = L^*$. Two lattices are *isomorphic* if one of them is obtained from the other by an orthogonal transformation of \mathbb{R}^n . The *theta series* of a coset $\lambda + L \in L^*/L$ is defined by

$$\Theta_{\lambda+L}(q) = \sum_{v \in \lambda+L} q^{(v,v)/2} = \sum_{m=0}^{\infty} |(\lambda+L)(m)| q^{m/2}.$$

We refer to [CS99] for the details about the lattices E_8 and D_{16}^+ .

Lemma 1.3 (cf. [Wi41, CS99]).

- (1) *The root lattice E_8 is the unique even unimodular lattice of rank 8, up to isomorphism.*
 (2) *There are exactly two even unimodular lattices of rank 16, up to isomorphism. They are isomorphic to E_8^2 and D_{16}^+ .*

1.4. Vertex operator algebras. In this subsection, we recall the basic definitions for vertex operator algebras (cf. [Bo86, FLM88, FHL93]). Throughout this article, all VOAs are defined over the field \mathbb{C} of complex numbers.

A *vertex operator algebra* (VOA) V is a $\mathbb{Z}_{\geq 0}$ -graded vector space $V = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_m$ that is equipped with a linear map $Y : V \rightarrow (\text{End}(V))[[z, z^{-1}]]$, $v \mapsto Y(v, z) = \sum_{i \in \mathbb{Z}} a_i z^{-i-1}$ and nonzero vectors $\mathbf{1}_V$ and ω_V satisfying a number of conditions ([Bo86, FLM88]). We often denote it by (V, Y) or V . Two VOAs (V, Y) and (V', Y') are said to be *isomorphic* if there exists a linear isomorphism g from V to V' such that $gY(v, z)w = Y'(gv, z)gw$ for all $v, w \in V$ and $g(\omega_V) = \omega_{V'}$.

An (ordinary) module (M, Y_M) for a VOA V is a \mathbb{C} -graded vector space $M = \bigoplus_{m \in \mathbb{C}} M_m$ equipped with a linear map $Y_M : V \rightarrow (\text{End}(M))[[z, z^{-1}]]$ satisfying a number of conditions ([FHL93]). We often denote it by M and its isomorphism class by $[M]$. The *weight* of a homogeneous vector $v \in M_k$ is k . The *character* of M is defined by

$$\text{ch}_M(q) = q^{-n/24} \sum_{m \in \mathbb{C}} \dim M_m q^m,$$

where n is the central charge of V . A module is *self-dual* if its contragredient module ([FHL93]) is isomorphic to itself. If M is irreducible, then there exists $h \in \mathbb{C}$ such that $M = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} M_{h+m}$ and $M_h \neq 0$. The number h is called the *lowest weight* of M .

A VOA V is of *CFT type* if $V_0 = \mathbb{C}\mathbf{1}$, and V is *C_2 -cofinite* if $V/\text{Span}_{\mathbb{C}}\{a_{-2}b \mid a, b \in V\}$ is finite-dimensional. A VOA V is *simple* if it is an irreducible V -module, and V is *rational* if any module is completely reducible. A rational simple VOA V is *holomorphic* if any irreducible module is isomorphic to V .

The following was proved in [DM04b]. For the lattice VOA V_L , see Section 1.5.

Lemma 1.4 ([DM04b]).

- (1) The lattice VOA V_{E_8} is the unique C_2 -cofinite holomorphic VOA of CFT type of central charge 8 up to isomorphism.
- (2) There are exactly two C_2 -cofinite holomorphic VOAs of CFT type of central charge 16 up to isomorphism. They are isomorphic to $V_{E_8^2}$ and $V_{D_{16}^+}$.

Let M_a, M_b, M_c be modules for a simple VOA V . Let $N_{M_a, M_b}^{M_c}$ denote the dimension of the space of all intertwining operators of type $M_a \times M_b \rightarrow M_c$, which is called the fusion rule ([FHL93]). By the definition of the fusion rules, $N_{M_a, M_b}^{M_c} = N_{N_a, N_b}^{N_c}$ if $M_x \cong N_x$ as V -modules for all $x = a, b, c$. Hence, the fusion rules are determined by the isomorphism classes of V -modules. Let $R(V)$ denote the set of all isomorphism classes of irreducible V -modules. For convenience, we use the following expression:

$$[M_a] \times [M_b] = \sum_{[M] \in R(V)} N_{M_a, M_b}^M [M],$$

which is also called the fusion rule.

Let $V(0)$ be a simple VOA. An irreducible $V(0)$ -module M^1 is called a *simple current* if for any irreducible $V(0)$ -module M^2 , there exists a unique irreducible $V(0)$ -module M^3 such that $[M^1] \times [M^2] = [M^3]$. A simple VOA V is called a *simple current extension* of $V(0)$ graded by a finite abelian group A if V is the direct sum of nonisomorphic irreducible simple current $V(0)$ -modules $\{V(\alpha) \mid \alpha \in A\}$ and for all $\alpha, \beta \in A$, the fusion rule $[V(\alpha)] \times [V(\beta)] = [V(\alpha + \beta)]$ holds. The uniqueness of simple current extensions was established as follows.

Proposition 1.5 ([DM04, Proposition 4.3] (cf. [Hö03b, Theorem 4.3])). *Let (V, Y) be a simple current extension of a simple VOA $V(0)$. Then the VOA structure of V as a simple current extension of $V(0)$ is uniquely determined by the $V(0)$ -module structure of V ; that is, if (\tilde{V}, \tilde{Y}) has a VOA structure with $V = \tilde{V}$ as vector spaces and $Y(v, z) = \tilde{Y}(v, z)$ for all $v \in V(0)$, then the VOAs (V, Y) and (\tilde{V}, \tilde{Y}) are isomorphic.*

When every irreducible V -module is a simple current, \times is a binary operation on $R(V)$. In addition, if \times satisfies the associative law, then $(R(V), \times)$ is a group, which we call the *fusion group* of V .

1.5. Constructions of binary codes, lattices and VOAs. First, let us consider the binary codes $\mathcal{C}(K)$ and $\mathcal{C}^+(K)$ of length $4m$ obtained from a Kleinian code K of length m [Hö03a, Section 7].

Construction A:

$$(1.1) \quad \mathcal{C}(K) = \hat{K} + d_4^m,$$

where $\hat{\cdot} : \mathbb{K}^m \rightarrow \mathbb{F}_2^{4m}$ is the map induced from $\mathbb{K} \rightarrow \mathbb{F}_2^4, 0 \mapsto (0000), a \mapsto (1100), b \mapsto (1010), c \mapsto (0110)$, and $d_4^m = \{(0000), (1111)\}^m$.

Construction B:

$$(1.2) \quad \mathcal{C}^+(K) = \hat{K} + (d_4^m)_0,$$

where $\hat{\cdot} : \mathbb{K}^m \rightarrow \mathbb{F}_2^{4m}$ is defined as before and $(d_4^m)_0$ is the subcode of d_4^m consisting of vectors of weight divisible by 8.

Lemma 1.6 (cf. [Hö03a, Lemmas 2 and 3]). *Let K be a Kleinian code of length m .*

- (1) If K is self-dual, then so is $\mathcal{C}(K)$.
- (2) If K is even, then both $\mathcal{C}(K)$ and $\mathcal{C}^+(K)$ are doubly-even.
- (3) The weight enumerators of $\mathcal{C}(K)$ and $\mathcal{C}^+(K)$ are given as follows:

$$\begin{aligned} W_{\mathcal{C}(K)}(X, Y) &= W_K(X^4 + Y^4, 2X^2Y^2), \\ W_{\mathcal{C}^+(K)}(X, Y) &= \frac{1}{2} (W_K(X^4 + Y^4, 2X^2Y^2) + (X^4 - Y^4)^m). \end{aligned}$$

Let us consider some examples.

Lemma 1.7.

- (1) The binary code $\mathcal{C}(\epsilon_2)$ is equivalent to e_8 .
- (2) The binary codes $\mathcal{C}(\epsilon_2^2)$ and $\mathcal{C}(\delta_4^+)$ are equivalent to e_8^2 and d_{16}^+ , respectively.
- (3) The binary codes $\mathcal{C}^+(\epsilon_2^2)$ and $\mathcal{C}^+(\delta_4^+)$ are equivalent.

Next, let us consider the lattices $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$ of rank n obtained from a binary code C of length n ([CS99]). Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be an orthogonal basis of \mathbb{R}^n of norm 2.

Construction A:

$$(1.3) \quad \mathcal{L}(C) = \sum_{i=1}^n \mathbb{Z}\alpha_i + \sum_{c \in C} \mathbb{Z} \frac{1}{2} \alpha_c,$$

where $\alpha_c = \sum_{i=1}^n c_i \alpha_i$ and $c_i \in \{0, 1\}$.

Construction B:

$$(1.4) \quad \mathcal{L}^+(C) = \sum_{1 \leq i, j \leq n} \mathbb{Z}(\alpha_i + \alpha_j) + \sum_{c \in C} \mathbb{Z} \frac{1}{2} \alpha_c,$$

where α_c is defined as before.

Lemma 1.8 ([CS99]). *Let C be a binary code of length n .*

- (1) If C is self-dual, then $\mathcal{L}(C)$ is unimodular.
- (2) If C is doubly-even, then both $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$ are even.
- (3) The theta series of $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$ are given as follows:

$$\begin{aligned} \Theta_{\mathcal{L}(C)}(q) &= W_C(\theta_3(q), \theta_2(q)), \\ \Theta_{\mathcal{L}^+(C)}(q) &= \frac{1}{2} (W_C(\theta_3(q), \theta_2(q)) + \theta_4(q)^n), \end{aligned}$$

where $\theta_3(q) = \sum_{i \in \mathbb{Z}} q^{i^2}$, $\theta_2(q) = \sum_{i \in \mathbb{Z}} q^{(i+1/2)^2}$ and $\theta_4(q) = \sum_{i \in \mathbb{Z}} (-1)^i q^{i^2}$.

Let us consider some examples. For (3), see [KKM91, (5.3.1)].

Lemma 1.9.

- (1) The even lattice $\mathcal{L}(e_8)$ is isomorphic to E_8 .
- (2) The lattices $\mathcal{L}(e_8^2)$ and $\mathcal{L}(d_{16}^+)$ are isomorphic to E_8^2 and D_{16}^+ , respectively.
- (3) The lattices $\mathcal{L}^+(e_8^2)$ and $\mathcal{L}^+(d_{16}^+)$ are isomorphic.

Finally, we consider the VOAs V_L and V_L^+ of central charge n obtained from an even lattice L of rank n ([Bo86, FLM88]).

Set $\mathfrak{h}_L = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend (\cdot, \cdot) to a symmetric \mathbb{C} -bilinear form on \mathfrak{h}_L . We view $\mathfrak{h}_L = \mathbb{C} \otimes_{\mathbb{Z}} L$ as an abelian Lie algebra. Let $\hat{\mathfrak{h}}_L = \mathfrak{h}_L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$ be its affine Lie algebra. Set $\hat{\mathfrak{h}}_L^- = \mathfrak{h}_L \otimes t^{-1}\mathbb{C}[t^{-1}]$ and let $S(\hat{\mathfrak{h}}_L^-)$ be the symmetric algebra of

$\hat{\mathfrak{h}}_L^-$. Then $S(\hat{\mathfrak{h}}_L^-) \cong \mathbb{C}[\alpha \otimes t^n \mid \alpha \in \mathfrak{h}_L, n < 0] \cdot \mathbf{1}$ is the unique irreducible $\hat{\mathfrak{h}}_L^-$ -module such that \mathbf{c} acts as 1 and $\alpha \otimes t^n \cdot \mathbf{1} = 0$ if $\alpha \in \mathfrak{h}_L$ and $n \geq 0$.

Let us consider the twisted group algebra of the additive group L . Let $\langle \kappa_L \rangle$ be a cyclic group of order 2 and let

$$(1.5) \quad 1 \rightarrow \langle \kappa_L \rangle \rightarrow \hat{L} \xrightarrow{\rho} L \rightarrow 1$$

be a central extension of L by $\langle \kappa_L \rangle$ with the commutator map $c_0(\alpha, \beta) = (\alpha, \beta) \pmod{2}$, $\alpha, \beta \in L$. Then we obtain the twisted group algebra $\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] / (\kappa_L + 1)$, where $\mathbb{C}[\hat{L}]$ is the usual group algebra of the group \hat{L} . The lattice VOA V_L associated with L is defined to be $V_L = S(\hat{\mathfrak{h}}_L^-) \otimes \mathbb{C}\{L\}$ as a vector space. We refer to [FLM88] for the definition of the vertex operator on V_L . Note that the central charge of V_L is n .

Let θ_{V_L} be an automorphism of V_L of order 2 which is a lift of the -1 -isometry of L . Let V_L^+ denote the subspace of V_L fixed by θ_{V_L} . Then V_L^+ is a vertex operator subalgebra of V_L . Note that the automorphism θ_{V_L} is not unique, but the isomorphism type of V_L^+ is independent of the choice of θ_{V_L} [DGH98, Appendix D].

As a summary, we obtain two VOAs of central charge n :

$$V_L = S(\hat{\mathfrak{h}}_L^-) \otimes \mathbb{C}\{L\} \quad \text{and} \quad V_L^+ = \{v \in V_L \mid \theta_{V_L}(v) = v\}.$$

We call the constructions of V_L and V_L^+ from an even lattice L *Construction A* and *Construction B* respectively.

Lemma 1.10 ([Do93, FLM88]). *Let L be an even lattice of rank n .*

- (1) *If L is unimodular, then V_L is holomorphic.*
- (2) *The characters of V_L and V_L^+ are given as follows:*

$$\begin{aligned} \text{ch}_{V_L}(q) &= \frac{\Theta_L(q)}{\eta(q)^n}, \\ \text{ch}_{V_L^+}(q) &= \frac{1}{2} \left(\frac{\Theta_L(q)}{\eta(q)^n} + \frac{\eta(q)^n}{\eta(2q)^n} \right), \end{aligned}$$

where $\eta(q) = q^{1/24} \prod_{i=1}^{\infty} (1 - q^i)$.

The following lemma was verified in [Sh06, Lemma 3.4].

Lemma 1.11. *The VOAs $V_{E_8}^+$ and $V_{D_{16}^+}^+$ are isomorphic.*

Now, we recall some facts about irreducible modules for V_L^+ ([FLM88, DN99, AD04]). Let V_L^- be the -1 -eigenspace of θ_{V_L} in V_L . Then V_L^- is an irreducible V_L^+ -module. For a coset $\lambda + L \in L^*/L$, $V_{\lambda+L} = S(\hat{\mathfrak{h}}_L^-) \otimes \mathbb{C}[\lambda + L]$ is an irreducible V_L^- -module. If $\lambda \notin L/2$, then $V_{\lambda+L}$ is also irreducible for V_L^+ . If $\lambda \in L/2$, then $V_{\lambda+L}$ is the direct sum of two irreducible V_L^+ -submodules $V_{\lambda+L}^+$ and $V_{\lambda+L}^-$. Set $J = \{a^{-1}\theta_{V_L}(a) \mid a \in \hat{L}\}$. Then J is a normal subgroup of \hat{L} . Let T_χ be an irreducible module for \hat{L}/J with character χ satisfying $\chi(\kappa J) = -1$. Then there are two irreducible V_L^+ -modules $V_L^{T_\chi, \pm}$ associated with T_χ .

It was shown in [DN99, AD04] that any irreducible V_L^+ -module is isomorphic to one of $V_{\lambda+L}^\pm$ ($\lambda \in L^* \cap (L/2)$), $V_{\mu+L}$ ($\mu \in L^* \setminus (L/2)$) and $V_L^{T_\chi, \pm}$. By the fusion rules for irreducible V_L^+ -modules [Ab01, ADL05], one can obtain the following lemma.

Lemma 1.12. *The irreducible V_L^+ -module $V_{\mu+L}$ ($\mu \in L^* \setminus (L/2)$) is not a simple current.*

The characters of $V_{\lambda+L}^\pm$ and $V_L^{T_X, \pm}$ are given as follows ([FLM88]):

$$(1.6) \quad \text{ch}_{V_{\lambda+L}^\pm}(q) = \frac{1}{2} \left(\frac{\Theta_{\lambda+L}(q)}{\eta(q)^n} \right) \quad (\lambda \notin L),$$

$$(1.7) \quad \text{ch}_{V_L^{T_X, \pm}}(q) = \frac{\dim T_X}{2} \left(\frac{\eta(q)^n}{\eta(q^{1/2})^n} \pm \frac{\eta(q^2)^n \eta(q^{1/2})^n}{\eta(q)^{2n}} \right).$$

2. ISOMORPHISM PROBLEMS FOR THE EVEN LATTICES $\mathcal{L}(C)$ AND $\mathcal{L}^+(C)$

2.1. A characterization of doubly-even binary codes obtained from Kleinian codes. In this subsection, we characterize the doubly-even binary codes obtained by Construction B from Kleinian codes, which will play an important role in Sections 2.3 and 2.4. Note that this is a code analogue of the characterization of the even lattices obtained by Construction B (see Proposition 3.2) given in [Sh06].

Let $u_i = (0^{4(i-1)}1^{40^{n-4i}}) \in \mathbb{F}_2^n$ for $1 \leq i \leq n/4$. If $m = n/4 \in \mathbb{Z}$, then $d_4^m = \text{Span}_{\mathbb{F}_2} \{u_i \mid 1 \leq i \leq m\}$ and $(d_4^m)_0 = \text{Span}_{\mathbb{F}_2} \{u_i + u_j \mid 1 \leq i, j \leq m\}$. For our purpose, we need the following lemma.

Lemma 2.1. *Let C be a doubly-even binary code of length $n = 4m \in 4\mathbb{Z}$.*

- (1) *If $d_4^m \subset C$, then $C \cong \mathcal{C}(K)$ for some even Kleinian code K of length m .*
- (2) *Assume that $(d_4^m)_0 \subset C$, $u_1 \notin C$ and $u_1 \in C^\perp$. Then $C \cong \mathcal{C}^+(K)$ and $C + \mathbb{F}_2 u_1 \cong \mathcal{C}(K)$ for some even Kleinian code K of length m .*

Proof. (1) Let $x = (x_1, x_2, \dots, x_m) \in C$, where $x_i \in \mathbb{F}_2^4$. Since C is doubly even, $\langle x, u_i \rangle = \langle x_i, (1111) \rangle = 0$; equivalently, $\text{wt}(x_i) \in 2\mathbb{Z}$. It follows from $\hat{\mathbb{K}} + \mathbb{F}_2(1111) = \{v \in \mathbb{F}_2^4 \mid \text{wt}(v) \in 2\mathbb{Z}\}$ that $x_i \in \hat{\mathbb{K}}$ or $x_i + (1111) \in \hat{\mathbb{K}}$. For the map $\hat{\cdot}$, see (1.1). Hence there is a unique $k_x \in \mathbb{K}^m$ such that $x \in \hat{k}_x + d_4^m$. Set $K = \{k_x \mid x \in C\}$. Then $C \subset \mathcal{C}(K)$. It follows from $d_4^m \subset C$ that $\hat{k}_x \in C$. Thus by (1.1) $C \cong \mathcal{C}(K)$. Since C is a doubly-even binary code, K is an even Kleinian code.

(2) Set $\tilde{C} = C + \mathbb{F}_2 u_1$. It follows from $u_1 \in C^\perp$ and $\text{wt}(u_1) = 4$ that \tilde{C} is doubly-even. Since $u_1 \notin C$, we have $C \subsetneq \tilde{C}$. Let $y = (y_1, y_2, \dots, y_m) \in C^\perp \setminus \tilde{C}^\perp$, where $y_i \in \mathbb{F}_2^4$. It follows from $\langle y, u_i \rangle \neq 0$ and $\langle y, u_i + u_j \rangle = 0$ for all i, j that $\text{wt}(y_i) = 1$ or 3 for all i . Since \tilde{C} is doubly-even, $\tilde{C} \subset \tilde{C}^\perp$. Hence $u_i \in \tilde{C}^\perp$ for any i , and we may assume that $\text{wt}(y_i) = 1$ for all i , and $y = (00010001 \dots 0001)$ up to coordinate permutation.

Let $x = (x_1, x_2, \dots, x_m) \in C$, where $x_i \in \mathbb{F}_2^4$. Since $u_i \in C^\perp$ for all i , we have $\langle x_i, (1111) \rangle = 0$. Hence there is $k_x \in \mathbb{K}^m$ such that $x \in \hat{k}_x + d_4^m$. Since $\langle x, y \rangle = \sum_{i=1}^m \langle x_i, (0001) \rangle = 0$ and $\langle \hat{k}_x, y \rangle = 0$, we have $x \in \hat{k}_x + (d_4^m)_0$. By the same argument as in (1), $K = \{k_x \mid x \in C\}$ is an even Kleinian code and $C \cong \mathcal{C}^+(K)$ (cf. (1.2)). Moreover, $C + \mathbb{F}_2 u_1 \cong \mathcal{C}(K)$.

Proposition 2.2. *Let C be a doubly-even binary code of length n . If there exists a coset $x + C \in C^\perp/C$ satisfying $|(x + C)(4)| \geq n/4 + |C(4)|$, then $C \cong \mathcal{C}^+(K)$ and $C + \mathbb{F}_2 x \cong \mathcal{C}(K)$ for some even Kleinian code K of length $n/4$.*

Proof. In order to apply Lemma 2.1 (2), we will show that the coset $x + C$ contains $\{u_i \mid 1 \leq i \leq n/4\}$. Assume $\{u_i \mid 1 \leq i \leq r\} \subset x + C$ and $r < n/4$. We claim that the coset $x + C$ contains $(0^{4r}, z)$ with $\text{wt}(z) = 4$.

Let $v = (v_1, v_2, \dots, v_r, z) \in (x + C)(4)$, where $v_i \in \mathbb{F}_2^4$ and $z \in \mathbb{F}_2^{n-4r}$. Then $v + u_i \in C$. It follows from $u_i \in C^\perp$ that $\langle v + u_i, u_i \rangle = \langle v_i, (1111) \rangle = 0$; equivalently,

$\text{wt}(v_i) \in 2\mathbb{Z}$. If $\text{wt}(v_i) = 4$ for some i , then $v = u_i$ since $\text{wt}(v) = 4$. Set

$$Y = \{(y_1, y_2, \dots, y_r, z) \in (x + C)(4) \mid \text{wt}(y_i) = 2 \text{ for some } 1 \leq i \leq r\}.$$

For $y \in Y$, let $m(y) = \min\{i \mid \text{wt}(y_i) = 2\}$. In order to give an upper bound for $|Y|$, we consider the map $\psi : Y \rightarrow C(4)$, $y \mapsto y + u_{m(y)}$ and show its injectivity. Assume $\psi(y) = \psi(y')$. Then $y + u_{m(y)} = y' + u_{m(y')}$; equivalently, $y + y' = u_{m(y)} + u_{m(y')}$. By the definition of u_i , $\text{wt}(u_{m(y)} + u_{m(y')}) = 0$ or 8 . If $\text{wt}(u_{m(y)} + u_{m(y')}) = 8$, then $m(y) \neq m(y')$. However, by $\text{wt}(y + y') = 8$, we have $\text{wt}(y_{m(y)}) = \text{wt}(y_{m(y')}) = \text{wt}(y'_{m(y)}) = \text{wt}(y'_{m(y')}) = 2$, and $m(y) = m(y')$, which is a contradiction. Hence $\text{wt}(u_{m(y)} + u_{m(y')}) = 0$; equivalently, $y = y'$. Thus ψ is injective. In particular, $|Y| \leq |C(4)|$.

By the definition of Y , we have

$$\{(0, \dots, 0, z) \in (x + C)(4) \mid z \in \mathbb{F}_2^{n-4r}\} = (x + C)(4) \setminus (Y \cup \{u_i \mid 1 \leq i \leq r\}).$$

By the upper bound of $|Y|$,

$$|(x + C)(4) \setminus (Y \cup \{u_i \mid 1 \leq i \leq r\})| \geq |(x + C)(4)| - |C(4)| - r,$$

and it is greater than 0 by $r < n/4$ and the claim follows. Hence there is an element $(0^{4r}, z) \in x + C$ with $\text{wt}(z) = 4$. We define it to be u_{r+1} up to coordinate permutation. Thus the coset $x + C$ contains $\{u_i \mid 1 \leq i \leq n/4\}$ up to coordinate permutation, and $n/4$ must be an integer. Moreover, $u_i + u_j \in C$ for all i, j . Hence $u_1 \in C^\perp$, $u_1 \notin C$, and $\text{Span}_{\mathbb{F}_2}\{u_i + u_j \mid 1 \leq i, j \leq m\} = (d_4^m)_0 \subset C$. Therefore the proposition follows from Lemma 2.1 (2).

Remark 2.3. If $C = \mathcal{C}^+(K)$, then $|((1^{40^{n-4}}) + C)(4)| = n/4 + |C(4)|$.

2.2. Even lattices $\mathcal{L}(C)$. In this subsection, we study the isomorphism classes of $\mathcal{L}(C)$. The following proposition is well known (cf. [CS99, Chapter 18, Proposition 4]). However, we give a sketch of a proof since it would be useful for understanding the arguments in the VOA case (see Section 3.1).

Proposition 2.4. *Let C and D be doubly-even binary codes of length n . Then $\mathcal{L}(C) \cong \mathcal{L}(D)$ if and only if $C \cong D$.*

Proof. The “if” part is obvious. For the “only” part, we assume that $\mathcal{L}(C) \cong \mathcal{L}(D)$. One can verify that the Weyl group of the norm 2 vectors in $\mathcal{L}(C)$ is transitive on the set of all orthogonal bases of norm 2 in $\mathcal{L}(C)$. Hence we may assume that the canonical bases $\{\alpha_i \mid 1 \leq i \leq n\}$ of $\mathcal{L}(C)$ and $\mathcal{L}(D)$ in (1.3) are the same up to automorphism. Since the binary code C can be recovered from $\{\alpha_i \mid 1 \leq i \leq n\}$, we obtain $C \cong D$.

2.3. Even lattices $\mathcal{L}^+(C)$. In this subsection, we study the isomorphism classes of $\mathcal{L}^+(C)$.

The following lemma can be easily proved by using the quadratic spaces of plus type over \mathbb{F}_2 associated to doubly-even binary codes of length $8k$ containing (1^{8k}) .

Lemma 2.5. *For any doubly-even binary code of length $8k$, there exists a doubly-even self-dual binary code of length $8k$ containing it.*

By (1.2), $(d_4^m)_0 \subset \mathcal{C}^+(K)$ and $(00010001 \dots 0001) \in \mathcal{C}^+(K)^\perp$. Hence by [KKM91, Lemma 3.3.2], we obtain the following lemma.

Lemma 2.6. *Let K be an even Kleinian code of length m . Let ρ be the orthogonal transformation of \mathbb{R}^{4m} defined by*

$$\rho(\alpha_j) = \begin{cases} \frac{1}{2}(\alpha_{4i-3} + \alpha_{4i-2} + \alpha_{4i-1} + \alpha_{4i}) & \text{if } j = 4i, \\ \frac{1}{2}(-\alpha_{4i-3} - \alpha_{4i-2} - \alpha_{4i-1} + \alpha_{4i}) + \alpha_j & \text{if } 4i - 3 \leq j \leq 4i - 1, \end{cases}$$

where $\{\alpha_i \mid 1 \leq i \leq 4m\}$ is the basis of \mathbb{R}^{4m} as defined in (1.3) and (1.4). Then ρ is an isomorphism of lattices from $\mathcal{L}(\mathcal{C}^+(K))$ to $\mathcal{L}^+(\mathcal{C}(K))$.

The following proposition was shown in [KKM91] if $n > 16$, and the case $n \leq 16$ is a finite problem. However, we give a proof here since it is useful for understanding the VOA case (Theorem 3.9).

Theorem 2.7 (cf. [KKM91, Theorem 3]). *Let C and D be doubly-even binary codes of length n . Then $\mathcal{L}^+(C) \cong \mathcal{L}^+(D)$ if and only if one of the following holds.*

- (1) C and D are equivalent.
- (2) C and D are doubly-even self-dual binary codes of length 16.

Proof. The “if” part follows from Lemmas 1.2 (2) and 1.9 (3). For the “only” part, we assume that $\mathcal{L}^+(C) \cong \mathcal{L}^+(D)$. Then by Lemma 1.8 (3) and the fact that θ_2 and θ_3 are algebraically independent ([Eb02, Section 2.9]), $W_C(X, Y) = W_D(X, Y)$. Since $\mathcal{L}^+(C) \cong \mathcal{L}^+(D)$, there is an isomorphism $\varphi : \mathcal{L}^+(D)^*/\mathcal{L}^+(D) \rightarrow \mathcal{L}^+(C)^*/\mathcal{L}^+(C)$ of their discriminant groups. Let $\lambda + \mathcal{L}^+(C) = \varphi(\alpha_1 + \mathcal{L}^+(D))$, where $\{\alpha_i \mid 1 \leq i \leq n\}$ is the canonical basis of \mathbb{R}^n as defined in (1.3) and (1.4). Without loss of generality, we assume that $\langle \lambda, \lambda \rangle = 2$. By (1.4),

$$\mathcal{L}^+(C)^* = \mathcal{L}(C^\perp) + \mathbb{Z}\frac{1}{4}\alpha_{(1^n)}.$$

For $c = (c_i) \in \mathbb{F}_2^n$, let ε_c be the orthogonal transformation defined by $\varepsilon_c(\alpha_i) = (-1)^{c_i}\alpha_i$. Then ε_c is an automorphism of $\mathcal{L}^+(C)$ if $c \in C^\perp$. By the actions of ε_c , we may assume that $\lambda = \alpha_1, \alpha_y/2$ ($y \in C^\perp(4) \setminus C$), $\alpha_{(1^n)}/4 - \alpha_1$ or $\alpha_{(1^n)}/4$.

- (i) Assume that $\lambda = \alpha_1$. Then $\mathcal{L}(C) \cong \mathcal{L}(D)$, and by Proposition 2.4, $C \cong D$.
- (ii) Assume that $\lambda = \alpha_y/2$. Comparing the numbers of vectors of norm 2 in $\alpha_1 + \mathcal{L}^+(D)$ and $\lambda + \mathcal{L}^+(C)$, we obtain $2^3|D(4)| + 2n = 2^3|(y + C)(4)|$. By $W_C(X, Y) = W_D(X, Y)$, we have $|C(4)| = |D(4)|$. Combining the equations, we obtain

$$2^3|C(4)| + 2n = 2^3|(y + C)(4)|.$$

Then by Proposition 2.2, $C \cong \mathcal{C}^+(K)$ and $C + \mathbb{F}_2y \cong \mathcal{C}(K)$ for some even Kleinian code K of length $n/4$. Hence

$$\mathcal{L}(D) = \mathcal{L}^+(D) \cup (\alpha_1 + \mathcal{L}^+(D)) \cong \mathcal{L}^+(C) \cup (\lambda + \mathcal{L}^+(C)) \cong \mathcal{L}^+(\mathcal{C}(K)) \cong \mathcal{L}(C),$$

where the last isomorphism is given in Lemma 2.6. Thus by Proposition 2.4, $C \cong D$.

- (iii) Assume that $\lambda = \alpha_{(1^n)}/4 - \alpha_1$. Since the norm of λ is 2, n must be 8. Since $2\lambda \in \mathcal{L}^+(C)$, C contains (1^8) . It follows from $W_C(X, Y) = W_D(X, Y)$ that D also has (1^8) and $\dim C = \dim D$. One easily sees that there are only four doubly-even binary codes of length 8 containing (1^8) up to equivalence, and those are determined by the dimensions. Hence $C \cong D$.

- (iv) Assume that $\lambda = \alpha_{(1^n)}/4$. Since the norm of λ is 2, n must be 16. Let k be the dimension of C . Then $C^\perp/C \cong \mathbb{Z}_2^{16-2k}$. By $|(\alpha_1 + \mathcal{L}^+(C))(2)| = |(\lambda + \mathcal{L}^+(C))(2)|$, we obtain $2^3|C(4)| + 2^5 = 2^k$; equivalently,

$$(2.1) \quad |C(4)| = 2^{k-3} - 4.$$

If C is self-dual, then so is D , and hence (2) holds. Assume that C is not self-dual. Then by $|C(4)| \geq 0$, we have $5 \leq k \leq 7$. By Lemma 2.5, there is a doubly-even self-dual binary code E of length 16 such that $C \subsetneq E$. By Lemma 1.3 (2), $E \cong d_{16}^+$ or e_8^2 , and $|E(4)| = 28$. Hence there exists a coset $x + C \in E/C$ such that

$$|(x + C)(4)| \geq \frac{|E(4)| - |C(4)|}{|E/C| - 1} = \frac{2^5 - 2^{k-3}}{2^{8-k} - 1} = 2^{k-3} = |C(4)| + \frac{16}{4}.$$

By Proposition 2.2, $C \cong C^+(K)$ for some even Kleinian code K of length 4. By $|C^+(K)| = 2^3|K|$, we have $|K| = 2^{k-3}$. By Lemma 1.6 (3) and (2.1), $|K(2)| = |C(4)|/2 = 2^{k-4} - 2$. Since $|K(4)| = |K| - |K(2)| - |K(0)| = 2^{k-4} + 1$, we obtain

$$(2.2) \quad W_K(X, Y) = (2^{k-4} + 1)X^4 + (2^{k-4} - 2)X^2Y^2 + Y^4.$$

Replacing C by D , we also have $C \cong D$, or $D \cong C^+(J)$ for some even Kleinian code J of length 4 with the same properties as K . One can verify that there are only four even Kleinian codes of length 4 satisfying (2.2) up to equivalence (cf. [Hö03a]). Two of them are characterized by the weight enumerators, and the others are the two even self-dual Kleinian codes ϵ_2^2 and δ_4^+ of length 4 (Lemma 1.1 (2)). By Lemma 1.7 (3), $C^+(\epsilon_2^2) \cong C^+(\delta_4^+)$. Thus we obtain $C \cong D$.

2.4. Even lattices $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$. In this subsection, we study the isomorphism classes of $\mathcal{L}(C)$ and $\mathcal{L}^+(C)$.

Theorem 2.8. *Let C and D be doubly-even binary codes of length n . Then $\mathcal{L}^+(C) \cong \mathcal{L}(D)$ if and only if there exists an even Kleinian code K such that $C \cong \mathcal{C}(K)$ and $D \cong C^+(K)$.*

Proof. The “if” part follows from Lemma 2.6. For the “only” part, we assume that $\mathcal{L}^+(C) \cong \mathcal{L}(D)$. Comparing the numbers of vectors of norm 2 in the lattices (Lemma 1.8 (3)), we obtain

$$8|C(4)| = 2n + 16|D(4)|.$$

Since $\mathcal{L}^+(C) \cong \mathcal{L}(D)$, there is an isomorphism $\varphi : \mathcal{L}^+(C)^*/\mathcal{L}^+(C) \rightarrow \mathcal{L}(D)^*/\mathcal{L}(D)$ of their discriminant groups. By $\mathcal{L}(D)^* = \mathcal{L}(D^\perp)$, there is an $x \in D^\perp \setminus D$ such that $\alpha_x/2 + \mathcal{L}(D) = \varphi(\alpha_1 + \mathcal{L}^+(C))$. Comparing the numbers of vectors of norm 2 in $\alpha_1 + \mathcal{L}^+(C)$ and $\alpha_x/2 + \mathcal{L}(D)$, we obtain

$$2n + 8|C(4)| = 16|(x + D)(4)|.$$

Combining the equations, we obtain

$$|(x + D)(4)| = n/4 + |D(4)|.$$

By Proposition 2.2, $D \cong C^+(K)$ for some even Kleinian code K . Then by Proposition 2.6 and an assumption,

$$\mathcal{L}^+(\mathcal{C}(K)) \cong \mathcal{L}(C^+(K)) \cong \mathcal{L}(D) \cong \mathcal{L}^+(C).$$

By Theorem 2.7, $C \cong \mathcal{C}(K)$ or both C and $\mathcal{C}(K)$ are (nonisomorphic) doubly-even self-dual binary codes of length 16. In the first case, we are done. In the latter case, Lemmas 1.2 (2) and 1.7 (2) show that $C \cong \mathcal{C}(K')$, where $\{K, K'\} = \{\epsilon_2^2, \delta_4^+\}$. By Lemma 1.7 (3), $C^+(K) \cong C^+(K')$. Hence we have $D \cong C^+(K')$ and $C \cong \mathcal{C}(K')$.

3. ISOMORPHISM PROBLEMS FOR THE VOAs V_L AND V_L^+

3.1. **VOAs V_L .** In this subsection, we study the isomorphism classes of V_L . The following proposition is well known. Its proof is similar to that of Proposition 2.4.

Proposition 3.1. *Let L and N be even lattices of rank n . Then $V_L \cong V_N$ if and only if $L \cong N$.*

Proof. The “if” part is obvious. For the “only” part, we assume $V_L \cong V_N$. By the VOA structure of V_L ([FLM88]), $S(\hat{\mathfrak{h}}_L^-)$ is a subVOA and $V_L \cong \bigoplus_{\alpha \in L} S(\hat{\mathfrak{h}}_L^-) \otimes e^\alpha$ as $S(\hat{\mathfrak{h}}_L^-)$ -modules. Note that $\mathfrak{h}_L(-1)\mathbf{1} = S(\hat{\mathfrak{h}}_L^-) \cap (V_L)_1$. Then for $h \in \mathfrak{h}_L$, $S(\hat{\mathfrak{h}}_L^-) \otimes e^\alpha$ is the eigenspace of the 0-th product of $h(-1)\mathbf{1}$ with eigenvalue (α, h) . By the same argument, $V_N \cong \bigoplus_{\beta \in N} S(\hat{\mathfrak{h}}_N^-) \otimes e^\beta$ as $S(\hat{\mathfrak{h}}_N^-)$ -modules, and $\mathfrak{h}_N(-1)\mathbf{1} = S(\hat{\mathfrak{h}}_N^-) \cap (V_N)_1$. Since $\dim(V_L)_0 = 1$, $(V_L)_1$ has a finite-dimensional reductive Lie algebra structure by the 0-th product. Using the fact that $V_L \cong V_N$, we may view both $\mathfrak{h}_L(-1)\mathbf{1}$ and $\mathfrak{h}_N(-1)\mathbf{1}$ as Cartan subalgebras of $(V_L)_1$. Hence there exists $g \in \langle \exp(v_0) \mid v \in (V_L)_1 \rangle \subset \text{Aut}(V_L)$ such that $g(\mathfrak{h}_N(-1)\mathbf{1}) = \mathfrak{h}_L(-1)\mathbf{1}$. Since $S(\hat{\mathfrak{h}}_N^-)$ is generated by $\mathfrak{h}_N(-1)\mathbf{1}$ as a VOA, we have $g(S(\hat{\mathfrak{h}}_N^-)) = S(\hat{\mathfrak{h}}_L^-)$. Hence for any $\beta \in N$, $g(S(\hat{\mathfrak{h}}_N^-) \otimes e^\beta) = S(\hat{\mathfrak{h}}_L^-) \otimes e^\alpha$ for some $\alpha \in L$. Set $\bar{g}(\beta) = \alpha$. Since g is an automorphism of a VOA, \bar{g} is a linear isomorphism from N to L , and $(\bar{g}(\beta), \bar{g}(\beta)) = (\beta, \beta)$ for all $\beta \in N$. Thus \bar{g} is an orthogonal transformation, and $N \cong L$.

3.2. **VOAs V_L^+ .** In this subsection, we study the isomorphism classes of V_L^+ .

The following proposition, which is similar to Proposition 2.2, was proved in [Sh06].

Proposition 3.2 ([Sh06, Theorem 2.2]). *Let L be an even lattice of rank n . If there exists $\lambda \in (L^* \cap L/2)$ such that $|(\lambda + L)(2)| \geq 2n + |L(2)|$, then $L \cong \mathcal{L}^+(C)$ and $L + \mathbb{Z}\lambda \cong \mathcal{L}(C)$ for some doubly-even binary code C .*

Next, we will consider an even lattice L such that $\sqrt{2}L^*$ is even.

Lemma 3.3. *Let L be an even lattice. If $\sqrt{2}L^*$ is even, then $2L^* \subset L$.*

Proof. Since $\sqrt{2}L^*$ is even, we have $\sqrt{2}L^* \subset (\sqrt{2}L^*)^* = L/\sqrt{2}$, and $2L^* \subset L$.

Remark 3.4. In [Sh04, Sh06], an even lattice L was said to be 2-elementary totally even if $2L^* \subset L$ and if $\sqrt{2}L^*$ is even. By the lemma above, if $\sqrt{2}L^*$ is even, then L is 2-elementary totally even.

The following lemma can be proved by using the quadratic spaces over \mathbb{F}_2 associated to 2-elementary totally even lattices of rank $8k$. Note that it is of plus type by [Ve79, Theorem 1].

Lemma 3.5. *Let L be an even lattice of rank $8k$ such that $\sqrt{2}L^*$ is even. Then there exists an even unimodular lattice of rank $8k$ containing L .*

Let us review some properties of irreducible modules for V_L^+ .

Lemma 3.6. *The following three conditions are equivalent.*

- (1) *Any irreducible V_L^+ -module is a self-dual simple current.*
- (2) *An irreducible V_L^+ -module $V_L^{T_{x,\varepsilon}}$ is self-dual.*
- (3) *$\sqrt{2}L^*$ is even.*

Proof. Obviously, (1) shows (2). By Lemma 3.3 and [ADL05], one can see that (3) implies (1).

Assume (2) and let $V_L^{T_{x,\varepsilon}}$ be a self-dual irreducible V_L^+ -module. Let M be the contragredient module of $V_L^{T_{x,\varepsilon}}$. Then by [ADL05, Proposition 3.7], $M \cong V_L^{T_{x',\varepsilon}}$ and $\chi'(s) = (-1)^{(\rho(s),\rho(s))/2}\chi(s)$ for $s \in Z(\hat{L}/J)$, where ρ is given in (1.5). Since $V_L^{T_{x,\varepsilon}}$ is self-dual, we have $\chi' = \chi$. It follows from $\rho(Z(\hat{L}/J)) = 2L^*/2L$ that $(v, v) \in 4\mathbb{Z}$ for all $v \in 2L^*$. Hence $\sqrt{2}L^*$ is even, and we obtain (3). \square

Note that the fusion rules of V_L^+ are associative [ADL05, Theorem 5.18].

Lemma 3.7 (cf. [ADL05]). *Let L be an even lattice such that $\sqrt{2}L^*$ is even. Then the fusion group of V_L^+ is isomorphic to an elementary abelian 2-group of order 2^{k+2} , where k is the integer satisfying $L^*/L \cong \mathbb{Z}_2^k$.*

A triality automorphism of $V_{\mathcal{L}^+(C)}^+$ which sends $V_{\mathcal{L}^+(C)}^-$ to $V_{\alpha_1+\mathcal{L}^+(C)}^+$ was given in [FLM88, Chapter 11]. The following lemma can be verified directly.

Lemma 3.8 (cf. [FLM88, Chapter 11]). *Let C be a doubly-even binary code. Then there is an isomorphism of VOAs from $V_{\mathcal{L}^+(C)}$ to $V_{\mathcal{L}(C)}^+$.*

Now, we prove the following. Its proof is similar to that of Theorem 2.7.

Theorem 3.9. *Let L and N be even lattices of rank n . Then the VOAs V_L^+ and V_N^+ are isomorphic if and only if one of the following holds.*

- (1) L and N are isomorphic.
- (2) L and N are even unimodular lattices of rank 16.

Proof. The “if” part follows from Lemmas 1.3 (2) and 1.11. For the “only” part, we assume that $V_L^+ \cong V_N^+$. By Lemma 1.10 (2), $\Theta_L(q) = \Theta_N(q)$, and hence $\text{ch}V_L^- = \text{ch}V_N^-$. We regard the irreducible V_N^+ -module V_N^- as an irreducible V_L^+ -module. Since V_N^- is a self-dual simple current, it is isomorphic to $V_L^-, V_{\lambda+L}^\varepsilon$ ($\lambda \notin L$) or $V_L^{T_{x,\varepsilon}}$ as V_L^+ -modules under the isomorphism of V_N^+ and V_L^+ by Lemma 1.12.

(i) Assume that $V_N^- \cong V_L^-$. Then $V_N \cong V_L^+ \oplus V_L^-$ is a simple current extension of V_L^+ . Hence by Proposition 1.5, $V_N \cong V_L$ as VOAs, and by Proposition 3.1, $L \cong N$.

(ii) Assume that $V_N^- \cong V_{\lambda+L}^\varepsilon$. Since $\text{Hom}(L^*/2L^*, \mathbb{Z}_2) \subset \text{Aut}(V_L^+)$ ([FLM88]), we may assume that $\varepsilon = +$ up to conjugation ([Sh04]). It follows from $\text{ch}V_L^- = \text{ch}V_N^-$ that $\text{ch}V_L^- = \text{ch}V_{\lambda+L}^+$. Comparing the dimensions of the weight 1 subspaces of V_L^- and $V_{\lambda+L}^+$ (Lemma 1.10 and (1.6)), we obtain $|L(2)| + 2n = |(\lambda + L)(2)|$. Hence by Proposition 3.2, $L \cong \mathcal{L}^+(C)$ and $L + \mathbb{Z}\lambda \cong \mathcal{L}(C)$ for some doubly-even binary code C . By Proposition 1.5, as VOAs,

$$V_N \cong V_L^+ \oplus V_N^- \cong V_L^+ \oplus V_{\lambda+L}^+ \cong V_{\mathcal{L}(C)}^+ \cong V_L,$$

where the last isomorphism is given in Lemma 3.8. By Proposition 3.1, we obtain $L \cong N$.

(iii) Assume that $V_N^- \cong V_L^{T_{x,-}}$. By Lemma 1.10 and (1.7), the lowest weights of V_N^- and $V_L^{T_{x,-}}$ are 1 and $n/16 + 1/2$, respectively. Hence $n = 8$. Since V_N^- is a self-dual, so is $V_L^{T_{x,-}}$. Hence $\sqrt{2}L^*$ is even by Lemma 3.6. Since $\sqrt{2}(\sqrt{2}L^*)^* = L$ is even, we have $\sqrt{2}L^* \subset E_8$ by Lemmas 1.3 (1) and 3.5. Hence $E_8 \subset (\sqrt{2}L^*)^* = L/\sqrt{2}$; equivalently, $\sqrt{2}E_8 \subset L$. Replacing N by L , we have (i), (ii) or (iii) for V_L^- . The first two cases show that $L \cong N$. Hence we may assume that $\sqrt{2}E_8 \subset N$ by the

same argument. One easily checks that there are only five even overlattices of $\sqrt{2}E_8$ up to isomorphism, and those are determined by the discriminant groups. Since the fusion groups of V_L^+ and V_N^+ are isomorphic, so are the discriminant groups of L and N by Lemma 3.7. Thus $L \cong N$.

(iv) Assume that $V_N^- \cong V_L^{Tx,+}$. By (1.7), the lowest weight of $V_L^{Tx,+}$ is $n/16$. By the same argument as in (iii), $\sqrt{2}L^*$ is even and $n = 16$. Comparing the dimensions of the weight 1 subspaces of V_N^- and $V_L^{Tx,+}$ (Lemma 1.10 and (1.7)), we obtain

$$(3.1) \quad |L(2)| = 2^{9-k} - 32,$$

where $|L^*/L| = 2^{2k}$. If L is unimodular, then so is N , and hence (2) holds. Assume that L is not unimodular. By Lemma 3.5 there exists an even unimodular lattice P of rank 16 such that $L \subsetneq P$. By Lemma 1.3 (2), $|P(2)| = 480$. Hence there exists a coset $\mu + L \in P/L$ such that

$$|(\mu + L)(2)| \geq \frac{|P(2)| - |L(2)|}{|P/L| - 1} = \frac{512 - 2^{9-k}}{2^k - 1} = 2^{9-k} = |L(2)| + 2 \times 16.$$

By Proposition 3.2, $L \cong \mathcal{L}^+(C)$ for some t -dimensional doubly-even binary code C of length 16. It follows from $|\mathcal{L}^+(C)^*/\mathcal{L}^+(C)| = 2^{18-2t}$ that $t = 9 - k$. By Lemma 1.8 and $n = 16$, $|\mathcal{L}^+(C)(2)| = 2^3|C(4)| + 2 \times 16$. Hence, by (3.1), we obtain

$$|C(4)| = 2^{t-3} - 4.$$

Since $|C(4)| \geq 0$, we have $5 \leq t \leq 8$. By the same arguments as in (iv) of Theorem 2.7 (see (2.1)), if C is not self-dual, then $C \cong \mathcal{C}^+(K)$ for some even Kleinian code K of length 4 with the weight enumerator (2.2).

Replacing L by N , we also obtain $L \cong N$ or $N \cong \mathcal{L}^+(D)$ for some doubly-even binary code D of length 16 with the same properties as C . If C is self-dual, then so is D , and $L \cong N$ by Lemma 1.9 (3). If not, $C \cong \mathcal{C}^+(K)$ and $D \cong \mathcal{C}^+(J)$ for some even Kleinian codes K and J of length 4. By Lemma 3.7, we have $L^*/L \cong N^*/N$, $\dim C = \dim D$, and $\dim K = \dim J$. Then by (2.2), both Kleinian codes K and J have the same weight enumerator. Thus $K \cong J$ or $\{K, J\} = \{\varepsilon_2^2, \delta_4^+\}$ (cf. [Hö03a]). Therefore by Lemma 1.7, $C \cong D$, and $L \cong \mathcal{L}^+(C) \cong \mathcal{L}^+(D) \cong N$.

3.3. VOAs V_L and V_L^+ . In this subsection, we study the isomorphism classes of V_L and V_L^+ . The proof of the following theorem is similar to that of Theorem 2.8.

Theorem 3.10. *Let L and N be even lattices of rank n . Then the VOAs V_L^+ and V_N are isomorphic if and only if there exists a doubly-even binary code C such that $L \cong \mathcal{L}(C)$ and $N \cong \mathcal{L}^+(C)$.*

Proof. Lemma 3.8 proves the “if” part. For the “only” part, we assume that $V_L^+ \cong V_N$. Comparing the dimensions of the weight 1 subspaces of V_L^+ and V_N (Lemma 1.10 (2)), we obtain

$$\frac{|L(2)|}{2} = n + |N(2)|.$$

Let $V_{\lambda+N}$ be an irreducible V_N -module such that $V_{\lambda+N} \cong V_L^-$ under the isomorphism of V_N and V_L^+ . Comparing the dimensions of the weight 1 subspaces of V_L^- and $V_{\lambda+N}$, we obtain

$$\frac{|L(2)|}{2} + n = |(\lambda + N)(2)|.$$

Combining the equations, we obtain

$$|N(2)| + 2n = |(\lambda + N)(2)|.$$

In addition, the fusion rule $[V_L^-] \times [V_L^-] = [V_L^+]$ ([ADL05]) shows that $[V_{\lambda+N}] \times [V_{\lambda+N}] = [V_N]$. Hence $2\lambda \in N$. By Proposition 3.2, $N \cong \mathcal{L}^+(C)$ for some doubly-even binary code C of length n . By Lemma 3.8, we obtain $V_L^+ \cong V_N \cong V_{\mathcal{L}^+(C)} \cong V_{\mathcal{L}(C)}^+$ as VOAs. By Theorem 3.9, $L \cong \mathcal{L}(C)$ or both L and $\mathcal{L}(C)$ are (nonisomorphic) even unimodular lattices of rank 16. In the first case, we are done. In the latter case, Lemmas 1.3 (2) and 1.9 (2) show that $L \cong \mathcal{L}(D)$, where $\{C, D\} = \{e_8^2, d_{16}^+\}$. By Theorem 2.7, $\mathcal{L}^+(C) \cong \mathcal{L}^+(D)$. Hence $N \cong \mathcal{L}^+(D)$ and $L \cong \mathcal{L}(D)$.

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