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CENTRAL VALUES OF THE SYMMETRIC SQUARE L-FUNCTIONS

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ABSTRACT. We establish a sharp bound for the square mean of the central values of the symmetric square L-functions associated to holomorphic cusp forms of level 1, as the weight k varies in the short interval $[K,\ K+K^{1/2+\epsilon}]$.

1. INTRODUCTION

Let $S_k(\Gamma_0(1))$ denote as usual the space of holomorphic cusp forms of weight k with respect to the modular group $\Gamma_0(1)$, and let H_k be the normalized Hecke basis for $S_k(\Gamma_0(1))$, where the normalizaton means that the first Fourier coefficient of each basis element $f \in H_k$ is 1.

Each $f \in H_k$ has the Fourier expansion

$$f(z) = \sum_{n\geq 1} a_f(n) n^{(k-1)/2} e(nz), \quad a_f = 1,$$

and the associated L-function

$$L(s,\ f) = \sum_{n\geq 1} a_f(n) n^{-s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1},\ \Re(s) > 1,$$

admits analytic continuation to the whole complex plane $\mathbf{C}$ and satisfies the functional equation

$$(2\pi)^{-s}\Gamma\left(\frac{k-1}{2}+s\right) L(s,\ f) = i^k (2\pi)^{1-s}\Gamma\left(\frac{k-1}{2}+1-s\right) L(1-s,\ f).$$

The symmetric square L-function associated to f is defined by

$$L(s,\ \mathrm{sym}^2(f)) = \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1},\ \Re(s) > 1,$$

and by the works of Shimura [7] and Gelbart-Jacquet [1], it admits analytic continuation to the whole complex plane $\mathbf{C}$ and satisfies the functional equation

$$\begin{aligned}\Lambda(s,\ \mathrm{sym}^2(f)) \quad &=: \quad \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right) L(s,\ \mathrm{sym}^2(f)) \\ &= \quad \Lambda(1-s,\ \mathrm{sym}^2(f)).\end{aligned}$$

Write

$$L_\infty(s) = L_\infty(s,\ \mathrm{sym}^2(f)) = \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)$$

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and note that

$$L(s,\ \mathrm{sym}^2(f)) = \zeta(2s)\sum_{n\geq 1} a_f(n^2)n^{-s},\ \ \Re(s) > 1.$$

To study the central L-value $L(1/2,\ \mathrm{sym}^2(f))$, the following approximate functional equation is a standard device (see, for example, [5, Lemma 2.2] for a detailed proof).

Lemma 1 ([5]). *We have*

$$L(1/2,\ sym^2(f)) = 2\sum_{n\geq 1}\frac{a_f(n^2)}{n^{1/2}}V_k(n),$$

where

$$V_k(\xi) = \frac{1}{2\pi i}\int_{(\sigma)}\frac{L_\infty(1/2+y)}{L_\infty(1/2)}\zeta(1+2y)\xi^{-y}\frac{dy}{y}\ \ (\sigma > 0),$$

and it satisfies the following bounds:

$$V_k(\xi) \ll_A \left(\frac{k}{\xi}\right)^A,\ \ \textit{for any } A > 0\ ,$$

and

$$V_k(\xi) \ll \log(2k/\xi),\ \ \textit{for } \xi \leq k.$$

The purpose of this paper is to show

Theorem 1. *For any $\epsilon > 0$ we have, as $K \to \infty$,*

$$\sum_{2|k,\ K\leq k\leq K+K^{1/2}}\ \sum_{f\in H_k}|L(1/2,\ sym^2(f))|^2 \ll_\epsilon K^{3/2+\epsilon}. \tag{1}$$

Remark. It seems to be an open problem (see, for instance, Conjecture 1.2 on p. 5 of [5]) to obtain the sharp bound for the square mean without average over the weight k. Previously the work [3] established a sharp bound for the second moment for $L(1/2,\ \mathrm{sym}^2(f))$, when f has *fixed* weight and large *level*, but its method unfortunately does not extend to the case of large weight. Our work is inspired by [6], which makes use of the Zagier kernel function Φ_s for $L(s,\ \mathrm{sym}^2(f))$ to derive a sharp bound for the first moment.

2. Zagier's kernel function

Let Δ be a discriminant; i.e., Δ is an integer such that $\Delta \equiv 0, 1 (\mathrm{mod} 4)$. Define

$$L(s,\ \Delta) = \begin{cases} \zeta(2s-1), & \text{if } \Delta = 0; \\ L_D(s)\sum_{d|f}\mu(d)\left(\frac{D}{d}\right)d^{-s}\sigma_{1-2s}\left(\frac{f}{d}\right), & \text{if } \Delta \neq 0, \end{cases}$$

where if $\Delta \neq 0$ we write $\Delta = Df^2$ with positive integer f and D the fundamental discriminant, $L_D(s)$ is the Dirichlet L-function associated to the Kronecker symbol $\left(\frac{D}{\cdot}\right)$, μ is the Möbius function, and $\sigma_\nu(m) = \sum_{d|m} d^\nu$ for a positive integer m and any complex number ν.

Moreover for t an integer with $\Delta < t^2$ and $s \in \mathbf{C}$ with $1/2 < \Re(s) < k$ we define (see [8])

$$\begin{aligned} I_k(\Delta, t; s) &= \int_0^\infty \int_{-\infty}^\infty \frac{y^{k+s-2}}{(x^2+y^2+ity-\frac{1}{4}\Delta)^k} dxdy \\ &= \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}}{(y^2+ity-\frac{1}{4}\Delta)^{k-1/2}} dy, \end{aligned}$$

where the second integral converges absolutely for $1-k < \sigma < k$ if $\Delta \neq 0$. For $\Delta = 0,\ \pm t > 0$, one has

$$I_k(0,\ t;\ s) = e^{\pm i\pi(s-k)/2} \frac{\Gamma(1/2)\Gamma(s-1/2)\Gamma(k-s)}{\Gamma(k)} |t|^{-k+s}.$$

We also have

$$2I_k(\Delta,\ 0,\ 1/2) = \frac{1}{m^{k-1/2}} \frac{\Gamma\left(\frac{k-1/2}{2}\right)^2 \Gamma(1/2)}{\Gamma(k)}.$$

Define

$$I_{k,s}(z) = \frac{2^{1-k}\Gamma(1/2)}{\Gamma(k-1/2)} \Gamma(k-1+s)\Gamma(k-s)(z^2-1)^{-(k-1)/2} P_{-s}^{1-k}(z),$$

for $1-k < \Re(s) < k,\ z \in \mathbf{C}\backslash(-\infty,\ 1]$, where $P_\nu^\mu(z)$ is the associated Legendre function of the first kind. Then for $\Delta < 0$, it follows that

$$I_k(\Delta,\ t;\ s) = (|\Delta|/4)^{(s-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,s}\left(\frac{it}{\sqrt{|\Delta|}}\right),$$

while for $\Delta > 0$, the following formulas are valid:

$$I_k(\Delta,\ t;\ s) = (\Delta/4)^{(s-k)/2} e^{i\pi(s-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,s}\left(\frac{t}{\sqrt{\Delta}}\right) \quad (\Delta > 0,\ t > 0);$$

$$I_k(\Delta,\ t;\ s) = (\Delta/4)^{(s-k)/2} e^{-i\pi(s-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,s}\left(\frac{|t|}{\sqrt{\Delta}}\right) \quad (\Delta > 0,\ t < 0).$$

We refer the reader to [8] for the proofs of these formulas for $I_k(\Delta, t; s)$. Now we are in a position to state the following important theorem due to Zagier, which is the basis for our study on the square mean of $L(1/2,\ \mathrm{sym}^2(f))$ via the kernel function Φ_s.

Theorem 2 ([8])**.** *Let $k \geq 4$ be an even integer. For a positive integer m and $s \in \mathbf{C}$ set*

$$\begin{aligned} c_m(s) &= m^{k-1} \sum_{t=-\infty}^{\infty} \left(I_k(t^2-4m,\ t;\ s) + I_k(t^2-4m,\ -t;\ s)\right) L(s,\ t^2-4m) \\ &\quad + \begin{cases} (-1)^{k/2} \frac{\Gamma(k+s-1)\zeta(2s)}{2^{2s+k-3}\pi^{s-1}\Gamma(k)} u^{k-s-1}, & \text{if } m = u^2,\ u > 0; \\ 0, & \text{if } m \text{ is not a perfect square.} \end{cases} \end{aligned}$$

Then:
(i) The series converges absolutely and uniformly for $2-k < \Re(s) < k-1$.
(ii) The function

$$\Phi_s(z) = \sum_{m=1}^{\infty} c_m(s) e(mz) \quad (z \in \mathbf{H},\ 2-k < \Re(s) < k-1)$$

is in $S_k(\Gamma_0(1))$.

(iii) Let $f \in H_k$. Then the Petersson product of Φ_s and f is given by

$$\langle \Phi_s,\ f\rangle \ = \ c_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} L(s,\ sym^2(f)),$$

where

$$c_k = \frac{(-1)^{k/2}\pi}{2^{k-3}(k-1)}.$$

In the above theorem, if m is a square, then the single term outside the sum has a simple pole at $s = 1/2$, while in this case the terms $t = \pm 2\sqrt{m}$ in the first sum involve the function

$$I_k(0,\ t,\ s) + I_k(0,\ t,\ s) = 2\pi(-1)^{k/2}\cos\frac{\pi s}{2}\frac{\Gamma(s-1/2)\Gamma(k-s)}{\Gamma(k)\Gamma(1/2)}|t|^{s-k},$$

which also has a simple pole at $s = 1/2$, and the two poles cancel, so that $c_m(s)$ is actually holomorphic in the region $2-k < \Re(s) < k-1$.

We infer that

$$\Phi_s(z) = c_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{f\in H_k} \frac{L(s,\ \mathrm{sym}^2(f))}{\langle f,\ f\rangle} f(z).$$

Taking the m-th Fourier coefficient of both sides, we have

$$c_m(s) = c_k \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{f\in H_k} \frac{L(s,\ \mathrm{sym}^2(f))}{\langle f,\ f\rangle} m^{(k-1)/2} a_f(m).$$

Specializing at $s = 1/2$ and replacing m by m^2, it follows from Lemma 1 that

$$2\sum_{m\geq 1} \frac{c_{m^2}(1/2)}{m^{k-1/2}} V_k(m) = c_k \frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}} \sum_{f\in H_k} \frac{|L(1/2,\ \mathrm{sym}^2(f))|^2}{\langle f,\ f\rangle}.$$

Recall that

$$\langle f,\ f\rangle = \frac{\Gamma(k)L(1,\ \mathrm{sym}^2(f))}{2^{2k-1}\pi^{k+1}}.$$

Hence
(2)

$$\sum_{f\in H_k} \frac{|L(1/2,\ \mathrm{sym}^2(f))|^2}{L(1,\ \mathrm{sym}^2(f))} = (-1)^{k/2}2^{-2}\pi^{-5/2}\left(\frac{\Gamma(k-1/2)}{2^k(k-1)\Gamma(k)}\right)^{-1}\sum_{m\geq 1}\frac{c_{m^2}(1/2)}{m^{k-1/2}}V_k(m).$$

3. The case $\Delta > 0$

We need the following integral representation of $P_\nu^{-\mu}(z)$ ([2, p. 1001, 8.713 (2)]):

$$P_\nu^{-\mu}(z) = \frac{(z^2-1)^{\mu/2}}{2^\nu\Gamma(\mu-\nu)\Gamma(\nu+1)} \int_0^\infty \frac{\sinh^{2\nu+1} r}{(z+\cosh r)^{\nu+\mu+1}} dr,$$

for $\Re(z) > -1,\ |\arg(z\pm 1)| < \pi,\ \Re(\nu+1) > 0,\ \Re(\mu-\nu) > 0.$

Consider first the case where $\Delta = t^2 - 4m^2 > 0$. With $z = \frac{|t|}{\sqrt{t^2-4m^2}}$ we deduce that

$$|P_{-1/2}^{1-k}(z)| \ll \frac{|z^2-1|^{(k-1)/2}}{\Gamma(k-1/2)} \int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}} dr.$$

Now for any $\delta > 0$,

$$\begin{aligned}
&\int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}}dr \\
&\ll \frac{1}{(z+1)^{k-1/2-\delta}}\int_0^\infty \frac{1}{(\cosh r)^\delta}dr \\
&\ll_\delta \frac{1}{(z+1)^{k-1/2-\delta}} \\
&\ll_\delta \frac{(t^2-4m^2)^{(k-1/2-\delta)/2}}{(|t|+\sqrt{t^2-4m^2})^{(k-1/2-\delta)}} \\
&\ll_\delta \begin{cases} \frac{1}{(1+1/\sqrt{2})^k}\frac{(t^2-4m^2)^{(k-1/2-\delta)/2}}{|t|^{(k-1/2-\delta)}} & \text{if } |t| > \sqrt{8}m, \\ \frac{(t^2-4m^2)^{(k-1/2-\delta)/2}}{|t|^{(k-1/2-\delta)}(1+\sqrt{1-2m/|t|})^{(k-1/2-\delta)}} & \text{if } 2m < |t| \le \sqrt{8}m \end{cases} \\
&\ll_\delta \frac{1}{\exp(k^{1/3})}\frac{(t^2-4m^2)^{(k-1/2)/2}}{|t|^{(k-1/2-\delta)}},
\end{aligned}$$

since in view of $|t| \ge 2m+1$, we have

$$(1+\sqrt{1-2m/|t|})^{(k-1/2-\delta)} \ge \left(1+\frac{1}{\sqrt{2m+1}}\right)^k \ge \left(1+\frac{1}{\sqrt{k^{1+2\epsilon}}}\right)^k \gg \exp(k^{1/2-\epsilon}).$$

Note for $\Delta = Df^2$ as before, the convexity bound yields

$$|L(1/2,\ \Delta)| \ll f^\epsilon |L_D(1/2)| \ll \Delta^{1/4+\epsilon}.$$

Furthermore,

$$\begin{aligned}
|I_{k,1/2}(z)| &\ll 2^{-k}\Gamma(k-1/2)|z^2-1|^{-(k-1)/2}|P^{1-k}_{-1/2}(z)| \\
&\ll 2^{-k}\int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}}dr, \\
|I_k(\Delta,\ t,\ 1/2)| &\ll (\Delta/4)^{(1/2-k)/2}\frac{\Gamma(k-1/2)}{\Gamma(k)}|I_{k,1/2}(|t|/\sqrt{\Delta})| \\
&\ll \Delta^{-(k-1/2)/2}\frac{\Gamma(k-1/2)}{\Gamma(k)}\int_0^\infty \frac{1}{(|t|/\sqrt{\Delta}+\cosh r)^{k-1/2}}dr \\
&\ll \frac{\Gamma(k-1/2)}{\exp(k^{1/3})|t|^{(k-1/2-\delta)}\Gamma(k)}.
\end{aligned}$$

Hence the contribution to the right-hand side of (2) from the terms with $\Delta = t^2-4m^2 > 0$ is

$$\begin{aligned}
&\ll k^{-10} + \left(\frac{\Gamma(k-1/2)}{2^k k\Gamma(k)}\right)^{-1}\sum_{m\ll k^{1+\epsilon}}\frac{m^{2k-2}}{m^{k-1/2}}\sum_{|t|>2m} \\
&\quad \times \left(I_k(t^2-4m^2,\ t;\ 1/2)+I_k(t^2-4m^2,\ -t;\ 1/2)\right)L(1/2,\ t^2-4m^2) \\
(3)\quad &\ll 1.
\end{aligned}$$

4. The case $\Delta \le 0$

Next we consider the terms in the right-hand side of (2) with $\Delta = t^2 - 4m^2 < 0$ and $z = \frac{it}{\sqrt{|\Delta|}}$. We have

$$\begin{aligned} I_k(\Delta,\ t;\ s) &= (|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} I_{k,1/2}\left(\frac{it}{\sqrt{|\Delta|}}\right) \\ &= (|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma(k-1/2)\Gamma(1/2)}{\Gamma(k)} \\ &\quad \times 2^{1-k}\Gamma(1/2)\Gamma(k-1/2)(z^2-1)^{-(k-1)/2} P^{1-k}_{-1/2}(z) \\ &= 2^{1-k}(|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma^2(k-1/2)\Gamma^2(1/2)}{\Gamma(k)} (z^2-1)^{-(k-1)/2} \\ &\quad \times \frac{(z^2-1)^{(k-1)/2}}{2^{-1/2}\Gamma(k-1/2))\Gamma(1/2)} \int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}} dr \\ &= 2^{3/2-k}\Gamma(1/2)(|\Delta|/4)^{(1/2-k)/2} \frac{\Gamma(k-1/2)}{\Gamma(k)} \int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}} dr \\ &= 2\Gamma(1/2)|\Delta|^{(1/2-k)/2} \frac{\Gamma(k-1/2)}{\Gamma(k)} \int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}} dr. \end{aligned}$$

Making the change of variable $u = \cosh t$, we have

$$\begin{aligned} \int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}} dr &= \int_1^\infty \frac{du}{\sqrt{u^2-1}(z+u)^{k-1/2}} \\ &= \int_0^\infty \frac{du}{\sqrt{u^2+2u}(z+1+u)^{k-1/2}}. \end{aligned}$$

Applying Cauchy's theorem we can replace the integral over the positive real line by the line $\{(1+z)v, v \ge 0\}$, and the above integral equals (we may assume $m \ll k^{1+\epsilon}$)

$$\begin{aligned} &\frac{1}{(1+z)^{k-1}} \int_0^\infty \frac{dv}{\sqrt{v}(1+v)^{k-1/2}\sqrt{2+(1+z)v}} \\ &\quad = \frac{1}{(1+z)^{k-1}} \left(\int_0^{k^{\epsilon-1}} \frac{dv}{\sqrt{v}(1+v)^{k-1/2}\sqrt{2+(1+z)v}} + O(k^{-A}) \right) \\ &\quad = \frac{1}{\sqrt{2}(1+z)^{k-1}} \left(\int_0^{k^{\epsilon-1}} \frac{dv}{\sqrt{v}(1+v)^{k-1/2}} + O(k^{-1+\epsilon}) \right) \\ &\quad = \frac{\sqrt{2}|\Delta|^{(k-1)/2}}{2^k m^{k-1}} \left(\frac{1+z}{|1+z|}\right)^{-(k-1)} \left(\int_0^{k^{\epsilon-1}} \frac{dv}{\sqrt{v}(1+v)^{k-1/2}} + O(k^{-1+\epsilon}) \right), \end{aligned}$$

in view of

$$|1+z| = \left(\frac{2m}{\sqrt{|\Delta|}}\right)$$

and

$$|\Delta| = 4m^2 - t^2 = (2m-t)(2m+t) > m.$$

Hence

$$\begin{aligned} I_k(\Delta,\ t;\ s) &= 2\Gamma(1/2)|\Delta|^{(1/2-k)/2}\frac{\Gamma(k-1/2)}{\Gamma(k)}\int_0^\infty \frac{1}{(z+\cosh r)^{k-1/2}}dr \\ &= 2\sqrt{2}\Gamma(1/2)|\Delta|^{-1/4}\frac{\Gamma(k-1/2)}{2^k\Gamma(k)}\frac{1}{m^{k-1}}\left(\frac{1+z}{|1+z|}\right)^{-(k-1)} \\ &\quad \times\left(\int_0^{k^{\epsilon-1}}\frac{dv}{\sqrt{v}(1+v)^{k-1/2}} + O(k^{-1+\epsilon})\right). \end{aligned}$$

Thus the contribution to the right-hand side of (2) from the terms with $\Delta = t^2 - 4m^2 < 0$ is

(4)

$$\begin{aligned} &O(k^{-10}) + (-1)^{k/2}2^{-2}\pi^{-5/2}\left(\frac{\Gamma(k-1/2)}{2^k(k-1)\Gamma(k)}\right)^{-1}\sum_{m\ll k^{1+\epsilon}}\frac{m^{2k-2}}{m^{k-1/2}}V_k(m) \\ &\times\sum_{|t|<2m}\left(I_k(t^2-4m^2,\ t;\ 1/2) + I_k(t^2-4m^2,\ -t;\ 1/2)\right)L(1/2,\ t^2-4m^2) \\ &= k^{-10} + (-1)^{k/2}2^{-1/2}\pi^{-2}(k-1)\sum_{m\ll k^{1+\epsilon}}m^{-1/2}V_k(m)\sum_{|t|<2m}(4m^2-t^2)^{-1/4} \\ &\times\left(\left(\frac{\sqrt{4m^2-t^2}+ti}{|\sqrt{4m^2-t^2}+ti|}\right)^{-(k-1)} + \left(\frac{\sqrt{4m^2-t^2}-ti}{|\sqrt{4m^2-t^2}-ti|}\right)^{-(k-1)}\right)L(1/2,\ t^2-4m^2) \\ &\times\left(\int_0^{k^{\epsilon-1}}\frac{dv}{\sqrt{v}(1+v)^{k-1/2}} + O(k^{-1+\epsilon})\right) \\ &= O(k^{1+\epsilon}) + (-1)^{k/2}2^{-1/2}\pi^{-2}(k-1)\left(\int_0^{k^{\epsilon-1}}\frac{dv}{\sqrt{v}(1+v)^{k-1/2}}\right) \\ &\times\sum_{m\ll k^{1+\epsilon}}m^{-1/2}V_k(m)\sum_{|t|<2m}(4m^2-t^2)^{-1/4} \\ &\times\left(\left(\frac{\sqrt{4m^2-t^2}+ti}{|\sqrt{4m^2-t^2}+ti|}\right)^{-(k-1)} + \left(\frac{\sqrt{4m^2-t^2}-ti}{|\sqrt{4m^2-t^2}-ti|}\right)^{-(k-1)}\right)L(1/2,\ t^2-4m^2). \end{aligned}$$

In the last step, we have inferred as follows:

$$\sum_{m\ll k^{1+\epsilon}} m^{-1/2} \sum_{|t|<2m} (4m^2-t^2)^{-1/4} L(1/2,\ t^2-4m^2)$$
$$\ll k^{\epsilon} \sum_{2^i\le k^{1+\epsilon}} \frac{1}{2^{i/2}} \sum_{f\le 2^{i+1}} \frac{1}{f^{1/2}} \sum_{|D|\le 2^{2i+2}/f^2} \frac{|L_D(1/2)|}{|D|^{1/4}}$$
$$\ll k^{\epsilon} \sum_{2^i\le k^{1+\epsilon}} \frac{1}{2^{i/2}} \sum_{f\le 2^{i+1}} \frac{1}{f^{1/2}} \left(\sum_{|D|\le 2^{2i+2}/f^2} |L_D(1/2)|^2 \right)^{1/2} \left(\sum_{|D|\le 2^{2i+2}/f^2} |D|^{-1/2} \right)^{1/2}$$
$$\ll k^{\epsilon} \sum_{2^i\le k^{1+\epsilon}} \frac{2^{3i/2}}{2^{i/2}} \sum_{f\le 2^{i+1}} \frac{1}{f^2}$$
$$\ll k^{1+3\epsilon},$$

where we have used Jutila's bound [4],

$$\sum_{|D|\le X} |L_D(1/2)|^2 \ll X \log^3 X.$$

Moreover the contribution to the right-hand side of (2) from the terms with $\Delta = t^2-4m^2 = 0$, i.e. $t=\pm 2m$, is

$$\ll \quad k^{-10} + \left(\frac{\Gamma(k-1/2)}{2^k k \Gamma(k)} \right)^{-1} \sum_{m\ll k^{1+\epsilon}} \frac{m^{2k-2}}{m^{k-1/2}} \frac{\Gamma(k-1/2)}{2^k m^{k-1/2} \Gamma(k)} \log^2 k$$
$$\ll \quad k \log^3 k, \tag{5}$$

in view of the bound

$$\frac{\Gamma^{'}(s)}{\Gamma(s)} = \log s + O(1/s), \quad \text{for } \Re(s) \ge 1.$$

5. The mean value in the short interval

The contribution from (4) is bounded trivially by $k^{3/2+\epsilon}$, but if we average over k with $K \le k \le K + K^{1/2}$, then the total contribution will be shown below to be $O(K^{3/2+\epsilon})$.

When $K \le k \le K+K^{1/2}$, the equality (4) can be replaced by (in light of the formula (2.7) in [5] for $V_k(\xi)$)

(6)

$$= O(K^{1+\epsilon}) + (-1)^{k/2} 2^{-1/2} \pi^{-2} K \left(\int_0^{K^{\epsilon-1}} \frac{dv}{\sqrt{v}(1+v)^{K-1/2}} \right)$$
$$\times \sum_{m\ll K^{1+\epsilon}} m^{-1/2} V_K(m) \sum_{|t|<2m} (4m^2-t^2)^{-1/4}$$
$$\times \left(\left(\frac{\sqrt{4m^2-t^2}+ti}{|\sqrt{4m^2-t^2}+ti|} \right)^{-(k-1)} + \left(\frac{\sqrt{4m^2-t^2}-ti}{|\sqrt{4m^2-t^2}-ti|} \right)^{-(k-1)} \right) L(1/2,\ t^2-4m^2).$$

Now we are summing over all even k in (6) with $K \le k \le K + K^{1/2}$. Denote

$$e^{i\theta_{t,m}} = \frac{\sqrt{4m^2 - t^2} + ti}{|\sqrt{4m^2 - t^2} + ti|},$$

and since

$$\sum_{2|k,\ K \le k \le K+K^{1/2}} (-e^{\pm 2i\theta_{t,m}})^{k/2} \ll \frac{1}{|1 + e^{2i\theta_{t,m}}|} \ll \frac{1}{\cos\theta_{t,m}} \ll \frac{m}{\sqrt{4m^2 - t^2}},$$

we have, with $t^2 - 4m^2 = Df^2$,

$$\begin{aligned}
&\sum_{m \ll K^{1+\epsilon}} m^{-1/2} V_K(m) \sum_{|t|<2m} (4m^2 - t^2)^{-1/4} \\
&\quad \times \left(\left(\frac{\sqrt{4m^2 - t^2} + ti}{|\sqrt{4m^2 - t^2} + ti|} \right)^{-(k-1)} + \left(\frac{\sqrt{4m^2 - t^2} - ti}{|\sqrt{4m^2 - t^2} - ti|} \right)^{-(k-1)} \right) L\left(\frac{1}{2},\ t^2 - 4m^2\right) \\
&\ll K^{\epsilon} \sum_{m \ll K^{1+\epsilon}} m^{1/2} \sum_{|t|<2m} (4m^2 - t^2)^{-3/4} |L(1/2,\ t^2 - 4m^2)| \\
&\ll K^{1/2+\epsilon} \sum_{m \ll K^{1+\epsilon}} \sum_{|t|<2m} (4m^2 - t^2)^{-3/4} |L_D(1/2)| \\
&\ll K^{1/2+\epsilon} \sum_{f \ll K^{1+\epsilon}} f^{-3/2} \sum_{|D| \ll K^{2+\epsilon}} |D|^{-3/4} |L_D(1/2)| \ll K^{1+\epsilon}.
\end{aligned}$$

Hence we conclude that

$$\sum_{2|k,\ K \le k \le K+K^{1/2}} \sum_{f \in H_k} |L(1/2,\ \mathrm{sym}^2(f))|^2 \ll_{\epsilon} K^{3/2+\epsilon}.$$

This completes the proof of Theorem 1.

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