PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 10, October 2012, Pages 3365–3375
S 0002-9939(2012)11195-5
Article electronically published on April 24, 2012

HIGHLY-TRANSITIVE ACTIONS OF SURFACE GROUPS

DANIEL KITROSER

(Communicated by Daniel Ruberman)

ABSTRACT. A group action is said to be highly-transitive if it is k-transitive for every $k \geq 1$. The main result of this thesis is the following:

Main Theorem. *The fundamental group of a closed, orientable surface of genus* > 1 *admits a faithful, highly-transitive action on a countably infinite set.*

From a topological point of view, finding a faithful, highly-transitive action of a surface group is equivalent to finding an embedding of the surface group into $\operatorname{sym}\mathbb{Z}$ with a dense image. In this topological setting, we use methods that were originally developed for densely embedding surface groups in locally compact groups.

1. INTRODUCTION

A permutation group $G \leq \operatorname{Sym}(X)$ is called *k-transitive* if it is transitive on ordered k-tuples of distinct elements and *highly-transitive* if it is k-transitive for every $k \in \mathbb{N}$. When G is given as an automorphism group, it can often be verified that the given action is highly-transitive. This is the case for any subgroup of $\operatorname{Sym}(X)$ which contains all finitely supported permutations, where X is any infinite set. Other examples include groups such as $\operatorname{Homeo}(S^2)$ and many of its subgroups. But for most groups the question, "what is the maximal k for which the group admits a faithful, k-transitive action?" is wide open.

It was shown by McDonough [4] that any non-abelian free group admits a highly transitive action. Another proof which is more useful from our point of view was later given by Dixon in [3], using a Baire Category type argument.

In this paper, we prove that a *surface group* (the fundamental group of a closed, orientable surface of genus greater than 1) admits a faithful, highly-transitive action of countably infinite degree.

1.1. **Topological interpretation.** Before we continue, we introduce some notation.

Notation. We denote by $S_\infty = \operatorname{Sym}(\mathbb{Z})$ the full symmetric group of a countable set which we identify with the integer numbers. The action of $G \leq S_\infty$ on $\mathbb{Z}$ is always from the right, and the image of an element $a \in \mathbb{Z}$ under $g \in G$ is denoted by a^g. If $n \in \mathbb{N}$, then $[n]$ will denote the set of all integers between $-n$ and n (including $-n$ and n). For $\phi \in S_\infty$ and $A \subset \mathbb{Z}$ we denote

$$U(\phi, A) = \left\{\psi \in S_\infty \;\middle|\; \psi\big|_A = \phi\big|_A \right\}.$$

Received by the editors December 15, 2010 and, in revised form, April 11, 2011.
2010 *Mathematics Subject Classification.* Primary 20B22, 20B35.
The author was partially supported by ISF grant 888/07.

We define a group topology on S_∞ where a basis is given by the sets $\{U(\phi,[n]) \mid \phi \in S_\infty, n \in \mathbb{N}\}$. It is easy to see that if $\phi \in S_\infty$ and $A \subset \mathbb{Z}$ is any *finite* set, then $U(\phi, A)$ is open in this topology. Notice that this topology is the restriction to S_∞ of the topology of pointwise convergence on the space of all functions $\mathbb{Z} \to \mathbb{Z}$. We now define a metric on S_∞: for all $\phi, \psi \in S_\infty$ we set $d(\phi,\psi) = 0$ if $\phi = \psi$ and $d(\phi,\psi) = 2^{-m}$ if $\phi \neq \psi$ and m is the largest natural number such that $\phi\big|_{[m]} = \psi\big|_{[m]}$ and $\phi^{-1}\big|_{[m]} = \psi^{-1}\big|_{[m]}$. A straightforward proof shows that this metric is complete and generates the topology on S_∞.

The problem of finding highly-transitive subgroups of S_∞ can now be approached via the following proposition.

Proposition 1.1. *Let $G \leq S_\infty$. Then G is highly-transitive if and only if G is dense in S_∞.*

Proof. Suppose G is dense in S_∞ and let $k \in \mathbb{N}$. We show that G is k-transitive. Let $(a_1, \ldots, a_k), (b_1, \ldots, b_n)$ be two k-tuples of distinct elements of $\mathbb{Z}$. Since S_∞ is k-transitive, there exists $\psi \in S_\infty$ such that $a_i^\psi = b_i$ for every $i = 1, \ldots, k$ and since G is dense in S_∞, there exists $\phi \in G \cap U(\psi, \{a_1, \ldots, a_k\})$. Thus, ϕ is an element of G that takes every a_i to b_i as required. Conversely, we assume that G is highly-transitive and show that G intersects every basic neighborhood in S_∞. Let $\psi \in S_\infty$ and $n \in \mathbb{N}$. Since G is $(2n+1)$-transitive, there exists $\phi \in G$ such that $i^\phi = i^\psi$ for all $i = -n, \ldots, n$ and so, $\phi \in G \cap U(\psi, [n])$. $\square$

1.2. **The main result.** In view of Proposition 1.1, the main result can be reformulated as follows.

Theorem 1.2 (Main Theorem). *Let $\Gamma = \pi_1(\Sigma_g)$ be the fundamental group of an orientable surface of genus ≥ 2. Then there exists a dense subgroup of S_∞ which is isomorphic to Γ.*

The methods used in this work to obtain dense embeddings of surface groups into S_∞ are analogous to those used in [2] and [1] to show that if a locally compact group contains a dense, free subgroup of every finite rank > 1, then it contains a dense surface group of every genus > 1. S_∞ is *not* locally compact but as we shall see, by proving that S_∞ has a dense free subgroup of every finite rank > 1 with the additional property that certain elements of that free group generate a non-discrete cyclic group, we can apply the same methods in our setting.

2. Dense embeddings of free groups

As a first step to proving Theorem 1.2 we prove the following.

Theorem 2.1. *Let $r \geq 2$, let $w_i = w_i(\tau_1, \ldots, \tau_r)$ $(i \in \mathbb{N})$ be cyclically reduced words in the free group on $\{\tau_1, \ldots, \tau_n\}$ and assume that w_i is not a power of τ_j for any i and j. Then, there exist $\tau_1, \ldots, \tau_r \in S_\infty$ such that $F = \langle \tau_1, \ldots, \tau_r \rangle$ is a dense, rank r free subgroup of S_∞ and such that $\langle w_i(\tau_1, \ldots, \tau_r)\rangle$ is non-discrete for every $i \in \mathbb{N}$.*

We prove the theorem using Baire Category Theorem.

Definition 2.2. Let X be a topological space. A subset of X which is a countable intersection of dense, open sets is called *residual* or *generic*.

Baire's theorem states that in a non-trivial, complete metric space, a residual set is dense. Since they are dense and closed under countable intersections, residual sets in a complete metric space can be thought of as being "large". Dixon showed in [3] that if we denote

$$U = \{(\tau_1, \ldots, \tau_r) \in S_\infty^r \mid \langle \tau_1, \ldots, \tau_r \rangle \text{ is dense in } S_\infty\},$$

then the set of elements in U that freely generate a free group is residual in $\overline{U}$. We use a somewhat different setting than Dixon in order to prove the existence of the desired dense free subgroups. Fix $\sigma \in S_\infty$ to be the *shift* permutation, i.e. $\forall a \in \mathbb{Z} : a^\sigma = a + 1$. We show that for every $n \geq 1$, the set of elements $(\tau_1, \ldots, \tau_n) \in S_\infty^n$ such that $\langle \sigma, \tau_1, \ldots, \tau_n \rangle$ has the properties stated in Theorem 2.1 is residual in S_∞^n.

Lemma 2.3. *Let $\gamma \in S_\infty$. $\langle \gamma \rangle \leq S_\infty$ is non-discrete if and only if the orbits of $\langle \gamma \rangle$ are all finite and of unbounded length.*

Proof. Assume first that $\langle \gamma \rangle$ has an infinite orbit Δ and let $a \in \Delta$. Then, $\langle \gamma \rangle \cap U(1, \{a\}) = \{1\}$ and so $\langle \gamma \rangle$ is discrete. Now assume that the lengths of the orbits of $\langle \gamma \rangle$ are uniformly bounded. Let m be the product of the lengths of the orbits of $\langle \gamma \rangle$. So, $\gamma^m = 1$; thus, $\langle \gamma \rangle$ is finite and hence discrete.

Conversely, suppose that the orbits of $\langle \gamma \rangle$ are finite and of unbounded length. To prove that $\langle \gamma \rangle$ is non-discrete it is enough to show that every basic neighborhood of the identity contains some non-trivial power of γ. Let $n \in \mathbb{N}$ and let $\Delta = \bigcup_{i=1}^k \Delta_i$ be a finite union of orbits of $\langle \gamma \rangle$ such that $[n] \subset \Delta$. By hypothesis, all the Δ_i's are finite. If we set $m = \prod_{i=1}^k |\Delta_i|$, then for every $a \in \Delta$ (and in particular for every $a \in [n]$) we have that $a^{\gamma^m} = a$; and since the orbit lengths of $\langle \gamma \rangle$ are unbounded, there is an orbit of $\langle \gamma \rangle$ which is longer than m, so γ^m is not the identity element. Thus, $\gamma^m \neq 1$ is an element of the pointwise stablilizer $U(1, [n])$. Since such stabilizers form a basis at the identity, we are finished. $\square$

Definition 2.4. Let $\gamma_1, \ldots, \gamma_n \in S_\infty$ and let $w = w(\gamma_1, \ldots, \gamma_n)$ be any word. If $w = w_1 w_2 \cdots w_n$, where $w_i \in \{\gamma_1^{\pm 1}, \ldots, \gamma_n^{\pm 1}\}$, then the *trace* of an element $a \in \mathbb{Z}$ under w is the ordered set

$$\mathrm{tr}_w(a) = \{a, a^{w_1}, a^{w_1 w_2}, \ldots, a^{w_1 w_2 \cdots w_n} = a^w\}.$$

Lemma 2.5. *Fix $n \in \mathbb{N}$ and let $w = w(\sigma, \tau_1, \ldots, \tau_n)$ be a reduced word which is not a conjugate of a power of σ. Then the following sets are residual:*

1. $\mathcal{F} = \{(\tau_1, \ldots, \tau_n) \in S_\infty^n \mid \langle \sigma, \tau_1, \ldots, \tau_n \rangle$ *is a free group of rank* $n\}$.
2. $\mathcal{D} = \{(\tau_1, \ldots, \tau_n) \in S_\infty^n \mid \langle \sigma, \tau_1, \ldots, \tau_n \rangle$ *is dense*$\}$.
3. $\mathcal{N} = \{(\tau_1, \ldots, \tau_n) \in S_\infty^n \mid \langle w(\sigma, \tau_1, \ldots, \tau_n,) \rangle$ *is non-discrete*$\}$.

Proof. (1) Let v be a reduced, non-trivial word on $n+1$ letters and consider the set

$$\mathcal{F}_v = \{(\tau_1, \ldots, \tau_n) \in S_\infty^n \mid v(\sigma, \tau_1, \ldots, \tau_n) \neq 1\}.$$

If we prove that $\mathcal{F}_v$ is open and dense in S_∞^n for every v as above, then $\mathcal{F}$ is residual since $\mathcal{F} = \bigcap_{v \neq 1} \mathcal{F}_v$. Obviously, $\mathcal{F}_v$ is open as the inverse image of the open set $S_\infty \setminus \{1\}$ under the continuous mapping $(\tau_1, \ldots, \tau_n) \mapsto v(\sigma, \tau_1, \ldots, \tau_n)$. To prove that $\mathcal{F}_v$ is dense, let $(\phi_1, \ldots, \phi_n) \in S_\infty^n$ and let $m \in \mathbb{N}$. We prove that there exists $(\tau_1, \ldots, \tau_n) \in \mathcal{F}_v$ such that $\tau_i\big|_{[m]} = \phi_i\big|_{[m]}$ for every $1 \leq i \leq n$. Write

$$v(\sigma, \tau_1, \ldots, \tau_n) = \sigma^{r_1} v_1 \sigma^{r_2} v_2 \cdots \sigma^{r_k} v_k \sigma^{r_{k+1}},$$

where $r_i \in \mathbb{Z}$ and $v_i \in \{\tau_1^{\pm 1}, \ldots, \tau_n^{\pm 1}\}$ for every i. We will define the permutations $\tau_1, \ldots, \tau_n$ in stages. First, choose $a_1, \ldots, a_{k+1} \in \mathbb{Z}$ such that the numbers $a_1, \ldots, a_{k+1}, a_1 + r_1, \ldots, a_{k+1} + r_{k+1}$ are all distinct and define $(a_i + r_i)^{v_i} = a_{i+1}$ (e.g. if $v_1 = \tau_5^{-1}$ define $(a_1 + r_1)^{\tau_5^{-1}} = a_2$ or equivalently, define $a_2^{\tau_5} = a_1 + r_1$). At this point, every τ_i is defined on a finite set of points. Note that since all the integers involved so far in the definition of the τ_i's are distinct and v is reduced, τ_i can be extended to a permutation of $\mathbb{Z}$, i.e. in our definition for it so far, τ_i does not send distinct points to the same point. We now have that $a_1^v = a_{k+1} + r_{k+1}$ and in particular, v is not trivial. We choose $a_1, \ldots, a_{k+1}$ so that τ_i sends $b \in [m]$ to b^{ϕ_i}. Explicitly, choose

$$a_1, \ldots, a_{k+1} \in \mathbb{Z} \smallsetminus \left([m] \cup \left(\bigcup_i [m]^{\phi_i} \right) \cup \left(\bigcup_j [m] - r_j \right) \cup \left(\bigcup_{i,j} [m]^{\phi_i} - r_j \right) \right).$$

Note that we can always choose $a_1, \ldots, a_{k+1}$ in the manner described since we are only excluding a finite set of integers that we cannot choose from. Now every τ_i is defined on $[m]$ and on some other elements $\{b_1, \ldots, b_\ell\} \subset \{a_1, \ldots, a_k, a_1 + r_1, \ldots, a_{k+1} + r_{k+1}\}$, sending them to $\{c_1, \ldots, c_\ell\}$ respectively. Finally, choose any bijection

$$f_i : \mathbb{Z} \smallsetminus \big([m] \cup \{b_1, \ldots, b_\ell\}\big) \to \mathbb{Z} \smallsetminus \big([m]^{\phi_i} \cup \{c_1, \ldots, c_\ell\}\big)$$

and define

$$x^{\tau_i} = \begin{cases} x^{\phi_i}, & x \in [m], \\ c_j, & x = b_j \text{ for some } j, \\ x^{f_i}, & \text{else.} \end{cases}$$

So we have that every τ_i is a permutation which lies in the basic neighbourhood of ϕ_i defined by $[m]$ such that $v(\sigma, \tau_1, \ldots, \tau_n)$ is non-trivial.

(2) Fix some $k \in \mathbb{N}$ and for every two k-tuples $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ of distinct integers consider the set

$$\mathcal{D}_{\mathbf{x},\mathbf{y}} = \left\{ (\tau_1, \ldots, \tau_n) \in S_\infty^n \;\middle|\; \begin{array}{l} \exists \phi \in \langle \sigma, \tau_1, \ldots, \tau_n \rangle : \\ \forall 1 \le j \le k : x_j^{\phi} = y_j \end{array} \right\}.$$

Notice that if $(\tau_1, \ldots, \tau_n) \in \bigcap_{\mathbf{x},\mathbf{y}} \mathcal{D}_{\mathbf{x},\mathbf{y}}$, where $\mathbf{x}$ and $\mathbf{y}$ range over all k-tuples of distinct integers, then $\langle \sigma, \tau_1, \ldots, \tau_n \rangle$ is k-transitive. Thus, by Proposition 1.1 we have that

$$\mathcal{D} = \bigcap_{k \in \mathbb{N}} \bigcap_{\mathbf{x},\mathbf{y}} \mathcal{D}_{\mathbf{x},\mathbf{y}}$$

and we are left to show that $\mathcal{D}_{\mathbf{x},\mathbf{y}}$ is open and dense for every $\mathbf{x}$ and $\mathbf{y}$ as above.

First, fix $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k)$ as above and let $(\tau_1, \ldots, \tau_n) \in \mathcal{D}_{\mathbf{x},\mathbf{y}}$. By definition there exists $\phi \in \langle \sigma, \tau_1, \ldots, \tau_n \rangle$ such that $x_i^{\phi} = y_i$ for every $i = 1, \ldots, k$. Let us write $\phi = v(\sigma, \tau_1, \ldots, \tau_n)$, where $v = v(\sigma, \tau_1, \ldots, \tau_n)$ is a word on $\{\sigma^{\pm 1}, \tau_1^{\pm 1}, \ldots, \tau_n^{\pm 1}\}$ and define $A = \bigcup_{i=1}^k \mathrm{tr}_v(x_i)$. A is finite and so the set

$$\mathcal{U} = \left\{ (\psi_1, \ldots, \psi_n) \in S_\infty^n \;\middle|\; \begin{array}{l} \psi_i\big|_A = \tau_i\big|_A,\ \psi_i^{-1}\big|_A = \tau_i^{-1}\big|_A \\ i = 1, \ldots, n \end{array} \right\}$$

is an open neighbourhood of $(\tau_1, \ldots, \tau_n)$ contained in $\mathcal{D}_{\mathbf{x},\mathbf{y}}$. Indeed, if $(\psi_1, \ldots, \psi_n) \in \mathcal{U}$ take $\xi = v(\sigma, \psi_1, \ldots, \psi_n) \in \langle \sigma, \psi_1, \ldots, \psi_n \rangle$. Then by the definition of $\mathcal{U}$, we have that ξ acts the same as $\phi = v(\sigma, \tau_1, \dots, \tau_n)$ on x_i and in particular, ξ sends x_i to y_i. This shows that $\mathcal{D}_{\mathbf{x},\mathbf{y}}$ is open.

Now we prove that $\mathcal{D}_{\mathbf{x},\mathbf{y}}$ is dense. Let $(\phi_1, \dots, \phi_n) \in S_\infty^n$ and $m \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $x_j + r \notin [m]$ and $y_j + r \notin [m]^{\phi_1}$ for every $1 \leq j \leq k$. Let

$$f : \mathbb{Z} \smallsetminus \big([m] \cup \{x_1 + r, \dots, x_k + r\}\big) \to \mathbb{Z} \smallsetminus \big([m]^{\phi_1} \cup \{y_1 + r, \dots, y_k + r\}\big)$$

be any bijection and define

$$x^{\tau_1} = \begin{cases} x^{\phi_1}, & x \in [m], \\ y_j + r, & x = x_j + r \text{ for some } 1 \leq j \leq k, \\ x^f, & \text{otherwise.} \end{cases}.$$

Now defining $\tau_i = \phi_i$ for every $2 \leq i \leq n$, we get that $(\tau_1, \dots, \tau_n)$ is an element of the basic neighbourhood of $(\phi_1, \dots, \phi_n)$ defined by $[m]$. Also, the permutation $\xi = \sigma^r \tau_1 \sigma^{-r} \in \langle \sigma, \tau_1, \dots, \tau_n \rangle$ sends each x_j to y_j; thus, $(\tau_1, \dots, \tau_n) \in \mathcal{D}_{\mathbf{x},\mathbf{y}}$.

(3) By Lemma 2.3 we can equivalently write

$$\mathcal{N} = \left\{ (\tau_1, \dots, \tau_n) \in S_\infty^n \;\middle|\; \begin{array}{l} \text{The orbits of } \langle w(\sigma, \tau_1, \dots, \tau_n) \rangle \text{ are all finite} \\ \text{and of unbounded length.} \end{array} \right\}.$$

Thus, if we define for every $t \in \mathbb{N}$ and $a \in \mathbb{Z}$:

$$\mathcal{U}_t = \{(\tau_1, \dots, \tau_n) \in S_\infty^n \mid \langle w(\sigma, \tau_1, \dots, \tau_n) \rangle \text{ has an orbit of length} \geq t\},$$
$$\mathcal{V}_a = \{(\tau_1, \dots, \tau_n) \in S_\infty^n \mid \text{the orbit of } a \text{ under } \langle w(\sigma, \tau_1, \dots, \tau_n) \rangle \text{ is finite}\},$$

we get that $\mathcal{N} = \big(\bigcap_{t \in \mathbb{N}} \mathcal{U}_t\big) \cap \big(\bigcap_{a \in \mathbb{Z}} \mathcal{V}_a\big)$.

We now show that $\mathcal{U}_t$ and $\mathcal{V}_a$ are open and dense for every $t \in \mathbb{N}$ and $a \in \mathbb{Z}$. Let $(\tau_1, \dots, \tau_n) \in \mathcal{U}_t$ and let $b \in \mathbb{Z}$ be an element belonging to an orbit of $\langle w(\sigma, \tau_1, \dots, \tau_n) \rangle$ of length $\geq t$. Thus, $b, b^w, b^{w^2}, \dots, b^{w^{t-1}}$ are all distinct. Let $\Delta = \bigcup_{i=1}^{k-1} \operatorname{tr}_{w^i}(b)$. The set

$$\left\{ (\psi_1, \dots, \psi_n) \in S_\infty^n \;\middle|\; \begin{array}{l} \psi_i\big|_\Delta = \tau_i\big|_\Delta,\ \psi_i^{-1}\big|_\Delta = \tau_i^{-1}\big|_\Delta \\ i = 1, \dots, n \end{array} \right\}$$

is an open neighbourhood of $(\tau_1, \dots, \tau_n)$ which is contained in $\mathcal{U}_t$; hence, $\mathcal{U}_t$ is open.

Now, take $(\tau_1, \dots, \tau_n) \in \mathcal{V}_a$ and let $\Delta = \{a_1, \dots, a_s\}$ be the finite orbit of $\langle w(\sigma, \tau_1, \dots, \tau_n) \rangle$ containing a. Similarly

$$\left\{ (\psi_1, \dots, \psi_n) \in S_\infty^n \;\middle|\; \begin{array}{l} \psi_i\big|_\Delta = \tau_i\big|_\Delta,\ \psi_i^{-1}\big|_\Delta = \tau_i^{-1}\big|_\Delta \\ i = 1, \dots, n \end{array} \right\}$$

is an open neighbourhood of $(\tau_1, \dots, \tau_n)$ which is contained in $\mathcal{V}_a$.

To prove that $\mathcal{U}_t$ is dense, let $(\phi_1, \dots, \phi_n) \in S_\infty^n$ and $m \in \mathbb{Z}$. In (1) we in fact showed that we can define $\tau_1, \dots, \tau_n \in S_\infty$ such that for every finite set $A \subset \mathbb{Z}$ we have that $\tau_i\big|_A$ acts in any way we please and there exists some $b \in \mathbb{Z}$ such that $b^w \neq b$, where $w = w(\sigma, \tau_1, \dots, \tau_n)$. By repeating the same argument we can find $\tau_1, \dots, \tau_n \in S_\infty$ such that $\tau_i\big|_{[m]} = \phi_i\big|_{[m]}$ for every $1 \leq i \leq n$ and such that there exists $b \in \mathbb{Z}$ such that $b, b^w, b^{w^2}, \dots, b^{w^{t-1}}$ are all distinct; i.e., $\langle w(\sigma, \tau_1, \dots, \tau_n) \rangle$ has an orbit of length $\geq t$.

Finally, we prove that $\mathcal{V}_a$ is dense. Since $\langle w(\sigma, \tau_1, \dots, \tau_n) \rangle$ has the same orbit structure as the cyclic group generated by any conjugate of w, we can assume without loss of generality that w is a cyclically reduced word that is not a power of σ. Let $(\phi_1, \dots, \phi_n) \in S_\infty^n$ and $m \in \mathbb{Z}$. We need to define permutations $\tau_1, \dots, \tau_n \in S_\infty$

such that $(\tau_1, \dots, \tau_n) \in \mathcal{V}_a$ and every τ_i agrees with ϕ_i on $[m]$. This condition can be thought of in the following way: τ_i is already defined on $[m]$ for every i and τ_i^{-1} is already defined on $[m]^{\phi_i}$ for every i (they act the same as ϕ_i and ϕ_i^{-1} respectively) and we are left to define τ_i on $\mathbb{Z} \smallsetminus [m]$ (and τ_i^{-1} on $\mathbb{Z} \smallsetminus [m]^{\phi_i}$) in such a way that the orbit of a under $\langle w(\sigma, \tau_1, \dots, \tau_n)\rangle$ will be finite. First we write $w(\sigma, \tau_1, \dots, \tau_n) = w_1 w_2 \cdots w_k$. Now, we start by applying the positive and negative powers of w to a letter by letter:

$$\xrightarrow{w_\ell} c \xrightarrow{w_{\ell+1}} \dots \xrightarrow{w_{n-1}} a_{-1} \xrightarrow{w_n} a = a_0 \xrightarrow{w_1} a_1 \xrightarrow{w_2} \dots \xrightarrow{w_{s-1}} b \xrightarrow{w_s},$$

where $b \in \mathbb{Z}$ is the first element such that we need to apply to it the permutation w_s and w_s is not yet defined on b; that is, $w_s = \tau_i$ for some i and $b \notin [m]$ or $w_s = \tau_i^{-1}$ for some i and $b \notin [m]^{\phi_i}$. Note that if such an element b does not exist, then, since by hypothesis at least one of the letters $w_1, \dots, w_k$ is not σ, the orbit of a under $\langle w(\sigma, \phi_1, \dots, \phi_n)\rangle$ is contained in $[m] \cup \left(\bigcup_{i=1}^n [m]^{\phi_i}\right)$, hence is finite, and we can just take $\tau_i = \phi_i$ for every i. We can thus assume that such a b exists. Similarly, $c \in \mathbb{Z}$ is the first element we reach when applying letter by letter the negative powers of w to a such that we need to apply to c the permutation w_ℓ^{-1}, and w_ℓ^{-1} is not yet defined on c. As before, we can assume without loss of generality that such an element c exists. By hypothesis, w is cyclically reduced and so the word $w_s w_{s+1} \cdots w_k w_1 \cdots w_\ell$ is reduced. Also, by their definition, $w_s, w_\ell \neq \sigma^{\pm 1}$. We wish to define $\tau_1, \dots, \tau_n$ in such a way that $b^{w_s w_{s+1} \cdots w_k w_1 \cdots w_\ell} = c$ (and of course fulfilling the condition that τ_i agrees with ϕ_i on $[m]$). By repeating the argument made in (1), we can find two distinct elements $d_1, d_2 \in \mathbb{Z}$ that do not lie in $[m]$ or any $[m]^{\phi_i}$ such that $w_{s+1} \cdots w_k w_1 \cdots w_{\ell-1}$ sends d_1 to d_2 and define $b^{w_s} = d_1$, $d_2^{w_\ell} = c$. From this we get that $b^{w_s \cdots w_k w_1 \cdots w_l} = d_1^{w_{s+1} \cdots w_k w_1 \cdots w_\ell} = d_2^{w_\ell} = c$l; thus the orbit of a is finite. Now, every τ_i is defined on $[m]$ and maybe on finitely many more elements. Again, exactly as we did in (1), we can extend the definition of τ_i to $\mathbb{Z}$ and get permutations $\tau_1, \dots, \tau_n$ satisfying the required conditions. □

Proof of Theorem 2.1. Let $j \in \{1, \dots, r\}$ and set $\tau_j = \sigma$. Then by Lemma 2.5, the set

$$W_i = \left\{ (\tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r) \in S_\infty^{r-1} \;\middle|\; \begin{array}{l} \langle \tau_1, \dots, \tau_r \rangle \text{ is a dense, rank } r \text{ free} \\ \text{subgroup and } \langle w_i(\tau_1, \dots, \tau_r)\rangle \\ \text{is non-discrete} \end{array} \right\}$$

is residual and so $\bigcap_{i\in\mathbb{N}} W_i$ is residual and, in particular, not empty. □

3. Eventually faithful sequences

Definition 3.1. Let G and H be groups. A sequence $\{f_n\}_{n=1}^\infty$ of homomorphisms from G to H is *eventually faithful* if for every $g \in G$ there exists $n_0 \in \mathbb{N}$ such that $g \notin \ker(f_n)$ for all $n \geq n_0$.

In order to prove the main theorem, we will need to produce eventually faithful sequences of homomorphisms from surface groups to free groups. The following constructions appear in [2] and [1]. Let $\Gamma = \Gamma_{2r}$ be the surface group of genus $2r$ $(r \geq 1)$. We have the following presentation for Γ:

$$\Gamma = \langle a_1, a_1', \dots, a_r, a_r', b_1, b_1', \dots, b_r, b_r' \mid [a_1, a_1'] \cdots [a_r, a_r'][b_r', b_r] \cdots [b_1', b_1]\rangle.$$

Let $x = [a_1, a'_1] \cdots [a_r, a'_r]$ and let $h : \Gamma \to \Gamma$ be the Dehn twist around x, i.e.

$$h(a_i) = a_i, \qquad h(b_i) = x b_i x^{-1},$$
$$h(a'_i) = a'_i, \qquad h(b'_i) = x b'_i x^{-1}.$$

Let F be the free group on $2r$ free generators $\{\phi_1, \phi'_1, \ldots, \phi_r, \phi'_r\}$ and let $k : \Gamma \to F$ be the homomorphism defined by

$$k(a_i) = k(b_i) = \phi_i,$$
$$k(a'_i) = k(b'_i) = \phi'_i.$$

Consider the map that folds the genus $2r$ surface that has Γ as its fundamental group across the curve corresponding to x (this curve separates the surface into two equal parts). The image of this map is a surface of genus r with one boundary component, so it has F as its fundamental group. k is the homomorphism induced on the fundamental groups by this folding map (see Figure 1). Denote $f_n = k \circ h^n$.

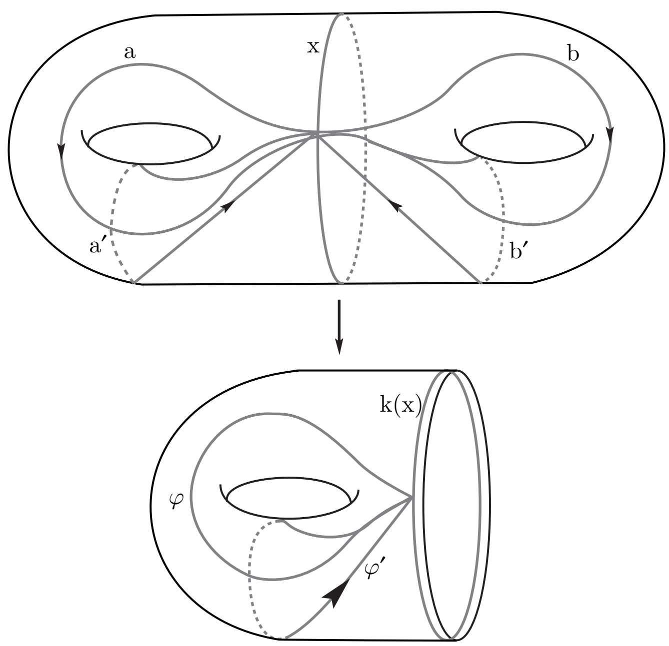

FIGURE 1

Lemma 3.2 (Breuillard, Gelander, Souto, Storm [2])**.** *The sequence* $\{f_n\}_{n=1}^{\infty}$ *is eventually faithful.*

We now construct an eventually faithful sequence of homomorphisms from an odd genus surface group into a free group. Let $\Gamma = \Gamma_{2r+1}$ be a surface group of genus $2r + 1$ ($r \geq 1$) with the presentation

$$\Gamma = \langle a_1, a'_1, \ldots, a_r, a'_r, b, b', c_1, c'_1, \ldots, c_r, c'_r \mid [a_1, a'_1] \cdots [a_r, a'_r][b', b][c'_1, c_1] \cdots [c'_r, c_r] \rangle.$$

Denote $x = [a_1, a'_1] \cdots [a_r, a'_r]b'$ and let F be the free group on $2r+1$ free generators $\{\phi_1, \phi'_1, \ldots, \phi_r, \phi'_r, \tau\}$. Let $\delta : \Gamma \to \Gamma$ and $\zeta : \Gamma \to \Gamma$ be Dehn twists around x and b' respectively, that is,

$$\begin{aligned} \delta(a_i) &= a_i & \zeta(a_i) &= a_i \\ \delta(a'_i) &= a'_i & \zeta(a'_i) &= a'_i \\ \delta(b) &= xb & \zeta(b) &= b(b')^{-1} \\ \delta(b') &= b' & \zeta(b') &= b' \\ \delta(c_i) &= xc_ix^{-1} & \zeta(c_i) &= c_i \\ \delta(c'_i) &= xc'_ix^{-1} & \zeta(c'_i) &= c'_i. \end{aligned}$$

Notice that δ and ζ commute.

Let $k : \Gamma \to F$ be the map induced by folding the $2r+1$ surface across the curves x and b' (these curves separate the surface into two surfaces of genus r and two boundary components). Explicitly,

$$\begin{aligned} k(a_i) &= \phi_i \\ k(a'_i) &= \phi'_i \\ k(b) &= 1 \\ k(b'_i) &= \tau \\ k(c_i) &= \phi_i \\ k(c'_i) &= \phi'_i \end{aligned}$$

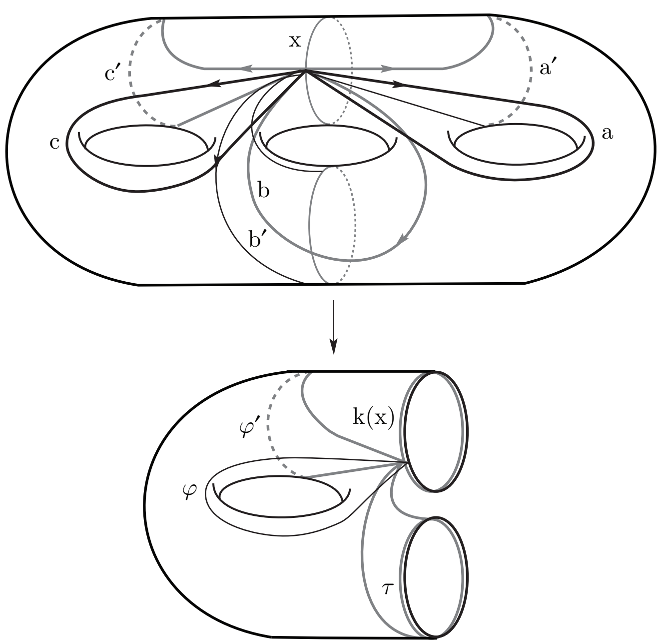

FIGURE 2

(see Figure 2). Finally, denote $\rho_n = k \circ (\delta \circ \zeta)^n$.

Lemma 3.3 (Barlev, Gelander [1])**.** *The sequence* $\{\rho_n\}_{n=1}^{\infty}$ *is eventually faithful.*

4. Proof of the main theorem

We will need the following results.

Lemma 4.1. *Let* $\phi, \psi \in S_\infty$ *be such that both* $\langle\phi\rangle$ *and* $\langle\psi\rangle$ *are non-discrete. Then* $\langle(\phi, \psi)\rangle \leq S_\infty^2$ *is non-discrete.*

Proof. We prove that every basic neighbourhood of $(1, 1)$ contains a non-trivial element of $\langle(\phi, \psi)\rangle$. Let $m \in \mathbb{N}$. Let $\Delta = \bigcup_{i=1}^{\ell} \Delta_i$ and $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$ be finite unions of orbits of $\langle\phi\rangle$ and $\langle\psi\rangle$ respectively such that $[m] \subset \Delta$ and $[m] \subset \Gamma$. From Proposition 2.3 we have that all the orbits of $\langle\phi\rangle$ and $\langle\psi\rangle$ are finite, and so we can define the number

$$n = \prod_{i=1}^{\ell} |\Delta_i| \cdot \prod_{i=1}^{k} |\Gamma_i|.$$

Notice that every element of Δ is fixed by ϕ^n and every element of Γ is fixed by ψ^n and in particular every $i \in [m]$ is fixed by ϕ^n and ψ^n. From Proposition 2.3 we also have that the lengths of the orbits of $\langle\phi\rangle$ and $\langle\psi\rangle$ are unbounded and in particular $\langle\phi\rangle$ and $\langle\psi\rangle$ both have an orbit of length greater than n, and so ϕ^n and ψ^n are non-trivial. Thus $(\phi, \psi)^n$ is a non-trivial element contained in the basic neighbourhood of $(1, 1)$ defined by $[m] \times [m]$. □

Lemma 4.2. *Let* G *be a Hausdorff topological group and let* $\gamma \in G$ *such that* $\langle\gamma\rangle$ *is non-discrete. Then, for every* $n_0 \in \mathbb{N}$ *the set* $\{\gamma^n \mid n \geq n_0\}$ *is dense in* $\overline{\langle\gamma\rangle}$*.*

Proof. First we notice that since $\langle\gamma\rangle$ is non-discrete, then also $\overline{\langle\gamma\rangle}$ is non-discrete. Let $U \subset \overline{\langle\gamma\rangle}$ be open. Then we have $\gamma^m \in U$ for some $m \in \mathbb{Z}$. If $m \geq n_0$ we are done, so assume $m < n_0$. Now, if we denote

$$U' = U\gamma^{-m},$$
$$U'' = U' \cap (U')^{-1},$$

then U'' is an open symmetric identity neighborhood and since $\overline{\langle\gamma\rangle}$ is non-discrete, U'' is not finite. Now take

$$\widetilde{U} = U'' \setminus \{\gamma^k \ : \ |k| < n_0 - m\}.$$

We have that $\widetilde{U}$ is open (since G is Hausdorff) and non-empty; thus there exists $n \in \mathbb{Z}$ such that $\gamma^n \in \widetilde{U}$. By the definition of $\widetilde{U}$ we have that $|n| \geq n_0 - m$. Also, since $\widetilde{U} \subset U''$ and U'' is symmetric it follows that $\gamma^n, \gamma^{-n} \in U'' \subset U'$. Hence, $\gamma^{n+m}, \gamma^{-n+m} \in U$, and since either $n + m \geq n_0$ or $-n + m \geq n_0$ we are done. □

4.1. **Proof of Theorem 1.2 for even genus.** By Theorem 2.1 there exists a subgroup $F \leq S_\infty$ such that F is dense, free with $2r$ free generators $\phi_1, \phi_1', \ldots, \phi_r, \phi_r' \in S_\infty$ and such that $\langle[\phi_1, \phi_1'] \cdots [\phi_r, \phi_r']\rangle$ is non-discrete. Denote $\gamma = [\phi_1, \phi_1'] \cdots [\phi_r, \phi_r']$ and $\Omega = \overline{\langle\gamma\rangle}$. Let $\Gamma = \Gamma_{2r}$ be a surface group of genus $2r$ ($r \geq 1$) with the presentation

$$\Gamma = \langle a_1, a_1', \ldots, a_r, a_r', b_1, b_1', \ldots, b_r, b_r' \,|\, [a_1, a_1'] \cdots [a_r, a_r'][b_r', b_r] \cdots [b_1', b_1]\rangle.$$

Define for every $\omega \in \Omega$ a homomorphism $f_\omega : \Gamma \to S_\infty$ by

$$f_\omega(a_i) = \phi_i,$$
$$f_\omega(a_i') = \phi_i',$$
$$f_\omega(b_i) = \omega\phi_i\omega^{-1},$$
$$f_\omega(b_i') = \omega\phi_i'\omega^{-1}.$$

Since every $\omega \in \Omega$ commutes with γ, this defines a homomorphism. Indeed, for every $\omega \in \Omega$:

$$\begin{aligned} &f_\omega([a_1, a_1'] \cdots [a_r, a_r'][b_r', b_r] \cdots [b_1', b_1]) \\ &= \underbrace{[\phi_1, \phi_1'] \cdots [\phi_r, \phi_r']}_{\gamma}\, \omega\, \underbrace{[\phi_r', \phi_r] \cdots [\phi_1', \phi_1]}_{\gamma^{-1}}\, \omega^{-1} \\ &= \gamma\omega\gamma^{-1}\omega^{-1} = 1. \end{aligned}$$

Every element of Ω corresponds to a homomorphism $\Gamma \to S_\infty$ whose image contains F; hence the image is dense. We are left to show that at least one of those homomorphisms is also faithful. The space Ω is a completely metrizable space, so it suffices to show that $\chi = \{\omega \in \Omega \mid f_\omega \text{ is faithful}\}$ is residual in Ω.

For every $g \in \Gamma \smallsetminus \{1\}$ denote $\chi_g = \{\omega \in \Omega \mid f_\omega(g) \neq 1\}$. Notice that

$$\chi = \bigcap_{g \in \Gamma \smallsetminus \{1\}} \chi_g.$$

From Lemma 3.2, the sequence $\{f_{\gamma^n}\}_{n\in\mathbb{N}}$ is eventually faithful, and so for every $g \in \Gamma \smallsetminus \{1\}$ there exists $n_0 \in \mathbb{N}$ such that $\gamma^n \in \chi_g$ for all $n \geq n_0$. By Lemma 4.2 we have that $\{\gamma^n \mid n \geq n_0\}$ is dense in Ω, and so χ_g is dense in Ω. χ_g is also open as the inverse image of the open set $S_\infty \smallsetminus \{1\}$ under the continuous map $\omega \to f_\omega(g)$. This shows that χ is residual in Ω. □

4.2. **Proof of Theorem 1.2 for odd genus.** Let $\Gamma = \Gamma_{2r+1}$ be a surface group of genus $2r+1$ $(r \geq 1)$ with the presentation

$$\Gamma = \langle a_1, a_1', \ldots, a_r, a_r', b, b', c_1, c_1', \ldots, c_r, c_r' \mid [a_1, a_1'] \cdots [a_r, a_r'][b', b][c_1', c_1] \cdots [c_r', c_r]\rangle.$$

Let $F \leq S_\infty$ be a dense, free subgroup on $2r+1$ free generators $\phi_1, \phi_1', \ldots, \phi_r, \phi_r', \tau \in S_\infty$ such that $\langle\tau\rangle$ and $\langle[\phi_1, \phi_1'] \cdots [\phi_r, \phi_r']\tau\rangle$ are non-discrete (the existence of such a free subgroup is assured by Theorem 2.1). Denote $\gamma = [\phi_1, \phi_1'] \cdots [\phi_r, \phi_r']\tau$ and $\Omega = \overline{\langle(\gamma, \tau)\rangle} \subset S_\infty^2$. For every $(\psi, \xi) \in \Omega$ we define a homomorphism $f_{(\psi,\xi)} : \Gamma \to S_\infty$ by setting

$$f_{(\psi,\xi)}(a_i) = \phi_i, \qquad f_{(\psi,\xi)}(b) = \psi\xi^{-1}, \qquad f_{(\psi,\xi)}(c_i) = \psi\phi_i\psi^{-1},$$
$$f_{(\psi,\xi)}(a_i') = \phi_i', \qquad f_{(\psi,\xi)}(b') = \tau, \qquad f_{(\psi,\xi)}(c_i') = \psi\phi_i'\psi^{-1}.$$

Since every $(\psi,\xi)\in\Omega$ commutes with (γ,τ) we have that $f_{(\psi,\xi)}$ is well defined as a homomorphism because

$$\begin{aligned}&f_{(\psi,\xi)}([a_1,a_1']\cdots[a_r,a_r'][b',b][c_1',c_1]\cdots[c_r',c_r])\\&=[\phi_1,\phi_1']\cdots[\phi_r,\phi_r'][\tau,\psi\xi^{-1}]\psi\underbrace{[\phi_1',\phi_1]\cdots[\phi_r',\phi_r]}_{=\tau\gamma^{-1}}\psi^{-1}\\&=\gamma\psi\xi^{-1}\tau^{-1}\xi\psi^{-1}\psi\tau\gamma^{-1}\psi^{-1}=\gamma\psi\xi^{-1}\tau^{-1}\xi\tau\gamma^{-1}\psi^{-1}\\&=\gamma\psi\xi^{-1}\xi\tau^{-1}\tau\gamma^{-1}\psi^{-1}=\gamma\psi\gamma^{-1}\psi^{-1}=1.\end{aligned}$$

Notice that $f_{(\gamma^n,\tau^n)}=\rho_n$ (as defined is Section 3) for every $n\in\mathbb{N}$, and so by Lemma 3.3, the sequence $f_{(\gamma^n,\tau^n)}$ is eventually faithful.

The image of every homomorphism $f_{(\gamma^n,\tau^n)}$ contains F; hence the image is dense. Finally, we show that $\chi=\{(\psi,\xi)\in\Omega\mid f_{(\psi,\xi)}\text{ is faithful}\}$ is residual in the completely metrizable space Ω and, in particular, $\chi\neq\varnothing$. For every $g\in\Gamma\smallsetminus\{1\}$ we denote $\chi_g=\{(\psi,\xi)\in\Omega\mid f_{(\psi,\xi)}(g)\neq 1\}$. As in the previous section, χ_g is open as the inverse image of an open set under a continuous map. Since $f_{(\gamma^n,\tau^n)}$ is eventually faithful there exists $n_0\in\mathbb{N}$ such that $f_{(\gamma^n,\tau^n)}\in\chi_g$ for all $n\geq n_0$. Since $\langle\tau\rangle$ and $\langle\gamma\rangle$ are non-discrete we get from Lemma 4.1 that Ω is non-discrete, and so by Lemma 4.2 we have that $\{(\gamma^n,\tau^n)\mid n\geq n_0\}$ is dense in Ω. This shows that χ_g is dense; hence

$$\chi=\bigcap_{g\in\Gamma\smallsetminus\{1\}}\chi_g$$

is residual.

Acknowledgement

This work is the author's M.Sc. thesis. He wishes to thank his advisor, Professor Yair Glasner, for his advice, patience and guidance.

References

1. J. Barlev and T. Gelander, *Compactifications and algebraic completions of limit groups*, Journal d'Analyse Mathématique 112 (2010), no. 1, 261–287. MR2763002
2. E. Breuillard, T. Gelander, J. Souto and P. Storm, *Dense embeddings of surface groups*, Geometry and Topology 10 (2006), 1373–1389. MR2255501 (2008b:22007)
3. J. D. Dixon, *Most finitely generated permutation groups are free*, Bull. London Math. Soc. 22 (1990), no. 3, 222–226. MR1041134 (91c:20005)
4. T. P. McDonough, *A permutation representation of a free group*, Quart. J. Math. Oxford Ser. 28 (1977), no. 3, 353–356. MR0453869 (56:12122)

Department of Mathematics, Ben-Gurion University of The Negev, Be'er Sheva, Israel

E-mail address: kitrosar@bgu.ac.il