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ON THE EXISTENCE OF J-CLASS OPERATORS ON BANACH SPACES

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Abstract. In this paper we answer in the negative the question raised by G. Costakis and A. Manoussos whether there exists a J-class operator on every non-separable Banach space. In particular we show that there exists a non-separable Banach space constructed by S. Argyros, A. Arvanitakis and A. Tolias such that the J-set of every operator on this space has empty interior for each non-zero vector. On the other hand, on non-separable spaces which are reflexive there always exists a J-class operator.

1. Preliminaries and the main result

Let X be a real or complex Banach space. If X is a real Banach space, then by $X_{\mathbb{C}}$ we denote the complexification of X. By $L(X)$ we mean the space of all bounded linear operators acting on X. If $T \in L(X)$, the symbol $\sigma(T)$ stands for the spectrum of T. Consider any subset C of X. The symbol C° denotes the interior of C in the norm topology of X. The symbol $\operatorname{orb}(T,x)$ denotes the orbit of x under T, i.e. $\operatorname{orb}(T,x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$. If X is separable and $\operatorname{orb}(T,x)$ is dense, then T is called hypercyclic, which is equivalent to T being topologically transitive; i.e. for every pair of non-empty open sets $U, V \subset X$, there exists a non-negative integer n such that $T^n(U) \cap V \neq \emptyset$. Following [7], by $J_T(x)$ we denote the J-set of x under T, i.e.

$$\begin{aligned} J_T(x) := \{y \in X :\ & \text{there exists a strictly increasing sequence} \\ & \text{of natural numbers } (k_n) \text{ and a sequence} \\ & (x_n) \text{ in } X, \text{ such that } x_n \to x \text{ and } T^{k_n} x_n \to y\}. \end{aligned}$$

If $J_T(x) = X$ for some $x \in X \backslash \{0\}$, then T is called a J-class operator. By A_T we denote the set of all $x \in X$ such that $J_T(x) = X$. On separable spaces every hypercyclic operator is J-class, but the converse is not true. It is known [5] that on l^{∞}, there does not exist a topological transitive operator. On the other hand there exist J-class operators such as the weighted backward shift $\lambda B : l^{\infty} \to l^{\infty}$, $\lambda B(x_1, x_2, \dots) := (\lambda x_2, \lambda x_3, \dots)$ for $|\lambda| > 1$. Therefore it is natural to ask whether every non-separable Banach space admits a J-class operator [7]. Our main result is the following:

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Theorem 1.1. *There exists a non-separable complex Banach space X on which the J-set of every operator has empty interior for every non-zero vector. Consequently there exists no J-class operator on X.*

The statement in Theorem 1.1 gives us a stronger result than the question raised by G. Costakis and A. Manoussos, since in general $(J_T(x))^\circ \neq \emptyset$ does not imply that $J_T(x) = X$, [4]. As is clear from our proof, the conclusion of Theorem 1.1 is satisfied for every complex non-separable Banach space, for which every $T \in L(X)$ is of the form $T = \lambda I + S$ with $\lambda \in \mathbb{C}$ and S a strictly singular operator with separable range. A real non-separable HI (Hereditarily Indecomposable) Banach space containing no reflexive subspace for which every $T \in L(X)$ takes the form $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and S a weakly compact operator with separable range has been constructed by S. Argyros, A. Arvanitakis and A. Tolias in [3]. The complexification of this space is easily shown to satisfy our requirements, and thus the conclusion of Theorem 1.1. In contrast we show in Theorem 1.18 that every non-separable reflexive Banach space admits a J-class operator.

Definition 1.2. Let X, Y be infinite dimensional Banach spaces. A linear and bounded operator $S : X \to Y$ is called strictly singular if for every infinite dimensional subspace $M \subset X$ the restriction $S_{|M} : M \to S(M)$ is not an isomorphism (linear homeomorphism).

Remark 1.3. If $X = Y$, then an immediate consequence of the above definition is that $0 \in \sigma(S)$. The spectrum of S is at most countable with 0 as the only possible accumulation point, [1].

The next two theorems can be found in [1].

Theorem 1.4. *Let X be a Banach space. The collection of all strictly singular operators is a closed subspace in $L(X)$, which is also a two-sided ideal.*

Proposition 1.5. *Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. A bounded operator $T : X \to X$ is strictly singular if and only if $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ is strictly singular.*

We show now that a certain class of operators on a complex Banach space can not be J-class. For this purpose we need the following proposition.

Proposition 1.6. *Let X be a complex Banach space and $T \in L(X)$. Suppose $(J_T(x))^\circ \neq \emptyset$ for some $x \in X \backslash \{0\}$. Then $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$.*

Proof. Assume that $\sigma(T) \cap \partial\mathbb{D} = \emptyset$. We decompose σ in $\sigma_1 := \{\lambda \in \sigma : |\lambda| > 1\}$ and $\sigma_2 := \{\lambda \in \sigma : |\lambda| < 1\}$. Then σ_1 and σ_2 are disjoint and closed. By the Riesz decomposition theorem we can decompose $X = M_1 \oplus M_2$, where M_1 and M_2 are closed and T-invariant subspaces and $\sigma_1 = \sigma(T_{|M_1})$, $\sigma_2 = \sigma(T_{|M_2})$. Assume now that there exists a non-zero vector $x \in X$, such that $J_T(x)$ has non-empty interior. We can write $x = x_1 + x_2$, with $x_1 \in M_1$ and $x_2 \in M_2$. Since $x \neq 0$, either x_1 or x_2 is not equal to zero. Consider the projection $P_1 : X \to M_1$ along M_2 onto M_1. Since $J_T(x) \subset J_{T|_{M_1}}(x_1) + J_{T|_{M_2}}(x_2)$ it follows that $P_1(J_T(x)) \subset J_{T|_{M_1}}(x_1)$. By the open mapping theorem we get that $P_1(J_T(x))$ has non-empty interior and hence $(J_{T|_{M_1}}(x_1))^\circ \neq \emptyset$. From the spectral radius formula we obtain that $\|T_1^n \widetilde{x}\| \leq a^n \|\widetilde{x}\|$ for some $a \in \mathbb{R}$ with $0 \leq a < 1$ and for all $\widetilde{x} \in M_1$. This implies that $J_{T|_{M_1}}(x_1) = \{0\}$ and therefore $M_1 = \{0\}$. So we get $x_1 = 0$. Again from the spectral radius formula we know that $\|T_1^n(\hat{x})\| \geq A^n \|\hat{x}\|$ for some $A > 1$ and all $\hat{x} \in M_2$. This implies $x_2 = 0$, which is a contradiction to our assumption that $x \neq 0$. $\square$

Theorem 1.7. *Let X be a complex infinite dimensional Banach space. Then for every operator of the form $T = \lambda I + S$, where S is strictly singular and $|\lambda| > 1$, and every $x \neq 0$, the set $J_T(x)$ has empty interior.*

Proof. By Remark 1.3 it follows that $\lambda \in \sigma(T)$ and it is the only possible accumulation point. We decompose the spectrum in $\sigma_1 := \{\mu \in \sigma(T) : |\mu| \leq 1\}$ and $\sigma_2 := \{\mu \in \sigma(T) : |\mu| > 1\}$. Clearly then $\lambda \in \sigma_2$. The set σ_1 is closed, and since $\lambda \in \sigma_2$, and λ is the only possible accumulation point, we conclude that σ_2 is also closed. Futhermore σ_1 and σ_2 are disjoint. By the Riesz decomposition theorem we can decompose $X = M_1 \oplus M_2$, where M_1 and M_2 are closed and T-invariant subspaces and $\sigma_1 = \sigma(T_{|M_1})$, $\sigma_2 = \sigma(T_{|M_2})$. Now assume that there exists a non-zero vector x with $(J_T(x))^\circ \neq \emptyset$. Let $x = x_1 + x_2$, where $x_1 \in M_1$ and $x_2 \in M_2$. Then since x is non-zero, either x_1 or x_2 is not equal to zero. We claim that M_1 is finite dimensional. Otherwise suppose M_1 is infinite dimensional. Then $S_{|M_1}$ is strictly singular and it follows from Remark 1.3 that $0 \in \sigma(S_{|M_1})$ and therefore $\lambda \in \sigma(T_{|M_1}) = \sigma_1$, which is not possible. Now consider the projection $P_1 : X \to M_1$ along M_2 onto M_1. Since $J_T(x) \subset J_{T_{|M_1}}(x_1) + J_{T_{|M_2}}(x_2)$, it follows that

$$P_1(J_T(x)) \subset J_{T_{|M_1}}(x_1).$$

By the open mapping theorem it follows that $P_1(J_T(x))$ has non-empty interior and so $J_{T_{|M_1}}(x_1)$ has non-empty interior. This is possible only if $x_1 = 0$, since M_1 is finite dimensional; see [7].

As above we conclude that $J_{T_{|M_2}}(x_2)$ has non-empty interior. Therefore since $\sigma(T_{|M_2}) = \sigma_2 \subset \mathbb{C}\backslash\overline{\mathbb{D}}$, it follows by Proposition 1.6 that $x_2 = 0$. Thus $x = x_1 + x_2 = 0$, which is a contradiction. $\square$

S. Argyros, A. Arvanitakis and A. Tolias constructed a non-separable real Banach space, on which every operator T has the form $T = \lambda I + S$, where S is strictly singular and has separable range; see [3].

Theorem 1.8 (Argyros, Arvanitakis, Tolias)**.** *There exists a real non-separable Banach space X_A, containing no reflexive subspace, on which every operator T is of the form $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and S a weakly compact operator (and hence of separable range).*

The fact that X_A contains no reflexive subspace implies that every weakly compact operator on X_A is strictly singular; thus every operator $T : X_A \to X_A$ is of the form $T = \lambda I + S$ with $\lambda \in \mathbb{R}$ and S a strictly singular operator with separable range. The proof of the following corollary is essentially contained in the proof of Lemma 4.3 in [9].

Corollary 1.9. *Consider $X := (X_A)_{\mathbb{C}}$. Then every operator $T \in L(X)$ is of the form $T = wI + S$ ($w \in \mathbb{C}$), where S is strictly singular and has separable range.*

Proof. Every operator $T \in L(X)$ can be written as $T = T_1 + iT_2$, where $T_1, T_2 \in L(X_A)$. By the previous theorem, $T_1 = \lambda I + S_1$ and $T_2 = \mu I + S_2$ ($\lambda, \mu \in \mathbb{R}$), where

$S_1, S_2 \in L(X_A)$ are strictly singular and have separable range. Therefore we get

$$\begin{aligned} T &= T_1 + iT_2 \\ &= \lambda I + S_1 + i(\mu I + S_2) \\ &= (\lambda + i\mu)I_{\mathbb{C}} + (S_1)_{\mathbb{C}} + i(S_2)_{\mathbb{C}}. \end{aligned}$$

By Proposition 1.5, $(S_i)_{\mathbb{C}}$ is strictly singular for $i \in \{1,2\}$ and by Theorem 1.4, $S := (S_1)_{\mathbb{C}} + i(S_2)_{\mathbb{C}}$ is strictly singular and has separable range. With $w := \lambda + i\mu$ we get $T = wI + S$. □

The next lemma can be found in [7].

Lemma 1.10. *Let X be a Banach space and $T \in L(X)$. If $J_T(x)$ has non-empty interior for some $x \neq 0$, then $T - \lambda I$ has dense range for each $|\lambda| \leq 1$.*

Theorem 1.11. *There exists a non-separable complex Banach space X on which the J-set of every operator has empty interior for every non-zero vector. In particular there does not exist a J-class operator on X.*

Proof. We condsider the space $X = (X_A)_{\mathbb{C}}$. Then every operator T is of the form $T = \lambda I + S$ by Corollary 1.9, where S is strictly singular and has separable range. If $|\lambda| > 1$, then it follows by Theorem 1.7 that the interior of $J_T(x)$ is empty for each non-zero vector x. Now consider $|\lambda| \leq 1$. Then by Lemma 1.10 the operator $T - \lambda I = S$ has dense range. This is not possible since S has separable range and X is non-separable. □

Our next aim is to show that on the space $Y := X \oplus X$, where $X = (X_A)_{\mathbb{C}}$, the J-set of every $T \in L(Y)$ also has empty interior for each non-zero vector in Y. The next lemma gives us some information about the form of the operators in $L(Y)$.

Lemma 1.12. *Consider $Y := X \oplus X$, where $X = (X_A)_{\mathbb{C}}$. Then for every operator $T \in L(Y)$ there exists an isomorphism $J \in L(Y)$, such that $J^{-1}TJ$ has one of the following two matrix representations:*

$$J^{-1}TJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

or

$$J^{-1}TJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

where $S_i \in L(X)$ are strictly singular and have separable range for $i \in \{1,2,3,4\}$.

Proof. Every operator $T \in L(Y)$ has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $T_i \in L(X)$ for $i \in \{1,2,3,4\}$. From Corollary 1.9 every $T_i = a_i I + \widetilde{S_i}$, where $\widetilde{S_i}$ is strictly singular with separable range. So we get

$$T = \overbrace{\begin{pmatrix} a_1 I & a_2 I \\ a_3 I & a_4 I \end{pmatrix}}^{A} + \overbrace{\begin{pmatrix} \widetilde{S_1} & \widetilde{S_2} \\ \widetilde{S_3} & \widetilde{S_4} \end{pmatrix}}^{\widetilde{S}}.$$

Applying the Jordan decomposition of matrices, there exists an isomorphism such that

$$J^{-1}AJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} \text{ or } J^{-1}AJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix}.$$

By Theorem 1.4, $S := J^{-1}\widetilde{S}J$ is also strictly singular and therefore there exist some $S_i \in L(X)$, $i \in \{1,2,3,4\}$ strictly singular with separable range (see [1]) such that

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

The desired statement now follows. □

The next theorem can be found in [1].

Theorem 1.13. *Assume that $S \in L(X)$ is strictly singular and that an operator $T \in L(X)$ has an at most countable spectrum. Then the spectrum of $S+T$ is at most countable and zero and the points of $\sigma(T)$ are the only possible accumulation points of $\sigma(S+T)$.*

Theorem 1.14. *Consider $Y = X \oplus X$ with $X = (X_A)_{\mathbb{C}}$ and $T \in L(Y)$. Then $(J_T((x,y)))^\circ = \emptyset$ for $(x,y) \in Y \backslash \{(0,0)\}$.*

Proof. We argue by contradiction. So suppose there exists a $T \in L(Y)$ such that $(J_T((x,y)))^\circ \neq \emptyset$ for some $(x,y) \in Y\backslash\{(0,0)\}$. By Lemma 1.12 there exists an isomorphism $D \in L(Y)$ such that

$$D^{-1}TD = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \quad (*)$$

or

$$D^{-1}TD = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \quad (**).$$

Then for $\widetilde{T} := D^{-1}TD$ the J-set $J_{\widetilde{T}}(D^{-1}(x,y))$ also has non-empty interior.

Case 1. $\widetilde{T} = (*)$.

If $|\lambda| \neq 1$, then we decompose $\sigma(\widetilde{T})$ in $\sigma_1 = \{\mu \in \sigma(\widetilde{T}) : |\mu| = 1\}$ and $\sigma_2 = \{\mu \in \sigma(\widetilde{T}) : |\mu| \neq 1\}$. The set σ_1 is closed and by Theorem 1.13, σ_2 is also closed, since λ and 0 are the only possible accumulation points of $\sigma(\widetilde{T})$ and hence of σ_2. Furthermore the corresponding $\widetilde{T}$-invariant closed subspace M_1 for σ_1, resulting from the Riesz decomposition theorem is finite dimensional; otherwise,

$$\widetilde{T}|_{M_1} - \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix}\Bigg|_{M_1} = \begin{pmatrix} S_1 & I + S_2 \\ S_3 & S_4 \end{pmatrix}\Bigg|_{M_1}$$

is not invertible and hence $\lambda \in \sigma(\widetilde{T}|_{M_1}) = \sigma_1$, which is not possible. The rest of the proof for $|\lambda| \neq 1$ is similar to Theorem 1.7.

Now consider $|\lambda| = 1$. Then $\widetilde{T} - \left(\begin{smallmatrix} \lambda I & 0 \\ 0 & \lambda I \end{smallmatrix}\right)$ does not have dense range, which is a contradiction to Lemma 1.10.

Case 2. $\widetilde{T} = (**)$.

If $\lambda_1 = \lambda_2$ the argumentation is almost similar to case 1. So suppose $\lambda_1 \neq \lambda_2$. Assume $|\lambda_1| = 1$ or $|\lambda_2| = 1$. Without loss of generality $|\lambda_1| = 1$. Then $\widetilde{T} - \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_1 I \end{pmatrix}$ does not have dense range, which is a contradiction as in Case 1.

For $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$ the argumentation is identical to Case 1. ☐

Remark 1.15. It is also possible with some more technicalities to prove the same result in Theorem 1.14 for $Y = \overbrace{X \oplus \ldots \oplus X}^{n \text{ times}}$, where $X = (X_A)_{\mathbb{C}}$.

We will now show that there is a large class of non-separable Banach spaces on which there always exists a J-class operator, namely the reflexive non-separable Banach spaces. The next theorem can be found in [7].

Theorem 1.16. *Let X be a Banach space and Y a separable Banach space. Consider $S \in L(X)$ with $\sigma(S) \subset \{\lambda : |\lambda| > 1\}$. Also let $T \in L(Y)$ be hypercyclic. Then:*

(1) *$S \times T : X \times Y \to X \times Y$ is a J-class operator, but not hypercyclic.*
(2) *$A_{S\times T} = \{0\} \times Y$.*

The next theorem by Lindenstrauss ([8]) gives us some information about the decomposition of reflexive non-separable Banach spaces.

Theorem 1.17 (Lindenstrauss)**.** *Let X be a non-separable reflexive Banach space and $Y \subset X$ a separable and closed subspace. Then there exists a separable closed subspace W of X that contains Y and a linear bounded projection $P_W : X \to W$ with $\|P_W\| = 1$.*

Theorem 1.18. *Let X be a non-separable reflexive Banach space. Then for every infinite dimensional separable and closed subspace Y and for every $\lambda \in (1, \infty)$ there exists a J-class operator T with $Y \subset A_T$ and $\|T\| = \lambda$.*

Proof. By Theorem 1.17 there exists a separable infinite dimensional subspace W that contains Y and a linear bounded projection $P_W : X \to W$ with $\|P_W\| = 1$. There exists a closed subspace U of X such that $X = U \oplus W$. For given $\epsilon > 0$ we can find a hypercyclic operator $T_1 : W \to W$, $T_1 := I_W + K$, with K compact and $\|K\| < \epsilon$; see ([2], [6]). Then by Theorem 1.16 the operator $T_\lambda := \lambda I_U \oplus T_1 = \lambda I + (1-\lambda)P_W + K \circ P_W$ is J-class for $\lambda > 1$. Furthermore $Y \subset W = A_T$. Now define the function $g : (1, \infty) \to \mathbb{R}$ by $g(\delta) := \|T_\delta\|$. Then it is easy to see that g is continuous. For given λ we choose $\delta > 1$ and $\epsilon > 0$ such that $2\delta + \epsilon < 1 + \lambda$. Therefore we get

$$\begin{aligned} g(\delta) = \|T_\delta\| &= \|\delta I + (1-\delta)P_W + K \circ P_W\| \\ &\leq \delta + |1-\delta|\,\|P_W\| + \|P_W\|\|\,K\| \leq 2\delta - 1 + \epsilon < \lambda. \end{aligned}$$

On the other hand we can find a $\mu > 1$ large enough such that $g(\mu) > \lambda$. By the intermediate value theorem there exists a $\xi \in [\delta, \mu]$ with $g(\xi) = \|T_\xi\| = \lambda$. ☐

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