ON THE EXISTENCE OF J-CLASS OPERATORS ON BANACH SPACES

AMIR BAHMAN NASSERI

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Abstract. In this paper we answer in the negative the question raised by G. Costakis and A. Manoussos whether there exists a J-class operator on every non-separable Banach space. In particular we show that there exists a non-separable Banach space constructed by S. Argyros, A. Arvanitakis and A. Tolias such that the J-set of every operator on this space has empty interior for each non-zero vector. On the other hand, on non-separable spaces which are reflexive there always exists a J-class operator.

1. Preliminaries and the main result

Let $X$ be a real or complex Banach space. If $X$ is a real Banach space, then by $X_C$ we denote the complexification of $X$. By $L(X)$ we mean the space of all bounded linear operators acting on $X$. If $T \in L(X)$, the symbol $\sigma(T)$ stands for the spectrum of $T$. Consider any subset $C$ of $X$. The symbol $C^o$ denotes the interior of $C$ in the norm topology of $X$. The symbol $\text{orb}(T, x)$ denotes the orbit of $x$ under $T$, i.e. $\text{orb}(T, x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$. If $X$ is separable and $\text{orb}(T, x)$ is dense, then $T$ is called hypercyclic, which is equivalent to $T$ being topologically transitive; i.e. for every pair of non-empty open sets $U, V \subset X$, there exists a non-negative integer $n$ such that $T^n(U) \cap V \neq \emptyset$. Following [7], by $J_T(x)$ we denote the J-set of $x$ under $T$, i.e.

$$J_T(x) := \{y \in X : \text{there exists a strictly increasing sequence}
$$

of natural numbers $(k_n)$ and a sequence $(x_n)$ in $X$, such that $x_n \to x$ and $T^{k_n} x_n \to y\}.$

If $J_T(x) = X$ for some $x \in X \setminus \{0\}$, then $T$ is called a J-class operator. By $A_T$ we denote the set of all $x \in X$ such that $J_T(x) = X$. On separable spaces every hypercyclic operator is J-class, but the converse is not true. It is known [4] that on $l^\infty$, there does not exist a topological transitive operator. On the other hand there exist J-class operators such as the weighted backward shift $\lambda B : l^\infty \to l^\infty$, $\lambda B(x_1, x_2, \ldots) := (\lambda x_2, \lambda x_3, \ldots)$ for $|\lambda| > 1$. Therefore it is natural to ask whether every non-separable Banach space admits a J-class operator [7]. Our main result is the following:
Theorem 1.1. There exists a non-separable complex Banach space \( X \) on which the \( J \)-set of every operator has empty interior for every non-zero vector. Consequently there exists no \( J \)-class operator on \( X \).

The statement in Theorem 1.1 gives us a stronger result than the question raised by G. Costakis and A. Manoussos, since in general \((J_T(x))^\circ \neq \emptyset\) does not imply that \( J_T(x) = \emptyset \). As is clear from our proof, the conclusion of Theorem 1.1 is satisfied for every complex non-separable Banach space, for which every \( T \in L(X) \) is of the form \( T = \lambda I + S \) with \( \lambda \in \mathbb{C} \) and \( S \) a strictly singular operator with separable range. A real non-separable HI (Hereditarily Indecomposable) Banach space containing no reflexive subspace for which every \( T \in L(X) \) takes the form \( T = \lambda I + S \) with \( \lambda \in \mathbb{R} \) and \( S \) a weakly compact operator with separable range has been constructed by S. Argyros, A. Arvanitakis and A. Tolias in [3]. The complexification of this space is easily shown to satisfy our requirements, and thus the conclusion of Theorem 1.1. In contrast we show in Theorem 1.18 that every non-separable reflexive Banach space admits a \( J \)-class operator.

Definition 1.2. Let \( X,Y \) be infinite dimensional Banach spaces. A linear and bounded operator \( S : X \to Y \) is called strictly singular if for every infinite dimensional subspace \( M \subset X \) the restriction \( S_M : M \to S(M) \) is not an isomorphism (linear homeomorphism).

Remark 1.3. If \( X = Y \), then an immediate consequence of the above definition is that \( 0 \in \sigma(S) \). The spectrum of \( S \) is at most countable with \( 0 \) as the only possible accumulation point.

The next two theorems can be found in [1].

Theorem 1.4. Let \( X \) be a Banach space. The collection of all strictly singular operators is a closed subspace in \( L(X) \), which is also a two-sided ideal.

Proposition 1.5. Let \( X \) be a real Banach space and \( X_C \) its complexification. A bounded operator \( T : X \to X \) is strictly singular if and only if \( T_C : X_C \to X_C \) is strictly singular.

We show now that a certain class of operators on a complex Banach space can not be \( J \)-class. For this purpose we need the following proposition.

Proposition 1.6. Let \( X \) be a complex Banach space and \( T \in L(X) \). Suppose \((J_T(x))^\circ \neq \emptyset \) for some \( x \in X \setminus \{0\} \). Then \( \sigma(T) \cap \partial D \neq \emptyset \).

Proof. Assume that \( \sigma(T) \cap \partial D = \emptyset \). We decompose \( \sigma \in \sigma_1 := \{ \lambda \in \sigma : |\lambda| > 1 \} \) and \( \sigma_2 := \{ \lambda \in \sigma : |\lambda| < 1 \} \). Then \( \sigma_1 \) and \( \sigma_2 \) are disjoint and closed. By the Riesz decomposition theorem we can decompose \( X = M_1 \oplus M_2 \), where \( M_1 \) and \( M_2 \) are closed and \( T \)-invariant subspaces and \( \sigma_1 = \sigma(T_{|M_1}) \), \( \sigma_2 = \sigma(T_{|M_2}) \). Assume now that there exists a non-zero vector \( x \in X \), such that \( J_T(x) \) has non-empty interior. We can write \( x = x_1 + x_2 \), with \( x_1 \in M_1 \) and \( x_2 \in M_2 \). Since \( x \neq 0 \), either \( x_1 \) or \( x_2 \) is not equal to zero. Consider the projection \( P_1 : X \to M_1 \) along \( M_2 \) onto \( M_1 \). Since \( J_T(x) \subset J_{T_{|M_1}}(x_1) + J_{T_{|M_2}}(x_2) \) it follows that \( P_1(J_T(x)) \subset J_{T_{|M_1}}(x_1) \). By the open mapping theorem we get that \( P_1(J_T(x)) \) has non-empty interior and hence \((J_{T_{|M_1}}(x_1))^\circ \neq \emptyset \). From the spectral radius formula we obtain that \( \|T_{|\overline{x}}\| \leq a^n \|\overline{x}\| \) for some \( a \in \mathbb{R} \) with \( 0 \leq a < 1 \) and for all \( \overline{x} \in M_1 \). This implies that \( J_{T_{|M_1}}(x_1) = \{0\} \) and therefore \( M_1 = \{0\} \). So we get \( x_1 = 0 \). Again
from the spectral radius formula we know that \([T^n_1(\hat{x})]\| \geq A^n \|\hat{x}\|\) for some \(A > 1\) and all \(\hat{x} \in M_2\). This implies \(x_2 = 0\), which is a contradiction to our assumption that \(x \neq 0\).

\[\square\]

**Theorem 1.7.** Let \(X\) be a complex infinite dimensional Banach space. Then for every operator of the form \(T = \lambda I + S\), where \(S\) is strictly singular and \(|\lambda| > 1\), and every \(x \neq 0\), the set \(J_T(x)\) has empty interior.

**Proof.** By Remark 1.3 it follows that \(\lambda \in \sigma(T)\) and it is the only possible accumulation point. We decompose the spectrum in \(\sigma_1 := \{\mu \in \sigma(T) : |\mu| \leq 1\}\) and \(\sigma_2 := \{\mu \in \sigma(T) : |\mu| > 1\}\). Clearly then \(\lambda \in \sigma_2\). The set \(\sigma_1\) is closed, and since \(\lambda \in \sigma_2\), and \(\lambda\) is the only possible accumulation point, we conclude that \(\sigma_2\) is also closed. Furthermore \(\sigma_1\) and \(\sigma_2\) are disjoint. By the Riesz decomposition theorem we can decompose \(X = M_1 \oplus M_2\), where \(M_1\) and \(M_2\) are closed and \(T\)-invariant subspaces and \(\sigma_1 = \sigma(T_{M_1}), \sigma_2 = \sigma(T_{M_2})\). Now assume that there exists a non-zero vector \(x\) with \((J_T(x))^o \neq \emptyset\). Let \(x = x_1 + x_2\), where \(x_1 \in M_1\) and \(x_2 \in M_2\). Then since \(x\) is non-zero, either \(x_1\) or \(x_2\) is not equal to zero. We claim that \(M_1\) is finite dimensional. Otherwise suppose \(M_1\) is infinite dimensional. Then \(S_{M_1}\) is strictly singular and it follows from Remark 1.3 that \(0 \in \sigma(S_{M_1})\) and therefore \(\lambda \in \sigma(T_{M_1}) = \sigma_1\), which is not possible. Now consider the projection \(P_1 : X \rightarrow M_1\) along \(M_2\) onto \(M_1\). Since \(J_T(x) \subset J_{T_{M_1}}(x_1) + J_{T_{M_2}}(x_2)\), it follows that

\[P_1(J_T(x)) \subset J_{T_{M_1}}(x_1)\).

By the open mapping theorem it follows that \(P_1(J_T(x))\) has non-empty interior and so \(J_{T_{M_1}}(x_1)\) has non-empty interior. This is possible only if \(x_1 = 0\), since \(M_1\) is finite dimensional; see \(\Box\).

As above we conclude that \(J_{T_{M_2}}(x_2)\) has non-empty interior. Therefore since \(\sigma(T_{M_2}) = \sigma_2 \subset \mathbb{C}\backslash \overline{\mathbb{D}}\), it follows by Proposition 1.6 that \(x_2 = 0\). Thus \(x = x_1 + x_2 = 0\), which is a contradiction. \(\square\)

S. Argyros, A. Arvanitakis and A. Tolias constructed a non-separable real Banach space, on which every operator \(T\) has the form \(T = \lambda I + S\), where \(S\) is strictly singular and has separable range; see \(\Box\).

**Theorem 1.8** (Argyros, Arvanitakis, Tolias). There exists a real non-separable Banach space \(X_A\), containing no reflexive subspace, on which every operator \(T\) is of the form \(T = \lambda I + S\) with \(\lambda \in \mathbb{R}\) and \(S\) a weakly compact operator (and hence of separable range).

The fact that \(X_A\) contains no reflexive subspace implies that every weakly compact operator on \(X_A\) is strictly singular; thus every operator \(T : X_A \rightarrow X_A\) is of the form \(T = \lambda I + S\) with \(\lambda \in \mathbb{R}\) and \(S\) a strictly singular operator with separable range. The proof of the following corollary is essentially contained in the proof of Lemma 4.3 in \(\Box\).

**Corollary 1.9.** Consider \(X := (X_A)\). Then every operator \(T \in L(X)\) is of the form \(T = wI + S\) (\(w \in \mathbb{C}\)), where \(S\) is strictly singular and has separable range.

**Proof.** Every operator \(T \in L(X)\) can be written as \(T = T_1 + iT_2\), where \(T_1, T_2 \in L(X_A)\). By the previous theorem, \(T_1 = \lambda I + S_1\) and \(T_2 = \mu I + S_2\) (\(\lambda, \mu \in \mathbb{R}\)), where
$S_1, S_2 \in L(X_A)$ are strictly singular and have separable range. Therefore we get

$$T = T_1 + iT_2$$

$$= \lambda I + S_1 + i(\mu I + S_2)$$

$$= (\lambda + i\mu)I + (S_1)_C + i(S_2)_C.$$  

By Proposition 1.5, $(S_i)_C$ is strictly singular for $i \in \{1, 2\}$ and by Theorem 1.4, $S := (S_1)_C + i(S_2)_C$ is strictly singular and has separable range. With $w := \lambda + i\mu$ we get $T = wI + S$. □

The next lemma can be found in [7].

**Lemma 1.10.** Let $X$ be a Banach space and $T \in L(X)$. If $J_T(x)$ has non-empty interior for some $x \neq 0$, then $T - \lambda I$ has dense range for each $|\lambda| \leq 1$.

**Theorem 1.11.** There exists a non-separable complex Banach space $X$ on which the $J$-set of every operator has empty interior for every non-zero vector. In particular there does not exist a $J$-class operator on $X$.

**Proof.** We consider the space $X = (X_A)_C$. Then every operator $T$ is of the form $T = \lambda I + S$ by Corollary 1.9, where $S$ is strictly singular and has separable range. If $|\lambda| > 1$, then it follows by Theorem 1.7 that the interior of $J_T(x)$ is empty for each non-zero vector $x$. Now consider $|\lambda| \leq 1$. Then by Lemma 1.10 the operator $T - \lambda I = S$ has dense range. This is not possible since $S$ has separable range and $X$ is non-separable. □

Our next aim is to show that on the space $Y := X \oplus X$, where $X = (X_A)_C$, the $J$-set of every $T \in L(Y)$ also has empty interior for each non-zero vector in $Y$. The next lemma gives us some information about the form of the operators in $L(Y)$.

**Lemma 1.12.** Consider $Y := X \oplus X$, where $X = (X_A)_C$. Then for every operator $T \in L(Y)$ there exists an isomorphism $J \in L(Y)$, such that $J^{-1}TJ$ has one of the following two matrix representations:

$$J^{-1}TJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

or

$$J^{-1}TJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix},$$

where $S_i \in L(X)$ are strictly singular and have separable range for $i \in \{1, 2, 3, 4\}$.

**Proof.** Every operator $T \in L(Y)$ has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

where $T_i \in L(X)$ for $i \in \{1, 2, 3, 4\}$. From Corollary 1.9 every $T_i = a_i I + \tilde{S}_i$, where $\tilde{S}_i$ is strictly singular with separable range. So we get

$$T = \begin{pmatrix} A & \tilde{S} \\ A \end{pmatrix} = \begin{pmatrix} a_1 I & a_2 I \\ a_3 I & a_4 I \end{pmatrix} + \begin{pmatrix} \tilde{S}_1 & \tilde{S}_2 \\ \tilde{S}_3 & \tilde{S}_4 \end{pmatrix},$$

where $A$ is a matrix of the form $A = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix}$. □
Applying the Jordan decomposition of matrices, there exists an isomorphism such that
\[ J^{-1}AJ = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} \] or \[ J^{-1}AJ = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix}. \]

By Theorem 1.4, \( S := J^{-1}J \) is also strictly singular and therefore there exist some \( S_i \in L(X) \), \( i \in \{1, 2, 3, 4\} \) strictly singular with separable range (see [1]) such that \( S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \).

The desired statement now follows. \( \square \)

The next theorem can be found in [1].

Theorem 1.13. Assume that \( S \in L(X) \) is strictly singular and that an operator \( T \in L(X) \) has an at most countable spectrum. Then the spectrum of \( S+T \) is at most countable and zero and the points of \( \sigma(T) \) are the only possible accumulation points of \( \sigma(S+T) \).

Theorem 1.14. Consider \( Y = X \oplus X \) with \( X = (X_A)_C \) and \( T \in L(Y) \). Then \( (J_T((x,y)))^\circ = \emptyset \) for \( (x,y) \in Y \backslash \{(0,0)\} \).

Proof. We argue by contradiction. So suppose there exists a \( T \in L(Y) \) such that \( (J_T((x,y)))^\circ \neq \emptyset \) for some \( (x,y) \in Y \backslash \{(0,0)\} \). By Lemma 1.12 there exists an isomorphism \( D \in L(Y) \) such that
\[ D^{-1}TD = \begin{pmatrix} \lambda I & I \\ 0 & \lambda I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \] (1)

or
\[ D^{-1}TD = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} + \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \] (2).

Then for \( \tilde{T} := D^{-1}TD \) the \( J \)-set \( J^{\tilde{T}}(D^{-1}(x,y)) \) also has non-empty interior.

Case 1. \( \tilde{T} = (1) \).

If \( |\lambda| \neq 1 \), then we decompose \( \sigma(\tilde{T}) \) in \( \sigma_1 = \{ \mu \in \sigma(\tilde{T}) : |\mu| = 1 \} \) and \( \sigma_2 = \{ \mu \in \sigma(\tilde{T}) : |\mu| \neq 1 \} \). The set \( \sigma_1 \) is closed and by Theorem 1.13, \( \sigma_2 \) is also closed, since \( \lambda \) and \( 0 \) are the only possible accumulation points of \( \sigma(\tilde{T}) \) and hence of \( \sigma_2 \).

Furthermore the corresponding \( \tilde{T} \)-invariant closed subspace \( M_1 \) for \( \sigma_1 \), resulting from the Riesz decomposition theorem is finite dimensional; otherwise,
\[ \tilde{T}|_{M_1} = \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} \bigg|_{M_1} = \begin{pmatrix} S_1 & I + S_2 \\ S_3 & S_4 \end{pmatrix} \bigg|_{M_1} \]
is not invertible and hence \( \lambda \in \sigma(\tilde{T}|_{M_1}) = \sigma_1 \), which is not possible. The rest of the proof for \( |\lambda| \neq 1 \) is similar to Theorem 1.7.

Now consider \( |\lambda| = 1 \). Then \( \tilde{T} - \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda I \end{pmatrix} \) does not have dense range, which is a contradiction to Lemma 1.10.
Case 2. \( \bar{T} = (**). \)

If \( \lambda_1 = \lambda_2 \) the argumentation is almost similar to case 1. So suppose \( \lambda_1 \neq \lambda_2 \). Assume \( |\lambda_1| = 1 \) or \( |\lambda_2| = 1 \). Without loss of generality \( |\lambda_1| = 1 \). Then \( \bar{T} - \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} \) does not have dense range, which is a contradiction as in Case 1.

For \( |\lambda_1| \neq 1 \) and \( |\lambda_2| \neq 1 \) the argumentation is identical to Case 1. \( \square \)

Remark 1.15. It is also possible with some more technicalities to prove the same result in Theorem 1.14 for \( Y = \bigoplus X \), where \( X = (X_A)_{\mathbb{C}} \).

We will now show that there is a large class of non-separable Banach spaces on which there always exists a \( J \)-class operator, namely the reflexive non-separable Banach spaces. The next theorem can be found in [7].

Theorem 1.16. Let \( X \) be a Banach space and \( Y \) a separable Banach space. Consider \( S \in L(X) \) with \( \sigma(S) \subset \{ \lambda : |\lambda| > 1 \} \). Also let \( T \in L(Y) \) be hypercyclic. Then:

1. \( S \times T : X \times Y \rightarrow X \times Y \) is a \( J \)-class operator, but not hypercyclic.
2. \( A_{S \times T} = \{0\} \times Y \).

The next theorem by Lindenstrauss ([8]) gives us some information about the decomposition of reflexive non-separable Banach spaces.

Theorem 1.17 (Lindenstrauss). Let \( X \) be a non-separable reflexive Banach space and \( Y \subset X \) a separable and closed subspace. Then there exists a separable closed subspace \( W \) of \( X \) that contains \( Y \) and a linear bounded projection \( P_W : X \rightarrow W \) with \( \|P_W\| = 1 \).

Theorem 1.18. Let \( X \) be a non-separable reflexive Banach space. Then for every infinite dimensional separable and closed subspace \( Y \) and for every \( \lambda \in (1, \infty) \) there exists a \( J \)-class operator \( T \) with \( Y \subset A_T \) and \( \|T\| = \lambda \).

Proof. By Theorem 1.17 there exists a separable infinite dimensional subspace \( W \) that contains \( Y \) and a linear bounded projection \( P_W : X \rightarrow W \) with \( \|P_W\| = 1 \). There exists a closed subspace \( U \) of \( X \) such that \( X = U \oplus W \). For given \( \epsilon > 0 \) we can find a hypercyclic operator \( T_1 : W \rightarrow W, T_1 := I_W + K \), with \( K \) compact and \( \|K\| < \epsilon \); see ([2], [3]). Then by Theorem 1.16 the operator \( T_1 := \lambda I_W + T_1 = \lambda I + (1 - \lambda)P_W + K \circ P_W \) is \( J \)-class for \( \lambda > 1 \). Furthermore \( Y \subset W = A_T \). Now define the function \( g : (1, \infty) \rightarrow \mathbb{R} \) by \( g(\delta) := \|T_3\| \). Then it is easy to see that \( g \) is continuous. For given \( \lambda \) we choose \( \delta > 1 \) and \( \epsilon > 0 \) such that \( 2\delta + \epsilon < 1 + \lambda \). Therefore we get

\[
g(\delta) = \|T_3\| = \|\delta I + (1 - \delta)P_W + K \circ P_W\| \\
\leq \delta + |1 - \delta| \|P_W\| + \|P_W\| \|K\| \leq 2\delta - 1 + \epsilon < \lambda.
\]

On the other hand we can find a \( \mu > 1 \) large enough such that \( g(\mu) > \lambda \). By the intermediate value theorem there exists a \( \xi \in [\delta, \mu] \) with \( g(\xi) = \|T_3\| = \lambda \). \( \square \)

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Fakultät für Mathematik, Technische Universität Dortmund, D-44221 Dortmund, Germany

E-mail address: amirbahman@hotmail.de