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INTERLACING PROPERTY OF THE ZEROS OF $j_n(\tau)$

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Abstract. We improve an estimate for $j_n(\tau)$ on the unit circle and use it to prove an interlacing property of the zeros of $j_n(\tau)$.

1. Introduction

Let $\mathbb{H}$ denote the upper half-plane and $M_k^!$ denote the space of weakly holomorphic modular forms of weight k for $\Gamma = SL_2(\mathbb{Z})$ and M_k (resp. S_k) denote the subspace of holomorphic modular forms (resp. cusp forms). For $\tau \in \mathbb{H}$, $q = e^{2\pi i\tau}$, if $f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n \in M_k^!$, then the n'th Hecke operator, $T(n)$, acts on f via

$$(f|_k T(n))(\tau) = n^{k-1} \sum_{A\in\mathcal{M}_n} d^{-k} f(A\tau) = \sum_{m\in\mathbb{Z}} \left(\sum_{0<d|(m,n)} d^{k-1} a_{\frac{mn}{d^2}} \right) q^m,$$

where $\mathcal{M}_n = \left\{ A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad = n, d > 0, -1 \le b \le d-2 \right\}$ is a set of representatives of matrices with determinant n modulo Γ.

Let $E_k \in M_k$ be the normalized Eisenstein series of weight k and $\Delta = \frac{1}{1728}(E_4^3 - E_6^2) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_k$ the discriminant function. For $n \ge 1$, let

$$j(\tau) := 1728 \frac{E_4^3}{E_4^3 - E_6^2} = \frac{E_4^3}{\Delta} = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n \in M_0^!, \tag{1}$$

$$j_n(\tau) := n\left(j(\tau) - 744\right)|_0 T(n) = \sum_{A\in\mathcal{M}_n} \left(j(A\tau) - 744\right) \in M_0^!. \tag{2}$$

It is well known that $M_0^! = \mathbb{C}[j]$. Furthermore $\{j_n(\tau)\}_{n\ge1}$ is (together with the constant function 1) the unique basis of $M_0^!$ with the property that $j_n(\tau) = q^{-n} + \sum_{m=1}^{\infty} c_n(m) q^m$ for some coefficients $c_n(m)$. $j_n(\tau)$ can be written as a polynomial in $j(\tau)$ of degree n, the so-called n'th Faber polynomial, denoted by $\phi_n \in \mathbb{Q}[x]$. The first few examples are:

$$\begin{aligned}
\phi_1(j) &= j - 744,\\
\phi_2(j) &= j^2 - 1488j + 159768,\\
\phi_3(j) &= j^3 - 2232j^2 + 1069956j - 36866976,\\
\phi_4(j) &= j^4 - 2976j^3 + 2533680j^2 - 561444608j + 8507424792.
\end{aligned}$$

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Let

$$\mathcal{C} := \left\{\tau \in \mathbb{H} \middle| |\tau| = 1 \text{ and } 0 \leq \Re(\tau) \leq \frac{1}{2}\right\} \tag{3}$$

denote the arc of the unit circle which is part of the strict fundamental domain of Γ, denoted by

$$\mathcal{F} = \left\{\tau \in \mathbb{H} \middle| -\frac{1}{2} < \Re(\tau) \leq \frac{1}{2} \text{ and } |\tau| > 1\right\} \cup \mathcal{C}. \tag{4}$$

Recall that $\tau \mapsto j(\tau)$ gives a 1-1 correspondence between $\mathcal{F}$ and $\mathbb{C}$. The inverse image of $\mathbb{R} = (-\infty, 0) \cup [0, 1728] \cup (1728, \infty)$ is given by $\{\frac{1}{2} + iy | y > \frac{\sqrt{3}}{2}\} \cup \mathcal{C} \cup \{iy | y > 1\}$. In particular $j(i) = 1728$, $j(\sigma) = 0$ (where $\sigma = e^{\frac{2\pi i}{6}}$) and $\mathcal{C}$ is mapped to the interval $[0, 1728]$. Consequently $j_n(\tau)$ takes real values for τ in $\mathcal{C}$.

The zeros of modular functions have been studied by several authors (see e.g. [RSD70], [Ran82], [AKN97], [DJ08], [Noz08], [KZ98]). The first result probably goes back to a paper [RSD70] by Rankin and Swinnerton-Dyer in which they prove that all the zeros of the Eisenstein series E_k in the fundamental domain are simple and lie on the unit circle using a basic estimate of E_k on the unit circle.

In 1997 Asai, Kaneko and Ninomiya [AKN97] showed that the same holds for j_n:

Theorem 1.1 ([AKN97])**.** *For each $n \geq 1$, all the zeros of $\phi_n(j)$ are simple and lie in the interval $(0, 1728)$ or equivalently the zeros of $j_n(\tau)$ in $\mathcal{F}$ are simple and lie on $\mathcal{C}$.*

To prove Theorem 1.1 the following key estimate for $j_n(\tau)$ on the unit circle was used:

Lemma 1.2 ([AKN97])**.** *Let $\tau = x + iy \in \mathcal{C}$. Then*

$$|j_n(\tau)e^{-2\pi n y} - 2\cos(2\pi n x)| < 2. \tag{5}$$

On the other hand, in 2008 Nozaki [Noz08] improved the results of Rankin and Swinnerton-Dyer ([RSD70]) and showed that the zeros of E_k and E_{k+12} interlace by improving the estimates for E_k on the unit circle.

Similarly we will improve the estimate in Lemma 1.2 for $j_n(\tau)$ on the unit circle (see Lemma 2.1) and use it to show Theorem 3.1:

Theorem. *The zeros of $j_n(\tau)$ and $j_{n+1}(\tau)$ on $\mathcal{C}$ interlace for $n \geq 1$.*

In this context the precise meaning of *interlacing* is given in equation (25) in Theorem 3.1.

2. Estimates for $j_n(\tau)$

For $x \in \left[0, \frac{1}{2}\right]$ we define the following functions:

$$v(x) := \sqrt{\frac{1+x}{1-x}} \in [1, \sqrt{3}], \tag{6}$$

$$R_n(x) := e^{-2\pi n v(x)\left(\frac{1}{2} - x\right)}. \tag{7}$$

The following estimate holds:

Lemma 2.1 (Key Lemma)**.** *Let $\tau = x + iy \in \mathcal{C}$, $n > 3$. Then:*

$$\left|j_n(\tau)e^{-2\pi n y} - 2\cos(2\pi n x) - (-1)^n R_n(x)\right| \leq e^{-\pi n y} n. \tag{8}$$

As an immediate consequence we get

Corollary 2.2.

$$
\begin{aligned}
(9)\qquad |j_n(\tau)e^{-2\pi ny} - 2\cos(2\pi nx)| &\leq e^{-2\pi n v(x)\left(\frac{1}{2}-x\right)} + e^{-\pi ny}n \\
&\leq e^{-2\pi n\left(\frac{1}{2}-x\right)} + e^{-\frac{\sqrt{3}}{2}\pi n}n < 1.1.
\end{aligned}
$$

Remark. If we enlarge the region from $\mathcal{C}$ to $\tilde{\mathcal{C}} := -\overline{\mathcal{C}} \cup \mathcal{C}$, then the following estimate holds (which is symmetric in x):

$$
(10)\qquad \left|j_n(\tau)e^{-2\pi ny} - 2\cos(2\pi nx) - (-1)^n\left(R_n(x) + R_n(-x)\right)\right| \leq e^{-\pi ny}n.
$$

We first make some observations before we begin the proof of Lemma 2.1:

By Theorem 1.1 $j_n(\tau)$ has exactly n simple zeros in $\mathcal{F}$, all lying on $\mathcal{C}$. Let $u_{n,k}$ for $k = 0, ..., n-1$ denote their real part (with decreasing real part for increasing k). As already shown in [AKN97], as a consequence of (5), we have the following estimate for $n \geq 4$:

$$
(11)\qquad |u_{n,k} - x_{n,k}| < \frac{1}{4n}, \text{ where}
$$

$$
(12)\qquad x_{n,k} := \frac{1}{2} - \frac{k}{2n} - \frac{1}{4n}
$$

are exactly the n distinct zeros of $2\cos(2\pi nx)$ on $\left[0, \frac{1}{2}\right]$. In particular the $u_{n,k}$'s are (uniformly) equidistributed on $\left[0, \frac{1}{2}\right]$. Using Corollary 2.2 we can show that in fact the following (slightly stronger) preliminary estimate for the location of the zeros $u_{n,k}$ holds:

Lemma 2.3 (Preliminary estimate)**.** *Let $n \geq 4$. Then:*

$$
(13)\qquad |u_{n,k} - x_{n,k}| < \frac{1}{11.0n}.
$$

We will need this technical estimate later for the proof of Theorem 3.1.

For the convenience of the reader we start by recalling quickly the proof of Theorem 1.1 and (11). The idea is to show that $2\cos(2\pi nx)$ determines the sign of $j_n(\tau)$ for $|x - x_{n,k}|$ large enough. The result then follows from the known behaviour of $2\cos(2\pi nx)$ around the zero $x_{n,k}$. Namely, if $x = x_{n,k} + r$, then

$$
2\cos(2\pi nx) = (-1)^{n+k} 2\sin(2\pi nr).
$$

This is an odd and strictly monotone function in r in the region $x_{n,k} - \frac{1}{4n} \leq x \leq x_{n,k} + \frac{1}{4n}$, and

$$
(14)\qquad 2\cos\left(2\pi n\left(x_{n,k} \pm \frac{1}{4n}\right)\right) = \pm(-1)^{n+k}2.
$$

Since $|j_n(\tau)e^{-2\pi ny} - 2\cos(2\pi nx)| < 2$ by (5) the function $j_n(\tau)$ must change sign at least once in each part of $\mathcal{C}$ with $x_{n,k} - \frac{1}{4n} < x < x_{n,k} + \frac{1}{4n}$ for $k = 0, ..., n-1$. Hence $j_n(\tau)$ has at least n distinct zeros on $\mathcal{C}$. By the valence formula those constitute all zeros. This shows Theorem 1.1 and the estimate (11).

To show (13) we replace 2 on the right-hand side of (14) by 1.1 and use Lemma 2.1 instead of Lemma 1.2 to get a slightly better estimate:

$$\begin{aligned}\left|2\cos\left(2\pi n\left(x_{n,k}\pm\frac{1}{11.0n}\right)\right)\right| &= 2\sin\left(\frac{2\pi}{11}\right)\geq 1.1\\ &> |\underbrace{j_n(\tau)e^{-2\pi ny}}_{=0\text{ for }x=u_{n,k}}-2\cos(2\pi n u_{n,k})|\\ &= |2\cos(2\pi n u_{n,k})|.\end{aligned}$$

Hence (by monotonicity) we indeed have $|u_{n,k}-x_{n,k}|<\frac{1}{11.0n}$.

Proof of Lemma 2.1. We will mainly follow the proof of [AKN97] for $\tau\in\tilde{\mathcal{C}}=-\overline{\mathcal{C}}\cup\mathcal{C}$. In [AKN97] only the case $\tau\in\mathcal{C}$ is considered but most arguments extend to $-\overline{\mathcal{C}}$.

First some notation: For $z\in\mathbb{H}$, let z^* denote the unique Γ-equivalent z in $\mathcal{F}$ and let $q_\tau:=e^{2\pi i\tau}$. From the Fourier series expansions we see that $|j(\tau)-744-q^{-1}|$ is bounded on $\mathcal{F}$. Let

$$M=\max_{\tau\in\overline{\mathcal{F}}}|j(\tau)-744-q^{-1}|=\max_{\tau\in\mathbb{H}}|j(\tau)-744-q_{\tau^*}^{-1}|. \tag{15}$$

$M<1335$ by [AKN97, pp. 95, 96].

If we directly apply the triangle inequality in the definition (2) of $j_n(\tau)$ we get the estimate

$$\begin{aligned}\left|j_n(\tau)-\sum_{A\in\mathcal{M}_n}q_{((A\tau)^*)}^{-1}\right| &\leq \sum_{A\in\mathcal{M}_n}\left|j\left(A\tau\right)-744-q_{((A\tau)^*)}^{-1}\right|\\ &\leq \sigma_1(n)M\leq n^2M,\end{aligned}\tag{16}$$

where $\sigma_k(n):=\sum_{d|n}d^k$ is the usual divisor sum. This is a very good approximation of $j_n(\tau)$. Unfortunately it is hard to give an explicit description for the terms $q_{((A\tau)^*)}^{-1}$ above. What we will do instead is to divide the sum over $A\in\mathcal{M}_n$ into two parts: $A\in\mathcal{M}_n$ such that $a,d\in\{1,n\}$, $b\in\{0,\pm1\}$ and the remaining part. For the terms of the first part we will give an explicit description of $q_{((A\tau)^*)}^{-1}$, and for the remaining parts it is possible to give a good enough estimate for $\left|q_{((A\tau)^*)}^{-1}\right|$.

More precisely we divide the sum over $A\in\mathcal{M}_n$ into six distinct pieces as follows:

$$\begin{aligned}\mathcal{M}_n=&\underbrace{\begin{pmatrix}n&0\\0&1\end{pmatrix}}_{A_i}\cup\underbrace{\begin{pmatrix}1&0\\0&n\end{pmatrix}}_{A_{ii}}\cup\underbrace{\begin{pmatrix}1&1\\0&n\end{pmatrix}}_{A_{iii}^+}\cup\underbrace{\begin{pmatrix}1&-1\\0&n\end{pmatrix}}_{A_{iii}^-}\cup\underbrace{\{A\in\mathcal{M}_n|a,d\neq 1,n\}}_{A_{iv}}\\ &\cup\underbrace{\left\{\begin{pmatrix}1&b\\0&n\end{pmatrix}\middle|2\leq b\leq n-2\right\}}_{A_v}.\end{aligned}$$

Here A_i, A_{ii} and $A_{iii}^{\pm}$ exactly constitute the first part and A_{iv}, A_v the remaining part. By the triangle inequality we have

$$\left|j_n(\tau)-\sum_{A\in A_i\cup A_{ii}\cup A_{iii}^{\pm}}q_{((A\tau)^*)}^{-1}\right|\leq 3M+\sum_{A\in A_{iv}\cup A_v}|j\left(A\tau\right)-744|\,. \tag{17}$$

The following lemma (essentially due to [AKN97]) gives the required estimates for the sum on the right-hand side of (17):

Lemma 2.4. *Let M be as in* (15), *let $n \geq 4$ and let $\tau = x + iy \in \tilde{\mathcal{C}}$. Then the following estimates hold:*

$$\left| \sum_{A \in A_{iv}} j(A\tau) - 744 \right| \leq \sigma_1(n) e^{\frac{2\pi n y}{3}} + \sigma_1(n) M, \tag{18}$$

$$\left| \sum_{A \in A_{v}} j(A\tau) - 744 \right| \leq (n-3) e^{\pi n y} + (n-3) M. \tag{19}$$

Proof of Lemma 2.4. The proof can essentially be found on p. 96 of [AKN97]. There it is given only for $\tau \in \mathcal{C}$ but it can easily be extended to $-\overline{\mathcal{C}}$. We just give a very rough sketch here.

Let $\tau \in \mathbb{H}$, $z = A\tau = \frac{a\tau+b}{d}$. From (15) we get

$$|j(z) - 744 - q_{z^*}^{-1}| \leq M, \qquad |j(z) - 744| \leq e^{2\pi \Im(z^*)} + M, \tag{20}$$

where $z^* = Bz = \frac{\alpha z + \beta}{\gamma z + \delta}$ with $B = B_z = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$ (with $\gamma \geq 0$) and $\Im(z^*) = \frac{ny}{L^2}$ with $L = |\gamma a \tau + \gamma b + \delta d|$.

In [AKN97] it is shown that for $A \in A_{iv}$, $L \geq \sqrt{3}$ and for $A \in A_v$, $L \geq \sqrt{2}$. This explains the corresponding exponents in the estimates (18) and (19). The factors $\sigma_1(n)$ (resp. $(n-3)$) are the number of terms in A_{iv} (resp. A_v).

Note that if we restrict to $\tau \in \mathcal{C}$ (resp. $\tau \in -\overline{\mathcal{C}}$), then the estimate $L \geq \sqrt{2}$ still holds for $A \in A_{iii}^{+}$ (resp. $A \in A_{iii}^{-}$) and we could include that case as one further term in (19). This will actually be needed later to show why the estimate (10) still holds if we drop the term corresponding to A_{iii}^{+} which gives us estimate (8). $\square$

To prove estimates (10) and (8), first note that

$$e^{-2\pi n y} \sum_{A \in A_i \cup A_{ii} \cup A_{iii}^{\pm}} q_{((A\tau)^*)}^{-1} = 2\cos(2\pi n x) + (-1)^n \left(R_n(x) + R_n(-x) \right). \tag{21}$$

To see this, we have

$$A_i \tau = n\tau, \qquad A_{ii}\tau = \frac{\tau}{n}, \qquad A_{iii}^{\mp}\tau = \frac{\tau \mp 1}{n}.$$

Since $n\tau - (n\tau)^* \in \mathbb{Z}$ and $-n\overline{\tau} - \left(\frac{\tau}{n}\right)^* = -\frac{n}{\tau} - \left(\frac{\tau}{n}\right)^* \in \mathbb{Z}$ we get

$$q_{((A_i\tau)^*)}^{-1} = e^{-2\pi (n\tau)^*} = e^{-2\pi n \tau}, \tag{22}$$

$$q_{((A_i i\tau)^*)}^{-1} = e^{-2\pi \left(\frac{\tau}{n}\right)^*} = e^{2\pi n \overline{\tau}}. \tag{23}$$

Set $B^{\mp} := \left(\begin{smallmatrix} \alpha & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$ with $\alpha = \mp\frac{n}{2}$ if n is even and $\alpha = \mp\frac{n}{2} + \frac{1}{2}$ if n is odd. Then we have

$$\begin{aligned} B^{\mp} A_{iii}^{\mp}\tau = x' + iy' &= \frac{\alpha\frac{\tau\mp 1}{n} - 1}{\frac{\tau\mp 1}{n}} = \alpha - n\frac{\overline{\tau}\mp 1}{|\tau \mp 1|^2}; \text{ hence:} \\ x' = \alpha - n\frac{x \pm 1}{(x\mp 1)^2 + y^2} &= \alpha \pm \frac{n}{2} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2} & \text{if } n \text{ is odd,} \end{cases} \\ y' = \frac{ny}{|\tau\mp 1|^2} &\geq \frac{n\sqrt{3}}{4} > 1 \text{ (for } n \geq 3). \end{aligned}$$

So $(A_{iii}^{\mp}\tau)^* = B^{\mp}A_{iii}^{\mp}\tau \in \mathcal{F}$. Note that $y\left(1 - \frac{1}{|\tau\mp 1|^2}\right) = v(\pm x)\left(\frac{1}{2} \mp x\right)$. Hence

$$\begin{aligned} q^{-1}_{(A_{iii}^{\mp}\tau)^*} e^{-2\pi n y} &= e^{-2\pi i x'} e^{-2\pi i\left(\frac{iny}{|\tau\mp 1|^2}\right)} e^{-2\pi n y} \\ &= (-1)^n e^{\left(\frac{2\pi n y}{|\tau\mp 1|^2}\right)} e^{-2\pi n y} = (-1)^n R_n(\pm x). \end{aligned} \tag{24}$$

Equation (21) now follows from equations (22) and (23) divided by $e^{2\pi n y}$ and by equation (24).

By (21), estimate (17) and Lemma 2.4 (divided by $e^{2\pi n y}$) we get the following bound:

$$\begin{aligned} &\left|j_n(\tau)e^{-2\pi n y} - 2\cos(2\pi n x) - (-1)^n\left(R_n(x) + R_n(-x)\right)\right| \\ &= e^{-2\pi n y}\left|j_n(\tau) - \sum_{A\in A_i\cup A_{ii}\cup A_{iii}^{\pm}} q^{-1}_{((A\tau)^*)}\right| \\ &\leq e^{-2\pi n y}\left(3M + \sum_{A\in A_{iv}\cup A_v} \left|j\left(A\tau\right) - 744\right|\right) \\ &\leq e^{-2\pi n y}\left(3M + \sigma_1(n)e^{\frac{2\pi n y}{3}} + \sigma_1(n)M + (n-3)e^{\pi n y} + (n-3)M\right) \\ &= e^{-\pi n y}(n-3) + e^{-\frac{4}{3}\pi n y}\sigma_1(n) + e^{-2\pi n y}\left(\sigma_1(n) + n + 1\right)M \\ &\leq e^{-\pi n y} n \text{ (for } n \geq 4). \end{aligned}$$

This proves estimate (10).

If we restrict ourselves to $\mathcal{C}$ (resp. $-\overline{\mathcal{C}}$) and drop the term $R_n(-x)$ corresponding to A_{iii}^{+} (resp. if we drop the term $R_n(x)$ corresponding to A_{iii}^{-}) we can include A_{iii}^{+} (resp. A_{iii}^{-}) as one further term in A_v (see the proof of Lemma 2.4). In the end the same (final) estimate as above holds, which shows estimate (8) and concludes the proof of Lemma 2.1. □

3. Interlacing property of zeros of $j_n(\tau)$

As a consequence of (11) we have seen that $j_n(\tau)$ has exactly n simple zeros in $\mathcal{F}$ which all lie on $\mathcal{C}$. Let $u_{n,k}$ for $k = 0, ..., n-1$ denote their real part (with decreasing real part for increasing k). Lemma 2.3 already gives a good (preliminary) estimate for their location. The main result of this paper is the following consequence of Lemma 2.1:

Theorem 3.1. *The zeros of $j_n(\tau)$ and $j_{n+1}(\tau)$ on $\mathcal{C}$ interlace for $n \geq 1$. More precisely, for $k = 0, ..., n-1$ we have*

$$u_{n+1,k+1} \overset{(a)}{<} u_{n,k} \overset{(b)}{<} u_{n+1,k}. \tag{25}$$

For $n \geq 10$ the distance between two zeros as above is at least $\frac{1}{20.0n(n+1)}$.

Remark. The point of the superscripts (a), (b) is to facilitate a later reference.

Proof of Theorem 3.1. We prove the theorem in two parts. For $n \leq 10$ the statement can easily be checked manually. So to simplify calculation let $n > 10$ from now on. We divide the interval $\left[0, \frac{1}{2}\right]$ into two parts:

$$\text{Part 1: } \mathcal{P}_1 := \left\{0 \leq x \leq \frac{1}{2} - \frac{\log n}{5n}\right\}, \qquad \text{then we have: } k \geq \frac{\log n}{3}; \tag{26}$$

$$\text{Part 2: } \mathcal{P}_2 := \left\{\frac{1}{2} - \frac{\log n}{2n} \leq x \leq \frac{1}{2}\right\}, \qquad \text{then we have: } k \leq \log n, \tag{27}$$

where the given (numerical) range of k in the two cases above follows from the original estimate and a simple calculation. First note that the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$ has length

$$\frac{1}{2} - \frac{\log n}{5n} - \frac{1}{2} + \frac{\log n}{2n} \geq \frac{3 \log 10}{10} \frac{1}{n} > \frac{1}{2n} + \frac{2}{11.0n}.$$

Therefore by Lemma 2.3 and the simple upper bound from Lemma 3.2, stated below, $\mathcal{P}_1 \cap \mathcal{P}_2$ contains at least 2 zeros. So Theorem 3.1 follows if we can show the interlacing properties (a) and (b) in each individual part. To this end we will need several (technical) lemmata whose proofs we postpone to the last section.

In Part 1 we approximate $j_n(\tau)e^{-2\pi n y}$ by $2\cos(2\pi n x)$. First note:

Lemma 3.2. *The zeros of $2\cos(2\pi n x)$ for $x \in \left[0, \frac{1}{2}\right]$ interlace; i.e., for $k = 0, ..., n-1$:*

(a) $\frac{1}{2n} \geq x_{n,k} - x_{n+1,k+1} = \dfrac{2(n-(k+1))+1}{4n(n+1)} \geq \dfrac{1}{4n(n+1)}$,

(b) $\frac{1}{2n} \geq x_{n+1,k} - x_{n,k} = \dfrac{2k+1}{4n(n+1)} \geq \dfrac{1}{4n(n+1)}$.

For Part 1 the approximation by $2\cos(2\pi n x)$ is good enough to transfer this property to the zeros of $j_n(\tau)$. For this we need the following lemma:

Lemma 3.3. *For $k \geq \frac{\log n}{3}$ (in particular for Part 1):*

$$|u_{n,k} - x_{n,k}| < \frac{1}{10.0n(n+1)}. \tag{28}$$

For $x \in \mathcal{P}_1$, Lemmas 3.2 and 3.3 imply (25) and the bound

$$\begin{aligned} u_{n,k} - u_{n+1,k+1} &\geq x_{n,k} - x_{n+1,k+1} - |u_{n,k} - x_{n,k}| - |u_{n+1,k+1} - x_{n+1,k+1}| \\ &> \frac{1}{4n(n+1)} - 2\frac{1}{10.0n(n+1)} \geq \frac{1}{20.0n(n+1)}, \\ u_{n+1,k} - u_{n,k}, &\geq ... > \frac{1}{20.0n(n+1)}, \end{aligned}$$

which finishes the proof of Theorem 3.1 for $x \in \mathcal{P}_1$. For $x \in \mathcal{P}_2$, the approximation by $2\cos(2\pi n x)$ is not good enough anymore since the summand $(-1)^n R_n(x)$ from

Lemma 2.1 becomes significant for x close to $\frac{1}{2}$. It causes a *shift* of the zeros of $2\cos(2\pi nx)$, which is inherited by the zeros of $j_n(\tau)e^{-2\pi ny}$. The additional term $R_n(x)$ is rather complicated but since x is close to $\frac{1}{2}$, i.e. $x \in \mathcal{P}_2$, we can approximate $j_n(\tau)e^{-2\pi ny}$ (resp. $R_n(x)$) by the following (simpler) functions:

$$f_n(x) := 2\cos(2\pi nx) + (-1)^n Q_n(x), \text{ where} \tag{29}$$

$$Q_n(x) := e^{-2\pi n\sqrt{3}\left(\frac{1}{2}-x\right)}. \tag{30}$$

Let $\hat{x}_{n,k}$ denote the zero of $f_n(x)$ around $x_{n,k}$ (there is clearly at most one zero around $x_{n,k}$). The idea of the proof is to show that the zeros of f_n and f_{n+1} interlace with a certain bound and that the distance of the zeros of f_n to the zeros of g_n (resp. to the zeros of j_n) is small enough to maintain a positive bound. For this we need the following three lemmata:

Lemma 3.4. *Let $k \leq \log n$. The zeros $\hat{x}_{n,k}$ are of the form*

$$\hat{x}_{n,k} = x_{n,k} + \frac{w_k}{n}, \textit{ where } w_k \textit{ is independent of } n \textit{ and satisfies } \left|\frac{w_k}{n}\right| \leq \frac{1}{12n}.$$

Lemma 3.5. *The zeros of $f_n(x)$ for $x \in \mathcal{P}_2$ interlace. For $k \leq \log n$:*

(a) $\hat{x}_{n,k} - \hat{x}_{n+1,k+1} > \dfrac{1}{7.0n}$,

(b) $\hat{x}_{n+1,k} - \hat{x}_{n,k} > \dfrac{1}{6n(n+1)}$.

Lemma 3.6. *For $k \leq \log n$ (in particular for Part 2),*

$$|u_{n,k} - \hat{x}_{n,k}| < \frac{1}{20.0n(n+1)}. \tag{31}$$

For $x \in \mathcal{P}_2$, Lemmas 3.5 and 3.6 imply (25) and the bound

$$|u_{n,k} - u_{n+1,k+1}| > \frac{1}{8.0n}, \qquad |u_{n+1,k} - u_{n,k}| > \frac{9.0}{10.0n(n+1)},$$

which finishes the proof of Theorem 3.1 for $x \in \mathcal{P}_2$, resp. for all $x \in \left[0, \frac{1}{2}\right]$. □

4. Proofs of Lemmata 3.2–3.6

The proof of Lemma 3.2 is an elementary calculation and is left to the reader.

Proof of Lemma 3.3. By contradiction we assume $\frac{1}{4n} > |u_{n,k} - x_{n,k}| \geq \frac{1}{10.0n(n+1)}$, where the first estimate is the original estimate. We then have

$$|2\cos(2\pi n u_{n,k})| \geq 2\sin\left(\frac{2\pi n}{10.0n(n+1)}\right) \geq \frac{4\pi}{10.0(n+1)} - \frac{8\pi^3}{3000(n+1)^3} > \frac{3\pi}{10.0n}.$$

On the other hand if we apply Corollary 2.2 for $x = u_{n,k}$ (which is a zero of $j_n(\tau)$) and if we use $k \geq \frac{\log n}{3}$, the following holds (for $n \geq 10$):

$$|2\cos(2\pi n u_{n,k})| \leq e^{-\pi k} + e^{-\pi n\frac{\sqrt{3}}{2}} n \leq n^{-\frac{\pi}{3}} + e^{-\pi n\frac{\sqrt{3}}{2}} n < \frac{3\pi}{10.0n}.$$

This gives a contradiction. Hence $|u_{n,k} - x_{n,k}| < \frac{1}{10.0n(n+1)}$. □

Proof of Lemma 3.4. We use the parametrization $x = x_{n,k} + \frac{w}{n}$. Then

$$2\cos(2\pi n x) = (-1)^{n+k} 2\sin(2\pi w),$$
$$Q_n(x) = e^{-\pi\sqrt{3}\left(k+\frac{1}{2}-2w\right)}.$$

From this we see that $(-1)^n f_n(x)$ is independent of n, so any zero is of the form $x_{n,k} + \frac{w_k}{n}$ as claimed. The estimate for w_k is obtained as in the proof of (11) and Lemma 2.3:

$$\left|2\cos\left(2\pi n\left(x_{n,k} \pm \frac{1}{12n}\right)\right)\right| = 1 > \left|Q_n\left(\hat{x}_{n,k}\right)\right|.$$

So $f_n(x)$ must change its sign in $x_{n,k} - \frac{1}{12n} < x < x_{n,k} + \frac{1}{12n}$ for $k \le \log n$. □

Proof of Lemma 3.5. For (the trivial) property (a) we use property (a) from Lemma 3.2 together with the assumption $k \le \log n$ and the estimate from Lemma 3.4:

$$\hat{x}_{n,k} - \hat{x}_{n+1,k+1} = x_{n,k} - x_{n+1,k+1} + \frac{w_k}{n} - \frac{w_k}{n+1} \ge \frac{2(n-(k+1))+1}{4n(n+1)} - \frac{1}{6n}$$
$$\ge \frac{\frac{10}{22}\left(1 - \frac{\log 10}{10} - \frac{1}{20}\right) - \frac{1}{6}}{n} \ge \frac{1}{7.0n}.$$

For property (b) we use property (b) from Lemma 3.2 and Lemma 3.4:

$$\hat{x}_{n+1,k} - \hat{x}_{n,k} = x_{n+1,k} - x_{n,k} - w_k\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$\ge \frac{1}{4n(n+1)} - \frac{1}{12}\frac{1}{n(n+1)} = \frac{1}{6n(n+1)}.$$

□

It remains to prove Lemma 3.6. This part is rather long and technical compared to its content. It is also the part of the proof where Lemma 2.1 is used instead of Corollary 2.2.

Proof of Lemma 3.6. Define, resp. recall, the following functions:

$$g_n(x) := 2\cos(2\pi n x) + (-1)^n R_n(x), \tag{32}$$
$$R_n(x) = e^{-2\pi n v(x)\left(\frac{1}{2}-x\right)},$$
$$v(x) = \sqrt{\frac{1+x}{1-x}} \in [1, \sqrt{3}].$$

Let $\tilde{x}_{n,k}$ denote the zero of $g_n(x)$ around $x_{n,k}$ (there is clearly at most one zero around $x_{n,k}$). We need the following technical bounds, whose proof we postpone to the end.

Let $k \le \log n$. The sign of the moving direction of the zeros is determined by the parity of k as follows:

$$\frac{1}{11.0n} > (-1)^k(x_{n,k} - u_{n,k}) > 0, \tag{33}$$

$$\frac{1}{12n} > (-1)^k(x_{n,k} - \tilde{x}_{n,k}) > (-1)^k(x_{n,k} - \hat{x}_{n,k}) > 0. \tag{34}$$

If moreover $|x - x_{n,k}| \leq \frac{1}{4n}$, then

$$0 \leq \sqrt{3} - v(x) \leq \frac{2}{\sqrt{3}} \frac{k+1}{n}, \tag{35}$$

$$0 \leq R_n(x) - Q_n(x) = g_n(x) - f_n(x) < \frac{2.2}{n}. \tag{36}$$

We will use these technical results to prove the following bounds (for $k \leq \log n$):

$$|\tilde{x}_{n,k} - \hat{x}_{n,k}| < \frac{1}{20.6n(n+1)}, \tag{37}$$

$$|u_{n,k} - \tilde{x}_{n,k}| < \frac{e^{-\pi n y}}{8.0}. \tag{38}$$

Lemma 3.6 follows from inequalities (37) and (38). □

Proof of inequalities (33) *to* (38). With absolute values the estimates (33) and the first inequality from (34) follow from Lemma 2.3 and by the same arguments as in the proof of Lemma 3.4 for $R_n(x)$ instead of $Q_n(x)$. Using $0 < Q_n(x) \leq R_n(x)$ and monotonicity the second inequality from (34) follows (with absolute values). For the signs in both estimates first note that since $k \leq \log n$ we can assume $x \geq x_{10,2} - \frac{1}{110} > 0.36$ for all relevant x. In this case we have

$$R_n(x) \geq Q_n(x) \geq e^{-20\pi\sqrt{3}\left(\frac{1}{2}-0.36\right)} > e^{-2\pi n y} n.$$

So terms $R_n(x)$, resp. $Q_n(x)$, dominate the remaining error term. So the sign of the moving direction of the zeros is determined by the sign in front of $R_n(x)$, resp. $Q_n(x)$, around $x_{n,k}$, which is equal to $(-1)^k$. This gives (33). The signs in (34) follow by similar arguments (in that case there is not even an error term present). This proves inequalities (33) and (34).

By the assumptions we have $\frac{k}{2n} \leq \frac{1}{2} - x \leq \frac{k+1}{2n}$. So

$$\begin{aligned} 0 \leq \frac{\sqrt{3} - v(x)}{\sqrt{3}} &\leq 1 - \frac{1}{\sqrt{3}} \sqrt{\frac{\frac{3}{2} - \left(\frac{1}{2} - x\right)}{\frac{1}{2} + \left(\frac{1}{2} - x\right)}} \leq 1 - \sqrt{\frac{1 - \frac{k+1}{3n}}{1 + \frac{k+1}{n}}} \\ &\leq 1 - \left(1 - \frac{k+1}{6n}\right)\left(1 - \frac{k+1}{2n}\right) \leq \frac{2}{3} \frac{(k+1)}{n}. \end{aligned}$$

This proves inequality (35), which in turn gives

$$\begin{aligned} 0 \leq R_n(x) - Q_n(x) &= e^{-2\pi n\sqrt{3}\left(\frac{1}{2}-x\right)} \left(e^{2\pi n\left(\sqrt{3}-v(x)\right)\left(\frac{1}{2}-x\right)} - 1\right) \\ &\leq e^{-\pi\sqrt{3}k} \left(e^{\pi \frac{2}{3} \frac{(k+1)^2}{n}} - 1\right) \leq e^{\pi \frac{2}{3} \frac{1}{n}} - 1 \leq \pi \frac{2}{3} \frac{1}{n} \left(\frac{e^{\pi \frac{2}{3} \frac{1}{10}} - 1}{\pi \frac{2}{3} \frac{1}{10}}\right) < \frac{2.2}{n}. \end{aligned}$$

Here we used the fact that $k+1$ is small compared to n (we have $k+1 < \log n$ with $n \geq 10$), so the contribution of this exponential term remains insignificant in $\mathcal{P}_2$ compared to $e^{-\pi\sqrt{3}k}$. Hence the third expression becomes maximal for $k = 0$. This proves equation (36).

Let $v_{n,k}$ be defined by $\tilde{x}_{n,k} =: \hat{x}_{n,k} + \frac{v_{n,k}}{n} = x_{n,k} + \frac{w_k + v_{n,k}}{n}$. To show (37) it remains to show that $|v_{n,k}| \leq \frac{1}{15.3n}$. By inequality (34) we have $|v_{n,k}|, |w_k|, |w_k + v_{n,k}| \leq \frac{1}{12}$. We use equation (36) for $x = \tilde{x}_{n,k}$, divide $f_n(\tilde{x}_{n,k})$ into three parts and use the mean value theorem for $2\sin(2\pi w)$:

$$\begin{aligned}
\frac{2.2}{n} \geq |f_n(\tilde{x}_{n,k})| &= \left|2\sin(2\pi(w_k+v_{n,k})) + (-1)^k e^{-\pi\sqrt{3}\left(k+\frac{1}{2}-2(w_k+v_{n,k})\right)}\right| \\
&= \Bigg| \left(\frac{2\sin(2\pi(w_k+v_{n,k})) - 2\sin(2\pi w_k)}{2\pi v_{n,k}}\right) 2\pi v_{n,k} \\
&\qquad + \underbrace{2\sin(2\pi w_k) + (-1)^k e^{-\pi\sqrt{3}\left(k+\frac{1}{2}-2w_k\right)}}_{=f_n(\hat{x}_{n,k})=0} \\
&\qquad -(-1)^k e^{-\pi\sqrt{3}\left(k+\frac{1}{2}-2w_k\right)} \left(e^{2\pi\sqrt{3}v_{n,k}} - 1\right)\Bigg| \\
&\geq \left(4\pi\cos\left(\frac{2\pi}{12}\right)\right) 2\pi|v_{n,k}| - \left(e^{2\pi\sqrt{3}|v_{n,k}|} - 1\right) \\
&\geq 4\sqrt{3}\pi^2|v_{n,k}| - 2\pi\sqrt{3}\,1.63|v_{n,k}| \geq 50.0|u_{n,k}|,
\end{aligned}$$

where at the end we used $e^x - 1 \leq \frac{e^{x_m}-1}{x_m}x \leq 1.63x$ with $x = 2\pi\sqrt{3}|v_{n,k}|$ and $x_m = \frac{2\pi\sqrt{3}}{12}$ (a maximal bound). Hence $|u_{n,k}| \leq \frac{2.2}{50.0n} < \frac{1}{20.6(n+1)}$. This proves inequality (37).

It remains to show inequality (38): Let $x = x_{n,k} + \frac{w}{n}$ with $|w| \leq \frac{1}{11}$. Note that $\tilde{x}_{n,k}$ and $u_{n,k}$ both satisfy this by inequalities (33) and (34). We first need an estimate for the minimum of the absolute value of the derivative of $g_n(x)$ in this region, denoted by m. A direct calculation using the usual assumptions and estimates shows:

$$\begin{aligned}
(-1)^{(n+k)}g_n'(x) &= n\left(4\pi\cos(2\pi w) + (-1)^k\left(\frac{1+2x-2x^2}{(1-x)\sqrt{1-x^2}}\right)\pi R_n(x)\right) \\
&\geq n\left(4\pi\cos\left(\frac{2\pi}{11}\right) - 2\sqrt{3}\pi R_n\left(x_{n,k} + \frac{1}{11.0n}\right)\right) \\
&\geq n\left(4\pi\cos\left(\frac{2\pi}{11}\right) - 2\sqrt{3}\pi e^{-\pi\left(\sqrt{3}-\frac{2}{10\sqrt{3}}\right)\left(\frac{1}{2}-\frac{2}{11}\right)}\right) \geq 8.0n.
\end{aligned}$$

So $m \geq 8.0n$. The calculation also shows that the sign of $g_n'(x)$ is equal to $(-1)^{(n+k)}$. Now let $x = \tilde{x}_{n,k} + r$ with $|r| \leq \frac{1}{11}$. Then $|g_n(x)| \geq m|r|$ and $(-1)^{(n+k)}g_n(x)$ is positive iff $r > 0$. We now apply Lemma 2.1: Choose x (resp. r) such that $8.0|r| \geq e^{-\pi n y}$ holds. Then we have

$$|g_n(x)| \geq m|r| \geq 8.0n|r| \geq e^{-\pi n y}n \geq |j_n(x)e^{-2\pi n y} - g_n(x)|.$$

Then $j_n(x)$ has the same sign as $g_n(x)$ and since $(-1)^{(n+k)}g_n(x)$ is positive iff $r > 0$ we have that $g_n(x)$ switches its sign for $r > 0$, so $u_{n,k} \in [\tilde{x}_{n,k} - |r|, \tilde{x}_{n,k} + |r|]$. By choosing $r = |r| = \frac{e^{-\pi n y}n}{8.0n}$ (as small as possible) we get $|u_{n,k} - \tilde{x}_{n,k}| \leq r \leq \frac{e^{-\pi n y}}{8.0}$. ☐

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