PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 10, October 2012, Pages 3409–3419
S 0002-9939(2012)11263-8
Article electronically published on February 2, 2012

C^*-ALGEBRAS ASSOCIATED WITH REAL MULTIPLICATION

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(Communicated by Marius Junge)

ABSTRACT. Noncommutative tori with real multiplication are the irrational rotation algebras that have special equivalence bimodules. Y. Manin proposed the use of noncommutative tori with real multiplication as a geometric framework for the study of abelian class field theory of real quadratic fields. In this paper, we consider the Cuntz-Pimsner algebras constructed by special equivalence bimodules of irrational rotation algebras. We shall show that the associated C^*-algebras are simple and purely infinite. We compute the K-groups of the associated C^*-algebras and show that these algebras are related to the solutions of Pell's equation and the unit groups of real quadratic fields. We consider the Morita equivalent classes of the associated C^*-algebras.

1. Introduction

Let θ be an irrational number. An irrational rotation algebra A_θ is the crossed product C^*-algebra for the action of the integers on the circle by powers of the rotation by angle $2\pi\theta$. It is simple and has a unique normalized trace τ_θ. These algebras are also called noncommutative tori and have been classified up to C^*-isomorphism and Morita equivalence [18],[19]. If A_{θ_1} is Morita equivalent to A_{θ_2}, then there exist $a, b, c, d \in \mathbb{Z}$ such that $\theta_1 = \frac{a\theta_2+b}{c\theta_2+d}$ and $ad - bc = \pm 1$ by Theorem 4 in [19].

Suppose that σ is a free and proper action of a locally compact group G on a locally compact Hausdorff space X. Then X/σ is a locally compact Hausdorff space and $C_0(X) \rtimes_\sigma G$ is Morita equivalent to $C_0(X/\sigma)$ (see [21]). The action of the integers on the circle by powers of the rotation by angle $2\pi\theta$ is free, but every orbit is dense and the orbit space $(\mathbb{R}/\mathbb{Z})/\mathbb{Z}\theta$ behaves badly. By the fact above, we may consider an irrational rotation algebra A_θ as a well-behaved quotient space of $(\mathbb{R}/\mathbb{Z})/\mathbb{Z}\theta$. This idea is due to A. Connes [4].

An elliptic curve E_ω over $\mathbb{C}$ can be described as the quotient $E_\omega \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$ of the complex plane by a 2-dimensional lattice $\mathbb{Z}+\mathbb{Z}\omega$, where we can take $\mathrm{Im}(\omega) > 0$. Most elliptic curves over $\mathbb{C}$ have only the multiplication-by-m endomorphisms. An elliptic curve E_ω over $\mathbb{C}$ has extra endomorphisms if and only if ω is in an imaginary quadratic field. In this case, an elliptic curve E_ω is said to have complex multiplication. Such curves have many special properties. One of the important properties is that the j-invariant and the torsion points of E_ω generate a maximal abelian extension of an imaginary quadratic field $\mathbb{Q}(\omega)$. For real quadratic fields,

Received by the editors June 11, 2009 and, in revised form, September 21, 2010 and April 1, 2011.

2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 11D09, 11R11.

Key words and phrases. Irrational rotation algebras, Morita equivalence, Cuntz-Pimsner algebras, real multiplication, real quadratic fields.

a similar description is not known. By the idea above and regarding $(\mathbb{R}/\mathbb{Z})/\mathbb{Z}\theta$ as $\mathbb{R}/(\mathbb{Z}+\mathbb{Z}\theta)$, Y. Manin proposed to use irrational rotation algebras corresponding to real quadratic irrationalities as a replacement of elliptic curves with complex multiplication. We shall explain that there exist analogous properties with complex multiplication in the irrational rotation algebras.

Two C^*-algebras A and B are Morita equivalent if there exists an A-B-equivalence bimodule. An A-B-equivalence bimodule is an A-B-bimodule $\mathcal{E}$ which is simultaneously a full left Hilbert A-module under an A-valued inner product ${}_A\langle\cdot,\cdot\rangle$ and a full right Hilbert B-module under a B-valued inner product $\langle\cdot,\cdot\rangle_B$, satisfying ${}_A\langle\xi,\eta\rangle\zeta=\xi\langle\eta,\zeta\rangle_B$ for any $\xi,\eta,\zeta\in\mathcal{E}$. We recall some definitions and elementary facts on the Picard groups of C^*-algebras introduced by Brown, Green and Rieffel in [2]. For A-A-equivalence bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$, we say that $\mathcal{E}_1$ is isomorphic to $\mathcal{E}_2$ as an equivalence bimodule if there exists a $\mathbb{C}$-linear one-to-one map Φ from $\mathcal{E}_1$ onto $\mathcal{E}_2$ with the properties such that $\Phi(a\xi b)=a\Phi(\xi)b$, ${}_A\langle\Phi(\xi),\Phi(\eta)\rangle={}_A\langle\xi,\eta\rangle$ and $\langle\Phi(\xi),\Phi(\eta)\rangle_A=\langle\xi,\eta\rangle_A$ for $a,b\in A$, $\xi,\eta\in\mathcal{E}_1$. The set of isomorphic classes $[\mathcal{E}]$ of the A-A-equivalence bimodules $\mathcal{E}$ forms a group under the product defined by $[\mathcal{E}_1][\mathcal{E}_2]=[\mathcal{E}_1\otimes_A\mathcal{E}_2]$. We call it the *Picard group* of A and denote it by $\mathrm{Pic}(A)$. The identity of $\mathrm{Pic}(A)$ is given by the A-A-bimodule $\mathcal{E}:=A$ with ${}_A\langle a,b\rangle=ab^*$ and $\langle a,b\rangle_A=a^*b$ for $a,b\in A$. The dual module $\mathcal{E}^*$ of an A-A-equivalence bimodule $\mathcal{E}$ is a set $\{\xi^*;\xi\in\mathcal{E}\}$ with the operations such that $\xi^*+\eta^*=(\xi+\eta)^*$, $\lambda\xi^*=(\overline{\lambda}\xi)^*$, $b\xi^*a=(a^*\xi b^*)^*$, ${}_A\langle\xi^*,\eta^*\rangle=\langle\eta,\xi\rangle_A$ and $\langle\xi^*,\eta^*\rangle_A={}_A\langle\eta,\xi\rangle$. Then $[\mathcal{E}^*]$ is the inverse element of $[\mathcal{E}]$ in the Picard group of A. Let α be an automorphism of A, and let $\mathcal{E}_\alpha=A$ with the obvious left A-action and the obvious left A-valued inner product. We define the right A-action on $\mathcal{E}_\alpha$ by $\xi\cdot a=\xi\alpha(a)$ for any $\xi\in\mathcal{E}_\alpha$ and $a\in A$, and the right A-valued inner product by $\langle\xi,\eta\rangle_A=\alpha^{-1}(\xi^*\eta)$ for any $\xi,\eta\in\mathcal{E}_\alpha$. Then $\mathcal{E}_\alpha$ is an A-A-equivalence bimodule. For $\alpha,\beta\in\mathrm{Aut}(A)$, $\mathcal{E}_\alpha$ is isomorphic to $\mathcal{E}_\beta$ if and only if there exists a unitary $u\in A$ such that $\alpha=ad\ u\circ\beta$. Moreover, $\mathcal{E}_\alpha\otimes\mathcal{E}_\beta$ is isomorphic to $\mathcal{E}_{\alpha\circ\beta}$. Hence we obtain a homomorphism ρ_A from $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ to $\mathrm{Pic}(A)$ and may regard A-A-equivalence bimodules as a generalization of automorphisms of C^*-algebras.

The equivalence bimodules of irrational rotation algebras are constructed by M. Rieffel. These bimodules are defined as completions of $C_c(\mathbb{R}\times\mathbb{Z}/c\mathbb{Z})$ or $S(\mathbb{R}\times\mathbb{Z}/c\mathbb{Z})$ with the certain actions of irrational rotation algebras [22], [23]. There exist special A_θ-A_θ-equivalence bimodules in the case where θ is a quadratic irrational number. In fact, K. Kodaka showed that if θ is not a quadratic irrational number, then the Picard group of A_θ is isomorphic to $\mathrm{Aut}(A_\theta)/\mathrm{Inn}(A_\theta)$ and that if θ is a quadratic number, then the Picard group of A_θ is isomorphic to a semidirect product of $\mathrm{Aut}(A_\theta)/\mathrm{Inn}(A_\theta)$ with $\mathbb{Z}$ [11]. We may consider that this is analogous to the complex multiplication of an elliptic curve. Y. Manin called this real multiplication and proposed the use of noncommutative tori with real multiplication as a geometric framework for the study of abelian class field theory of real quadratic fields [12]. But we have no idea of how to solve this problem.

In this paper, we consider the Cuntz-Pimsner algebras constructed by the equivalence bimodules of irrational rotation algebras that are not generated by automorphisms. In the theory of C^*-algebras, the crossed product algebras help us to understand the automorphisms. Since the Cuntz-Pimsner construction is a generalization of the crossed product construction, the Cuntz-Pimsner algebras constructed by special equivalence bimodules help us to understand special bimodules of irrational rotation algebras and may give us a hint for studying the abelian class field theory of real quadratic fields by using irrational rotation algebras. In fact, we show that these algebras are related to the solutions of Pell's equation and the unit groups of real quadratic fields. This shows that there exists a deep relation between irrational rotation algebras and real quadratic fields. (See, for example, [3] and [7] for the relation between the algebraic number theory of quadratic fields and Pell's equation.)

In Section 2 we consider the A_θ-A_θ-equivalence bimodules. K. Kodaka considered the A_θ-A_θ-equivalence bimodules as the automorphisms of $A_\theta \otimes \mathbb{K}$, where $\mathbb{K}$ consists of the C^*-algebras of all compact operators on a countably infinite dimensional Hilbert space, and determined the Picard groups of the irrational rotation algebras in [11]. We consider the equivalence bimodules of irrational rotation algebras neither as the completions of $C_c(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$ $(S(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}))$ nor as the automorphisms of $A_\theta \otimes \mathbb{K}$ in this paper. If A is unital, then an A-A-equivalence bimodule is a finitely generated projective A-module as a right A-module. Hence an A-A-equivalence bimodule is isomorphic to qA^n as a right Hilbert A-module, where q is a projection in $M_n(A)$. Note that we need to be careful with a left action. We consider the A_θ-A_θ-equivalence bimodules from this viewpoint. But we analyze the A_θ-A_θ-equivalence bimodules by the similar arguments in [11]. We show that $\tau_\theta \otimes Tr(q)$ is an invertible element of an order of a real quadratic field. The unit group (invertible elements) of real quadratic fields is an important invariant in algebraic number theory. Hence the A_θ-A_θ-equivalence bimodules are deeply related to the real quadratic fields.

In Section 3 we construct C^*-algebras associated with the bimodules considered in Section 2. We show that these algebras are purely infinite, simple, nuclear and in the UCT class. Therefore these C^*-algebras are classifiable by K-groups [9], [16].

In Section 4 we compute the K-groups of the associated C^*-algebras and show that these groups are related to the solutions of Pell's equation, where Pell's equation is $x^2 - Dy^2 = \pm 4$ for a given integer D, to be solved in integers x and y. Moreover, we consider the Morita equivalent classes of the associated C^*-algebras. We show that the Morita equivalent classes of the associated C^*-algebras depend on the discriminants of irrational numbers. Let θ_1 and θ_2 be quadratic irrational numbers such that the discriminant of θ_1 is equal to the discriminant of θ_2. In general A_{θ_1} is not Morita equivalent to A_{θ_2}. We denote by D_{θ_1} the discriminant of θ_1. If either $D_{\theta_1} \equiv 1 \pmod 4$ is square free or $D_{\theta_1} \equiv 8$ or 12 modulo 16 and $D_{\theta_1}/4$ is square free and the class number of $\mathbb{Q}(\theta_1)$ is equal to 1, then A_{θ_1} is Morita equivalent to A_{θ_2}. See [3] and [7] for basic facts of the class numbers of real quadratic fields. But there exists no relation between A_{θ_1} and A_{θ_2} in general. Hence this result is very surprising.

2. Real multiplication

In this section, we shall consider the A_θ-A_θ-equivalence bimodules and review the result in [11] from a slightly different perspective. We refer the reader to [1] for basic facts of C^*-algebras and equivalence bimodules.

Let A be a unital C^*-algebra. Then an A-A-equivalence bimodule $\mathcal{E}$ is a finitely generated projective A-module as a right module [20]. Hence $\mathcal{E}$ is isomorphic to qA^n as a right Hilbert A-module, where q is a projection in $M_n(A)$. A right Hilbert A-module qA^n has a structure of a $qM_n(A)q$-A-equivalence bimodule with the obvious left $qM_n(A)q$-action and the $qM_n(A)q$-valued inner product such that ${}_{qM_n(A)q}\langle q(a_i)_i, q(b_i)_i\rangle = q(a_jb_k^*)_{j,k}q$ for any $a_i, b_i \in A$. Since $\mathcal{E}$ is an A-A-equivalence bimodule, $qM_n(A)q$ is isomorphic to A. Note that the left A-action on qA^n and the left A-valued inner product are dependent on an isomorphism ϕ from A to $qM_n(A)q$. Therefore we need to be careful with an isomorphism from A to $qM_n(A)q$.

We shall consider the condition of q that qA_θ^n is an A_θ-A_θ-equivalence bimodule. We need the following well-known fact [22].

Lemma 2.1. *Let q_1 and q_2 be projections in A_θ. If $\tau_\theta(q_1) \geq \tau_\theta(q_2)$ (resp. $\tau_\theta(q_1) = \tau_\theta(q_2)$), then there exists a unitary element w in A_θ such that $q_1 \geq w^*q_2w$ (resp. $q_1 = w^*q_2w$).*

Let $Tr_\theta := \tau_\theta \otimes Tr$ be the unnormalized trace on $M_n(A_\theta)$, where Tr is the usual trace on $M_n(\mathbb{C})$. We have the following lemma (see [10] and [13]).

Lemma 2.2. *If q is a proper projection in $M_n(A_\theta)$ such that $Tr_\theta(q) = k(c\theta + d)$, where k is a natural number and c, d are integers such that $\gcd(c, d) = 1$, then $qM_n(A_\theta)q \cong M_k(A_{\frac{a\theta+b}{c\theta+d}})$ for any $a, b \in \mathbb{Z}$ such that $ad - bc = \pm 1$.*

Since qA_θ^n is a $qM_n(A_\theta)q$-A_θ-equivalence bimodule, the condition of q is equivalent to $Tr_\theta(q) = c\theta + d$, where there exist integers a and b such that $\frac{a\theta+b}{c\theta+d} = \theta$ by the lemma above. We shall consider the case where θ is not a quadratic number. The following proposition is Corollary 9 in [11].

Proposition 2.3. *Let θ be an irrational number and $\mathcal{E}$ an A_θ-A_θ-equivalence bimodule. Assume that θ is not a quadratic number. Then there exists an automorphism α such that $\mathcal{E}$ is isomorphic to $\mathcal{E}_\alpha$.*

We shall consider the case where θ is a quadratic irrational number. We may assume that θ satisfies $k\theta^2 + l\theta + m = 0$ with a natural number k and integers l,m such that $\gcd(k, l, m) = 1$. The equation is uniquely determined. Let $D_\theta = l^2 - 4km$ be the discriminant of θ. The following fact is well known (see [6] and [10, Remark 7]).

Fact 2.4. Let θ be a quadratic number. Then there exists a real number ϵ_0 such that

$$\{c\theta + d \ : \ \frac{a\theta + b}{c\theta + d} = \theta, a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1\} = \{\pm\epsilon_0^n \ : \ n \in \mathbb{Z}\}.$$

The number ϵ_0 is only dependent on D_θ.

We may assume $\epsilon_0 > 1$. A real number ϵ_0 is called the fundamental unit. It is known that the set $\{\pm\epsilon_0^m; m \in \mathbb{Z}\}$ is the group of units (invertible elements) of $\mathbb{Z}[k\theta]$ which is an order of a real quadratic field $\mathbb{Q}(\theta)$. Especially, if either $D_\theta \equiv 1$ (mod 4) is square free or $D_\theta \equiv 8$ or 12 modulo 16 and $D_\theta/4$ is square free, then $\mathbb{Z}[k\theta]$ is a ring of integers of $\mathbb{Q}(\theta)$. We refer the reader to [3] and [7] for basic facts of the quadratic number fields. We obtain the following proposition.

Proposition 2.5. *Let θ be a quadratic number with $k\theta^2 + l\theta + m = 0$, and let q be a projection in $M_n(A_\theta)$ for $n \in \mathbb{N}$. Then qA^n is an A_θ-A_θ-equivalence bimodule if and only if $Tr_\theta(q)$ is a unit (an invertible element) in $\mathbb{Z}[k\theta]$ which is an order of a real quadratic field $\mathbb{Q}(\theta)$.*

Since $Tr_\theta(q)$ is a positive number, we have

$$\{Tr_\theta(q) \ : \ qA_\theta^n \text{ is an } A_\theta\text{-}A_\theta\text{-equivalence bimodule}\} = \{\epsilon_0^m \ : \ m \in \mathbb{Z}\}.$$

We shall consider the relation between the Picard groups of the irrational rotation algebras and the unit groups of the real quadratic fields. We shall show that the A_θ-A_θ-equivalence bimodules are of simple form.

Lemma 2.6. *Let $\mathcal{E}$ be an A_θ-A_θ-equivalence bimodule. Then there exist a projection $q \in A_\theta$ and an isomorphism ϕ of A_θ to $qA_\theta q$ such that $\mathcal{E}$ is isomorphic to qA_θ with the obvious right A_θ-action, the obvious right A_θ-valued inner product, the left A_θ-action such that $a \cdot qb = \phi(a)b$ and the left A_θ-valued inner product such that ${}_{A_\theta}\langle qa, qb\rangle = \phi^{-1}(qab^*q)$ or $A_\theta q$ with the obvious left A_θ-action, the obvious left A_θ-valued inner product, the right A_θ-action such that $bq \cdot a = b\phi(a)$ and the right A_θ-valued inner product such that $\langle aq, bq\rangle_{A_\theta} = \phi^{-1}(qa^*bq)$ for any $a, b \in A_\theta$.*

Proof. There exist a natural number n and a projection $q \in M_n(A_\theta)$ such that $\mathcal{E}$ is isomorphic to qA_θ^n. We denote by ρ an isomorphism of A_θ to $qM_n(A_\theta)q$ such that the left A_θ-action of qA_θ^n is defined by $a \cdot \xi = \rho(a)\xi$ for any $a \in A_\theta$ and $\xi \in qA_\theta^n$.

(1) The case $Tr_\theta(q) \leq 1$.

By Lemma 2.1, there exists a unitary element w in $M_n(A_\theta)$ such that $w^*qw = q' \otimes e_{11}$, where e_{11} is a rank one projection in $M_n(\mathbb{C})$ and q' is a projection in A_θ. Since $w^*qwA_\theta^n$ where the left A_θ-action is defined by $a \cdot \xi = w^*\rho(a)w\xi$ for any $a \in A_\theta$ and $\xi \in w^*qwA_\theta^n$ is isomorphic to qA_θ^n, $\mathcal{E}$ is isomorphic to $(q' \otimes e_{11})A_\theta^n$. It is easy to see that $(q' \otimes e_{11})A_\theta^n$ is isomorphic to $q'A_\theta$.

(2) The case $Tr_\theta(q) \geq 1$.

By Lemma 2.1, there exists a unitary element w in $M_n(A_\theta)$ such that $w^*qw \geq 1 \otimes e_{11}$. It is easy to see that $w^*qwM_n(A_\theta)(1 \otimes e_{11})$ is isomorphic to $w^*qwA_\theta^n$. Hence $\mathcal{E}$ is isomorphic to $w^*qwM_n(A_\theta)(1 \otimes e_{11})$. Since $w^*qw(1 \otimes e_{11}) = 1 \otimes e_{11}$, $\mathcal{E}$ is isomorphic to $w^*qwM_n(A_\theta)w^*qw(1 \otimes e_{11})$. Define a map ψ from A_θ to $w^*qwM_n(A_\theta)w^*qw$ by $\psi(a) = w^*\rho(a)w$ for any $a \in A_\theta$. Then ψ is an isomorphism of A_θ to $w^*qwM_n(A_\theta)w^*qw$. Therefore $\mathcal{E}$ is isomorphic to $A_\theta\psi^{-1}(1 \otimes e_{11})$, where the right A_θ-action is defined by $b\psi^{-1}(1 \otimes e_{11}) \cdot a = b\psi^{-1}(a \otimes e_{11})$ for any $a, b \in A_\theta$. □

Remark 2.7. (1) By the proof above, $A_\theta q_1$ is isomorphic to $q_2A_\theta^n$ such that $Tr_\theta(q_2) = \tau_\theta(q_1)^{-1}$.

(2) Lemma 2.6 is important in the next section.

(3) Let A be a unital simple C^*-algebra with a unique normalized trace τ. Assume that the normalized trace on A separates equivalence classes of projections. Then A satisfies the conclusion of Lemma 2.6.

We denote by $\mathcal{E}_{\theta,\phi,q}$ (resp. $\mathcal{F}_{\theta,\phi,q}$) the A_θ-A_θ-equivalence bimodule qA_θ with the left A_θ-action such that $a \cdot qb = \phi(a)b$ (resp. $A_\theta q$ with the right A_θ-action such that $bq \cdot a = b\phi(a)$) for any $a, b \in A_\theta$. We call $\mathcal{E}_{\theta,\phi,q}$ and $\mathcal{F}_{\theta,\phi,q}$ *real multiplication* in the case $q \neq 1$. We shall consider the Picard groups of the irrational rotation algebras.

Proposition 2.8. *Let q_1 and q_2 be projections in A_θ such that A_θ is isomorphic to $q_1A_\theta q_1$ and $q_2A_\theta q_2$. Assume that ϕ_1 (resp. ϕ_2) is an isomorphism of A_θ to $q_1A_\theta q_1$ (resp. $q_2A_\theta q_2$). Then $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_2,q_2}$ is isomorphic to $\mathcal{E}_{\theta,\phi_2\circ\phi_1,q}$, where q is a projection in A_θ such that $\tau_\theta(q) = \tau_\theta(q_1)\tau_\theta(q_2)$. Moreover, $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{F}_{\theta,\phi_1,q_1}$ and $\mathcal{F}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_1,q_1}$ are isomorphic to A_θ with the obvious actions and the obvious inner products.*

Proof. Let $q = \phi_2(q_1)$. Then $\tau_\theta(q) = \tau_\theta(q_1)\tau_\theta(q_2)$ and $q \leq q_2$. Define a map F from $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_2,q_2}$ to $\mathcal{E}_{\theta,\phi_2\circ\phi_1,q}$ by $F(q_1 a \otimes q_2 b) = q\phi_2(a)b$ for any $a, b \in A_\theta$ and extend it by the universality. Since $qq_2 = q$, if $q_1 \otimes q_2 a = q_1 \otimes q_2 b$, then $qa = qb$. Hence F is well-defined. Easy computations show that F is an isomorphism of equivalence bimodules. Since $\mathcal{F}_{\theta,\phi_1,q_1}$ is a dual module of $\mathcal{E}_{\theta,\phi_1,q_1}$, $\mathcal{E}_{\theta,\phi_1,q_1} \otimes \mathcal{F}_{\theta,\phi_1,q_1}$ and $\mathcal{F}_{\theta,\phi_1,q_1} \otimes \mathcal{E}_{\theta,\phi_1,q_1}$ are isomorphic to A_θ with the obvious actions and the obvious inner products. $\square$

K. Kodaka showed that if θ is a quadratic number, then the Picard group of A_θ is isomorphic to a semidirect product of $\mathrm{Aut}(A_\theta)/\mathrm{Inn}(A_\theta)$ with $\mathbb{Z}$ [11]. By Remark 2.7 and Proposition 2.8, the $\mathbb{Z}$ part is related to the unit group of $\mathbb{Z}[k\theta]$ which is an order of a real quadratic field $\mathbb{Q}(\theta)$. This is shown by another method in [14].

3. Associated C^*-algebras

In this section, we shall construct C^*-algebras associated with real multiplication. We show that these algebras are simple and purely infinite. We recall Cuntz-Pimsner algebras [17]. Let A be a C^*-algebra and $\mathcal{E}$ a right Hilbert A-module. For $\xi, \eta \in \mathcal{E}$, the "rank one operator" $\Theta_{\xi,\eta}$ is defined by $\Theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle_A$ for any $\zeta \in \mathcal{E}$. We denote by $B_A(\mathcal{E})$ the algebra of the adjointable operators on $\mathcal{E}$ and by $K_A(\mathcal{E})$ the closure of the linear span of rank one operators of $\mathcal{E}$. We say that $\mathcal{E}$ is a *Hilbert bimodule* over A if $\mathcal{E}$ is a right Hilbert A-module with a homomorphism $\phi : A \to B_A(\mathcal{E})$. We assume that $\mathcal{E}$ is full and ϕ is injective. Define $I_\mathcal{E} = \phi^{-1}(K_A(\mathcal{E}))$. The Cuntz-Pimsner algebra $\mathcal{O}_\mathcal{E}$ is the universal C^*-algebra generated by A and $\{S_\xi : \xi \in \mathcal{E}\}$ with the relation

$$S_{\alpha\xi+\beta\eta} = \alpha S_\xi + \beta S_\eta, \quad aS_\xi b = S_{\phi(a)\xi b}, \quad S_\xi^* S_\eta = \langle \xi, \eta\rangle_A$$

for any $a, b \in A,\ \xi, \eta \in \mathcal{E},\ \alpha, \beta \in \mathbb{C}$ and

$$i_K(\phi(a)) = a$$

for $a \in I_\mathcal{E}$, where $i_K : I_\mathcal{E} \to \mathcal{O}_\mathcal{E}$ is defined by $i_K(\Theta_{\xi,\eta}) = S_\xi S_\eta^*$. We shall consider the Cuntz-Pimsner algebras generated by the equivalence bimodules of irrational rotation algebra that are not generated by automorphisms. By Lemma 2.6, an equivalence bimodule of an irrational rotation algebra A_θ is isomorphic to $\mathcal{E}_{\theta,\phi,q}$ or $\mathcal{F}_{\theta,\phi,q}$. It is easy to see that the Cuntz-Pimsner algebra generated by $\mathcal{E}_{\theta,\phi,q}$ is isomorphic to the Cuntz-Pimsner algebra generated by $\mathcal{F}_{\theta,\phi,q}$. We denote by $\mathcal{O}_{qA_\theta,\phi}$ the Cuntz-Pimsner algebra generated by $\mathcal{E}_{\theta,\phi,q}$. Since $\mathcal{E}_{\theta,\phi,q}$ is an equivalence bimodule, $K_{A_\theta}(\mathcal{E}_{\theta,\phi,q})$ is isomorphic to A_θ and $S_\xi S_\eta^* = {}_{A_\theta}\langle \xi, \eta\rangle$ for $\xi, \eta \in \mathcal{E}_{\theta,\phi,q}$. Hence $\mathcal{O}_{qA_\theta,\phi}$ is the universal C^*-algebra generated by A_θ and S_q with the following relation: for $a \in A_\theta$,

$$aS_q = S_q\phi(a), \quad S_q^* S_q = \langle q, q\rangle_{A_\theta} = q, \quad S_q S_q^* = {}_{A_\theta}\langle q, q\rangle = \phi^{-1}(q) = 1.$$

Note that ϕ is regarded as an endomorphism such that $\phi : A_\theta \to qA_\theta q \subseteq A_\theta$. An endomorphism ρ on a unital C^*-algebra A is called a corner endomorphism if its image is equal to the corner $\rho(1)A\rho(1)$ of A. It is easy to see that $\mathcal{O}_{qA_\theta,\phi}$ is isomorphic to the corner endomorphism crossed product $A_\theta \rtimes_\phi \mathbb{N}$. (See [15] and [24].)

Theorem 3.1. *Let $\mathcal{E}_{\theta,q,\phi}$ be an A_θ-A_θ-equivalence bimodule that is not generated by an automorphism. Then the Cuntz-Pimsner algebra $\mathcal{O}_{qA_\theta,\phi}$ generated by $\mathcal{E}_{\theta,q,\phi}$ is purely infinite, simple, nuclear and in the UCT class.*

Proof. By [24] (Theorem 3.1), $\mathcal{O}_{qA_\theta,\phi}$ is simple and purely infinite because A_θ is a simple unital C^*-algebra of real rank zero and with the property of Lemma 2.1. Since A_θ is nuclear and in the UCT class, $\mathcal{O}_{qA_\theta,\phi}$ is nuclear and in the UCT class. □

The isomorphism class of the C^*-algebra $\mathcal{O}_{qA_\theta,\phi}$ is completely determined by the K-groups together with the class of the unit by the classification theorem by Kirchberg-Phillips [9], [16]. Especially, the Morita equivalence class of C^*-algebra $\mathcal{O}_{qA_\theta,\phi}$ is completely determined by the K-groups.

4. K-groups

In this section, we shall compute the K-groups of $\mathcal{O}_{qA_\theta,\phi}$. Let $D_{\mathcal{E}_{\theta,q,\phi}}$ be the linking algebra of $\mathcal{E}_{\theta,q,\phi}$. The linking algebra of $D_{\mathcal{E}_{\theta,q,\phi}}$ has the following form:

$$D_{\mathcal{E}_{\theta,q,\phi}} = \{\begin{pmatrix} x & \xi \\ \eta & y \end{pmatrix} : x \in K_{A_\theta}(\mathcal{E}_{\theta,q,\phi}) = qA_\theta q,\ y \in A_\theta,\ \xi \in \mathcal{E}_{\theta,q,\phi},\ \eta \in \mathcal{E}^*_{\theta,q,\phi}\},$$

where $\mathcal{E}^*_{\theta,q,\phi}$ is the dual module of $\mathcal{E}_{\theta,q,\phi}$. The linking algebra $D_{\mathcal{E}_{\theta,q,\phi}}$ has the unique normalized trace Tr_D such that $Tr_D(\begin{pmatrix} x & \xi \\ \eta & y \end{pmatrix}) = \frac{\tau_\theta(x)+\tau_\theta(y)}{1+\tau_\theta(q)}$. The natural embeddings are denoted by

$$i_{K_{A_\theta}(\mathcal{E}_{\theta,q,\phi})} : K_{A_\theta}(\mathcal{E}_{\theta,q,\phi}) \to D_{\mathcal{E}_{\theta,q,\phi}},\ i_{A_\theta} : A_\theta \to D_{\mathcal{E}_{\theta,q,\phi}},\ i : I_{\mathcal{E}_{\theta,q,\phi}} \to A_\theta.$$

By [8] (Proposition B.3), the inclusion $i_{A_\theta} : A_\theta \to D_{\mathcal{E}_{\theta,q,\phi}}$ induces an isomorphism of the K-groups. We can define a map $K_*([\mathcal{E}_{\theta,q,\phi}]) : K_*(I_{\mathcal{E}_{\theta,q,\phi}}) \to K_*(A_\theta)$ by the composition of the map $K_*(\phi)$ induced by the restriction of ϕ to $I_{\mathcal{E}_{\theta,q,\phi}}$, the map $K_*(i_{K_{A_\theta}(\mathcal{E}_{\theta,q,\phi})})$ induced by $i_{K_{A_\theta}(\mathcal{E}_{\theta,q,\phi})}$ and the inverse of the isomorphism $K_*(i_{A_\theta})$. We have the following exact sequence [8]:

$$\begin{array}{ccccc} K_0(I_{\mathcal{E}_{\theta,q,\phi}}) & \xrightarrow{K_0(i)-K_0([\mathcal{E}_{\theta,q,\phi}])} & K_0(A_\theta) & \longrightarrow & K_0(\mathcal{O}_{qA_\theta,\phi}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_{qA_\theta,\phi}) & \longleftarrow & K_1(A_\theta) & \xleftarrow[K_1(i)-K_1([\mathcal{E}_{\theta,q,\phi}])]{} & K_1(I_{\mathcal{E}_{q,\phi}}). \end{array}$$

Since $\mathcal{E}_{\theta,q,\phi}$ is an equivalence bimodule, $K_*(i_{K_{A_\theta}(\mathcal{E}_{\theta,q,\phi})})$ is an isomorphism and $I_{\mathcal{E}_{\theta,q,\phi}} = A_\theta$. Therefore $K_*([\mathcal{E}_{\theta,q,\phi}]) \in GL(2,\mathbb{Z})$ because $K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbb{Z}^2$ and $K_*(\phi)$ is an isomorphism. For any $g \in GL(2,\mathbb{Z})$, there exists an automorphism α of A_θ such that $K_1(\alpha) = g$ [5]. Hence $K_1([\mathcal{E}_{\theta,q,\phi}])$ depends on ϕ and can be any element in $GL(2,\mathbb{Z})$. Let $B_{\theta,q,\phi} = K_1(i) - K_1([\mathcal{E}_{\theta,q,\phi}])$. We shall consider $K_0([\mathcal{E}_{\theta,q,\phi}])$.

Proposition 4.1. *Let $\mathcal{E}_{\theta,q,\phi}$ be an A_θ-A_θ-equivalence bimodule such that $\tau_\theta(q) = c\theta + d$ for $c, d \in \mathbb{Z}$ and $0 < \theta < 1$. Then there exist integers a and b such that $\frac{a\theta+b}{c\theta+d} = \theta$ and $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ for some basis of $\mathbb{Z}^2$.*

Proof. We can choose $\langle [1], [p] \rangle$, where $\tau_\theta(p) = \theta$ as a basis of $K_0(A_\theta)$ (see [18] and [19]). Since $K_0(i_{A_\theta})$ is an isomorphism, $\langle [i_{A_\theta}(1)], [i_{A_\theta}(p)] \rangle$ is a basis of $K_0(D_{\mathcal{E}_{\theta,q,\phi}})$. Easy computations show that $Tr_D(i_{K_{A_\theta}(\mathcal{E}_{\theta,q,\phi})}(\phi(1))) = \frac{\tau_\theta(q)}{1+\tau_\theta(q)}$, $Tr_D(i_{K_{A_\theta}(\mathcal{E}_{\theta,q,\phi})}(\phi(p))) = \frac{\theta\tau_\theta(q)}{1+\tau_\theta(q)}$, $Tr_D(i_{A_\theta}(1)) = \frac{1}{1+\tau_\theta(q)}$ and $Tr_D(i_{A_\theta}(p)) = \frac{\theta}{1+\tau_\theta(q)}$. We denote by $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ the matrix $K_0([\mathcal{E}_{\theta,q,\phi}])$ with respect to the basis $\langle [i_{A_\theta}(1)], [i_{A_\theta}(p)] \rangle$. Then we have $\frac{\tau_\theta(q)}{1+\tau_\theta(q)} = \frac{x_{11}+x_{21}\theta}{1+\tau_\theta(q)}$ and $\frac{\theta\tau_\theta(q)}{1+\tau_\theta(q)} = \frac{x_{12}+x_{22}\theta}{1+\tau_\theta(q)}$. By the discussion in Section 2, there exist integers a and b such that $\frac{a\theta+b}{c\theta+d} = \theta$. Therefore we have $x_{11} = d$, $x_{12} = b$, $x_{21} = c$ and $x_{22} = a$. □

By the proposition above, we shall show some examples.

Example 4.2. Let $\theta = \frac{-1+\sqrt{5}}{2}$ and q be a projection in A_θ such that $\tau_\theta(q) = \frac{-1+\sqrt{5}}{2} = \theta$. Then $\frac{-\theta+1}{\theta} = \theta$ and $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. By the exact sequence, the K-groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}).$$

Example 4.3. Let $\theta = \frac{5+\sqrt{5}}{10}$ and q be a projection in A_θ such that $\tau_\theta(q) = \frac{-1+\sqrt{5}}{2} = 5\theta - 3$. Then $\frac{2\theta-1}{5\theta-3} = \theta$ and $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$. By the exact sequence, the K-groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}).$$

Example 4.4. Let $\theta = \frac{-1+\sqrt{5}}{2}$ and q be a projection in A_θ such that $\tau_\theta(q) = \sqrt{5} - 2 = 2\theta - 1$. Then $\frac{-3\theta+2}{2\theta-1} = \theta$ and $K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$. By the exact sequence, the K-groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}).$$

We shall consider the condition of $\theta_1, \theta_2, q_1, q_2$ that $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$ is Morita equivalent to $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$. The following proposition is natural.

Proposition 4.5. *Let $\mathcal{E}_{\theta_1,q_1,\phi_1}$ be an A_{θ_1}-A_{θ_1}-equivalence bimodule and $\mathcal{E}_{\theta_2,q_2,\phi_2}$ an A_{θ_2}-A_{θ_2}-equivalence bimodule such that A_{θ_1} is Morita equivalent to A_{θ_2} and $\tau_{\theta_1}(q_1) = \tau_{\theta_2}(q_2)$. Assume that $\mathbb{Z}^2/\mathrm{Im}(B_{\theta_1,q_1,\phi_1})$ is isomorphic to $\mathbb{Z}^2/\mathrm{Im}(B_{\theta_2,q_2,\phi_2})$. Then $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$ is Morita equivalent to $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$.*

Proof. We may assume that $0 < \theta_1, \theta_2 < 1$. Since A_{θ_1} is Morita equivalent to A_{θ_2}, there exists $g \in GL(2,\mathbb{Z})$ such that $\theta_1 = g\theta_2$ by Theorem 4 in [19]. There exist $c_1, c_2, d_1, d_2 \in \mathbb{Z}$ such that $\tau_{\theta_1}(q_1) = \tau_{\theta_2}(q_2) = c_1\theta_1 + d_1 = c_2\theta_2 + d_2$. The discussion in Section 2 shows that there exist $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ such that $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\theta_1 = \theta_1$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\theta_2 = \theta_2$. By the computation, we have $g\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}g^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Hence $\mathrm{Ker}(K_0(i) - K_0([\mathcal{E}_{\theta_1,q_1,\phi_1}]))$ is isomorphic to $\mathrm{Ker}(K_0(i) - K_0([\mathcal{E}_{\theta_2,q_2,\phi_2}]))$, and $\mathrm{Im}(K_0(i) - K_0([\mathcal{E}_{\theta_1,q_1,\phi_1}]))$ is isomorphic to $\mathrm{Im}(K_0(i) - K_0([\mathcal{E}_{\theta_2,q_2,\phi_2}]))$ by Proposition 4.1. Therefore the K-groups of $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$ are isomorphic to the K-groups of $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$. Consequently $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$ is Morita equivalent to $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$ by Theorem 3.1 and the classification theorem by Kirchberg-Phillips [9], [16]. □

We shall show that the K_0-group of $\mathcal{O}_{qA_\theta,\phi}$ is related to Pell's equation. We shall review some facts of elementary number theory. Let θ be a quadratic irrational number. We may assume that θ satisfies $k\theta^2 + l\theta + m = 0$ with a natural number k and integers l, m such that $\gcd(k, l, m) = 1$. The equation is uniquely determined. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element in $GL(2,\mathbb{Z})$ such that $\frac{a\theta+b}{c\theta+d} = \theta$. Then the $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{t+lu}{2} & ku \\ -mu & \frac{t-lu}{2} \end{pmatrix},$$

where t, u are integers such that

$$t^2 - D_\theta u^2 = 4 \quad \text{if } ad - bc = 1,$$
$$t^2 - D_\theta u^2 = -4 \quad \text{if } ad - bc = -1.$$

The equation above is called Pell's equation. We see that $c\theta + d = \frac{t+u\sqrt{D_\theta}}{2}$. Especially, if $t, u > 0$ are minimum integers satisfying one of the equations above, then $\frac{t+u\sqrt{D_\theta}}{2}$ is the fundamental unit. We shall determine the K-groups of $\mathcal{O}_{qA_\theta,\phi}$.

Theorem 4.6. *Let $\mathcal{E}_{\theta,q,\phi}$ be an A_θ-A_θ-equivalence bimodule such that $\tau_\theta(q) = \frac{t+u\sqrt{D_\theta}}{2}$, where D_θ is the discriminant of θ. Then $B_{\theta,q,\phi}$ can be any element in $\{1-g; g \in GL(2,\mathbb{Z})\}$ by the choice of ϕ and the K-groups have the following form:*
(1) If $t^2 - D_\theta u^2 = 4$, then

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/\frac{2-t}{s_1}\mathbb{Z} \oplus \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}),$$

where s_1 is a maximal integer such that $2-t = es_1^2$ and $u = fs_1$ for some $e, f \in \mathbb{Z}$.
(2) If $t^2 - D_\theta u^2 = -4$, then

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}/s_2\mathbb{Z} \oplus \mathbb{Z}/\frac{t}{s_2}\mathbb{Z} \oplus \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}),$$

where s_2 is a maximal integer such that $t = es_2^2$ and $u = fs_2$ for some $e, f \in \mathbb{Z}$.

Proof. We may assume $0 < \theta < 1$. Let $\theta' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\theta$ and q' be a projection in $A_{\theta'}$ such that $\tau_{\theta'}(q') = \tau_\theta(q)$. Then there exists $g \in GL(2,\mathbb{Z})$ such that $K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}]) = g^{-1}(K_0(i) - K_0([\mathcal{E}_{\theta',q',\phi}]))g$ by Proposition 4.5. It is easy to see that $m(\theta')^2 + (l+m)\theta' + k + l + m = 0$ and $0 < \theta' < 1$.

Proof of (1). The equation $t^2 - D_\theta u^2 = 4$ gives $e(2+t) = -f^2 D_\theta$. It is easy to see that es_1 is an odd number if and only if t is an odd number, and D_θ is an odd number if and only if l is an odd number. Hence we see that $\frac{es_1 - lf}{2}$ and $\frac{es_1+lf}{2}$ are integers by elementary computations. Therefore there exist integers $x_{11}, x_{12}, x_{21}, x_{22}$ such that

$$K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}]) = \begin{pmatrix} 1 - \frac{t-lu}{2} & -ku \\ mu & 1 - \frac{t+lu}{2} \end{pmatrix} = s_1 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

The greatest common divisor of the matrix elements of $K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}])$ is equal to the greatest common divisor of the matrix elements of $K_0(i) - K_0([\mathcal{E}_{\theta',q',\phi}])$ because $K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}]) = g^{-1}(K_0(i) - K_0([\mathcal{E}_{\theta',q',\phi}]))g$. Since we have

$$K_0(i) - K_0([\mathcal{E}_{\theta',q',\phi}]) = \begin{pmatrix} 1 - \frac{t-(l+m)u}{2} & -mu \\ (k+l+m)u & 1 - \frac{t+(l+m)u}{2} \end{pmatrix}$$

and $\gcd(k, m, k+l+m) = 1$, we see that $\gcd(x_{11}, x_{12}, x_{21}, x_{22}) = 1$. An easy computation shows that $\det(K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}])) = 2 - t$. Hence there exist $g_1, g_2 \in GL(2, \mathbb{Z})$ such that $\mathrm{Im}(g_1(K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}]))g_2) = \mathrm{Im}(\begin{pmatrix} s_1 & 0 \\ 0 & \frac{2-t}{s_1} \end{pmatrix})$. By the exact sequence, we obtain the conclusion. □

Proof of (2). By an easy computation, $\det(K_0(i) - K_0([\mathcal{E}_{\theta,q,\phi}])) = -t$. The rest is proved similarly as above. □

We shall show some examples.

Example 4.7. Let $\theta = \frac{-1+\sqrt{5}}{2}$ and q be a projection in A_θ such that $\tau_\theta(q) = \frac{3-\sqrt{5}}{2}$. Then the K-groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}).$$

Example 4.8. Let $\theta = \sqrt{5} - 2$ and q be a projection in A_θ such that $\tau_\theta(q) = \sqrt{5} - 2 = \frac{-4+\sqrt{20}}{2}$. Then the K-groups have the following form:

$$K_0(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \ker(B_{\theta,q,\phi}),\ K_1(\mathcal{O}_{qA_\theta,\phi}) \cong \mathbb{Z}^2/\mathrm{Im}(B_{\theta,q,\phi}).$$

Therefore $\mathcal{O}_{qA_\theta,\phi}$ is not Morita equivalent to the C^*-algebra in Example 4.4.

The following corollary is an extension of Proposition 4.5.

Corollary 4.9. *Let $\mathcal{E}_{\theta_1,q_1,\phi_1}$ be an A_{θ_1}-A_{θ_1}-equivalence bimodule and $\mathcal{E}_{\theta_2,q_2,\phi_2}$ an A_{θ_2}-A_{θ_2}-equivalence bimodule such that $D_{\theta_1} = D_{\theta_2}$ and $\tau_\theta(q_1) = \tau_\theta(q_2)$. Assume that $\mathbb{Z}^2/\mathrm{Im}(B_{\theta_1,q_1,\phi_1})$ is isomorphic to $\mathbb{Z}^2/\mathrm{Im}(B_{\theta_2,q_2,\phi_2})$. Then $\mathcal{O}_{q_1A_{\theta_1},\phi_1}$ is Morita equivalent to $\mathcal{O}_{q_2A_{\theta_2},\phi_2}$.*

Remark 4.10. Let $\theta_1 = \sqrt{10} - 3$ and $\theta_2 = \frac{2+\sqrt{10}}{3}$. Then $D_{\theta_1} = D_{\theta_2} = 40$ and A_{θ_1} is not Morita equivalent to A_{θ_2}. (This fact is related to the fact that the class number of $\mathbb{Q}(\sqrt{10})$ is two.) Therefore Corollary 4.9 is an extension of Proposition 4.5. Since there exists no relation between A_{θ_1} and A_{θ_2}, this result is very surprising.

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