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3-MANIFOLDS WITH POSITIVE FLAT CONFORMAL STRUCTURE

REIKO AIYAMA AND KAZUO AKUTAGAWA

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ABSTRACT. In this paper, we consider a closed 3-manifold M with flat conformal structure C. We will prove that if the Yamabe constant of (M, C) is positive, then (M, C) is Kleinian.

1. INTRODUCTION AND MAIN THEOREM

In 1988, Schoen and Yau [19] gave a final resolution for the *Yamabe Problem* (cf. [3, 15, 18]). In [19, Proposition 3.3], they also proved that *any closed n-manifold with flat conformal structure of positive Yamabe constant is Kleinian, provided that* $n \geq 4$. Moreover, under the assumption that an extended Positive Mass Theorem holds (but a proof has not yet appeared), they showed that the above assertion still holds even when $n = 3$ (see [19, Proposition 4.4′] and the paragraph just before it). On the other hand, there are enormous examples of closed 3-manifolds with flat conformal structures which are not Kleinian (see [8, Remark 7.4]).

The purpose of this brief note is to prove the above assertion for the remaining case $n = 3$.

Theorem 1.1. *Let M be a closed 3-manifold with flat conformal structure C. If its Yamabe constant is positive, then (M, C) is Kleinian.*

This assertion can be obtained by an argument in the proof of [1, the second assertion of Theorem 1.4], which is a combination of a result [19, Proposition 4.2], a positive mass theorem [1, the first assertion of Theorem 1.4] (different from the one Schoen and Yau mentioned in [19]) and a classification of 3-manifolds with positive scalar curvature [7, 10, 11]. Here, we will explicitly give a proof of it (see also Remark 2.2 below).

The remaining sections are organized as follows. Section 2 contains some necessary definitions and preliminary geometric results. Section 3 is devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

Let M be a closed 3-manifold, that is, a compact 3-manifold without boundary. To simplify the presentation and the argument, we always assume that dim $M = 3$

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throughout this paper. For each conformal class C on M, the *Yamabe constant* $Y(M,C)$ of (M,C) is defined by

$$Y(M,C) := \inf_{g\in C} E(g), \qquad E(g) := \frac{\int_M R_g d\mu_g}{\mathrm{Vol}_g(M)^{1/3}},$$

where R_g, μ_g and $\mathrm{Vol}_g(M)$ denote respectively the scalar curvature, the volume element of g and the volume of (M,g). It is a finite-valued conformal invariant of C. The Yamabe constant $Y(M,C)$ is positive if and only if there exists a positive scalar curvature metric $g \in C$ (cf. [3]). A remarkable theorem [22, 20, 2, 17, 19] of Yamabe, Trudinger, Aubin and Schoen asserts that each conformal class C contains a minimizer $\check{g}$ of $E|_C$, called a *Yamabe metric* (or a *solution of the Yamabe Problem*), which is of constant scalar curvature

$$R_{\check{g}} = Y(M,C)\cdot \mathrm{Vol}_{\check{g}}(M)^{-2/3}.$$

Let M_∞ be an infinite covering of M. We shall say that the fundamental group $\pi_1(M)$ of M has a *descending chain of finite index subgroups tending to* $\pi_1(M_\infty)$ if it satisfies the following: There exists a family of subgroups $\{\Gamma_i\}_{i\geq 1}$ of $\pi_1(M)$ such that

(i) each Γ_i is of finite index in $\pi_1(M)$ with $\Gamma_i \supset \pi_1(M_\infty)$,
(ii) $\pi_1(M) = \Gamma_1 \supsetneqq \Gamma_2 \supsetneqq \cdots \supsetneqq \Gamma_i \supsetneqq \Gamma_{i+1} \supsetneqq \cdots$,
(iii) $\bigcap_{i=1}^{\infty} \Gamma_i = \pi_1(M_\infty)$.

Assume that $Y(M,C) > 0$. Take a positive scalar curvature metric $g \in C$ and any point $p \in M$. Then, there exists the *normalized Green's function* G_p for L_g with a pole at p, that is,

$$L_g G_p = c_0 \cdot \delta_p \quad \text{on} \quad M \quad \text{and} \quad \lim_{q\to p} \mathrm{dist}(q,p) G_p(q) = 1.$$

Here, $L_g := -8\Delta_g + R_g, c_0 > 0$ and δ_p stand respectively for the *conformal Laplacian*, a specific universal positive constant and the Dirac δ-function at p. Assume also that the covering $P_\infty : M_\infty \to M$ is normal. Let g_∞ denote the lift of g to M_∞, and p_∞ a point in M_∞ with $P_\infty(p_\infty) = p$. Then, there exists uniquely also a *normalized minimal positive Green's function* G_∞ on M_∞ for $L_{g_\infty} := -8\Delta_{g_\infty} + R_{g_\infty}$ with pole at p_∞ (cf. [19]), which satisfies the following:

$$(P_\infty)^* G_p = \sum_{\gamma\in\mathcal{G}} G_\infty \circ \gamma \quad \text{on} \quad M_\infty.$$

Here, $\mathcal{G}$ stands for the group of deck transformations for the normal covering $M_\infty \to M$. Set

$$g_{\infty,AF} := G_\infty^4 \cdot g_\infty \quad \text{on} \quad M_\infty^* := M_\infty - \{p_\infty\}.$$

Then, $g_{\infty,AF}$ defines a scalar-flat, asymptotically flat metric on M_∞^* (cf. [15]). Note that this asymptotically flat 3-manifold $(M_\infty^*, g_{\infty,AF})$ has *infinitely many* singularities created by the ends of M_∞^*. However, the mass $\mathfrak{m}_{ADM}(g_{\infty,AF})$ of $(M_\infty^*, g_{\infty,AF})$ can be defined in the usual way (cf. [4]). Note also that the positive mass theorem for asymptotically flat 3-manifolds with singularities does not always hold (see [1, Remark 1.5-(2)] for instance).

Once this is understood, the following positive mass theorem holds as a special case of [1, the first assertion of Theorem 1.4]:

Proposition 2.1. *Let (M, C) be a closed 3-manifold with $Y(M, C) > 0$. Let (M_∞, g_∞) be a normal infinite Riemannian covering of (M, g) such that $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$, where $g \in C$ is a positive scalar curvature metric and g_∞ is its lift to M_∞. For any point $p_\infty \in M_\infty$, let G_∞ denote the normalized minimal positive Green's function on M^*_∞ with pole at p_∞. Then, the asymptotically flat 3-manifold $(M^*_\infty, g_{\infty,AF})$ has nonnegative mass*

$$\mathfrak{m}_{ADM}(g_{\infty,AF}) \geq 0.$$

Remark 2.2. Assume that $M = \#\ell(S^1 \times S^2)$ for $\ell \geq 2$ and M_∞ is its universal covering. Note that M_∞ is spin. For each small $\sigma > 0$, consider the complete metric $g_{\sigma,AF} := (G_\infty + \sigma)^4 \cdot g_\infty$ with $R_{g_{\sigma,AF}} \geq 0$ on M^*_∞ (cf. [19, Proposition 4.4′]). Then, only one end of $(M^*_\infty, g_{\sigma,AF})$ is asymptotically flat and the other infinitely many ends are merely complete. Gilles Carron and the referee kindly pointed out that Witten's approach [21] (cf. [16]) to the Positive Mass Theorem is still valid for the family $\{(M^*_\infty, g_{\sigma,AF})\}_{0<\sigma<1}$. It implies that a more general positive mass theorem than Proposition 2.1 is a *folk theorem* for experts in this field, and Theorem 1.1 is too. But Proposition 2.1 itself is a complete form, and hence, by using it, we will give here an explicit and self-contained proof of Theorem 1.1.

A conformal 3-manifold (M, C) is said to be *locally conformally flat* if, for any point $p \in M$, there exists a metric $\overline{g} \in C$ such that $\overline{g}$ is flat on some neighborhood of p. A conformal class C on M is called a *flat conformal structure* if (M, C) is locally conformally flat. In [14], Kuiper proved that, for a simply connected locally conformally flat 3-manifold (X, C'), there is a conformal immersion into (S^3, C_0) called a *developing map*, which is unique up to composition with a Möbius transformation of (S^3, C_0). Therefore, the universal covering of a locally conformally flat manifold (M, C) admits a developing map. Here, (S^3, C_0) denotes the 3-sphere S^3 with the conformal class $C_0 := [g_0]$ of the standard metric g_0 of constant curvature one. (M, C) is called *Kleinian* if (M, C) is conformal to Ω/Γ for some open set Ω of S^3 and some discrete subgroup Γ of the conformal transformation group $\mathrm{Conf}(S^3, C_0)$, which leaves Ω invariant and acts freely and properly discontinuously on Ω. Note that, if the developing map of the universal covering of a locally conformally flat manifold (M, C) is injective, then (M, C) is Kleinian.

With this understanding, the following criterion also holds as a special case of [19, Proposition 4.2]:

Proposition 2.3. *Let (M, C) be a closed 3-manifold with $Y(M, C) > 0$, and $(\widetilde{M}, \widetilde{g})$ be the universal Riemannian covering of (M, g), where $g \in C$ is a positive scalar curvature metric. For any point $\widetilde{p} \in \widetilde{M}$, let $\widetilde{G}$ denote the normalized minimal positive Green's function on $\widetilde{M}$ for $L_{\widetilde{g}}$ with pole at $\widetilde{p}$, and $(\widetilde{M} - \{\widetilde{p}\}, \widetilde{g}_{AF} = \widetilde{G}^4 \cdot \widetilde{g})$ the asymptotically flat 3-manifold as above. If the mass $\mathfrak{m}_{ADM}(\widetilde{g}_{AF})$ is nonnegative, then the developing map of $(\widetilde{M}, [\widetilde{g}])$ is injective. In particular, (M, C) is Kleinian.*

Remark 2.4. We remark that the mass $\mathfrak{m}_{ADM}(\widetilde{g}_{AF})$ is equal to the ADM energy E of $(\widetilde{M} - \{\widetilde{p}\}, \widetilde{g}_{AF})$ appearing in [19, page 64] up to a positive constant.

3. Proof of main theorem

Proof of Theorem 1.1. Consider the universal covering $\widetilde{M}$ of M and denote the lift of the flat conformal structure C by $\widetilde{C}$. If $|\pi_1(M)| < \infty$, then $(\widetilde{M}, \widetilde{C})$ is conformal to (S^3, C_0) by Kuiper's Theorem [14]. Hence, (M, C) is Kleinian. From now on, we assume that $|\pi_1(M)| = \infty$, that is, the degree of the covering map $P : \widetilde{M} \to M$ is infinite.

Take a unit-volume Yamabe metric $g \in C$, and consider its lift $\widetilde{g} \in \widetilde{C}$ to $\widetilde{M}$. Note that $R_{\widetilde{g}} = R_g = Y(M, C) > 0$. Take any base points $p \in M, \widetilde{p} \in \widetilde{M}$ satisfying $P(\widetilde{p}) = p$, and fix them. Then, let $\widetilde{G}$ denote the normalized minimal positive Green function on $\widetilde{M}$ for $L_{\widetilde{g}}$ with pole at $\widetilde{p}$, and the mass $\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF})$ of the asymptotically flat 3-manifold $(\widetilde{M} - \{\widetilde{p}\}, \widetilde{g}_{AF} := \widetilde{G}^4 \cdot \widetilde{g})$.

Suppose that

$$\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF}) \geq 0.$$

Recall that we can choose the base point $\widetilde{p} \in \widetilde{M}$ arbitrarily. It then follows from Proposition 2.3 that the developing map of $(\widetilde{M}, \widetilde{C})$ is injective, and hence (M, C) is Kleinian. In this case, especially $\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF}) = 0$. Therefore, it is enough to show that $\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF}) \geq 0$.

By combining [7, Theorem 8.1] (cf. [9]) with $Y(M, C) > 0$ (replacing M by its orientable double covering if necessary), M can be decomposed uniquely into *prime* closed 3-manifolds

$$M = N_1 \# \cdots \# N_{\ell_1} \# \ell_2 (S^1 \times S^2),$$

where $\pi_1(N_i)$ is finite for $i = 1, \ldots, \ell_1$ and ℓ_1, ℓ_2 are nonnegative integers. By applying the C-prime decomposition theorem for closed 3-manifolds with flat conformal structures [10, 11] to (M, C), there exists a flat conformal structure C_i on each N_i $(i = 1, \ldots, \ell_1)$. Then, Kuiper's Theorem [14] again implies that each (N_i, C_i) is a nontrivial quotient of (S^3, C_0). After taking an appropriate finite covering M' of M, we have

$$M' = \# \ell (S^1 \times S^2) \quad \text{for some} \quad \ell \geq 1.$$

Recall that $\widetilde{M}$ is the infinite universal covering of M. Then, there exists (uniquely) an infinite universal covering $\widetilde{M} \to M'$. Moreover, since $\pi_1(M')$ is a finitely generated free group, it has a descending chain of finite index subgroups tending to $\pi_1(\widetilde{M}) = \{e\}$. Let g' be the lifting of g to M'. Applying Proposition 2.1 to the normal infinite Riemannian covering $(\widetilde{M}, \widetilde{g}) \to (M', g')$, we have that

$$\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF}) \geq 0.$$

This completes the proof of Theorem 1.1. □

Remark 3.1. Even if we replace the positivity $Y(M, C) > 0$ in Theorem 1.1 by the nonnegativity $Y(M, C) \geq 0$, it seems that the same conclusion still holds. More precisely, we propose the following (cf. [5, 13]).

Conjecture. *Let M be a closed 3-manifold with flat conformal structure C. If its Yamabe constant is zero, then either (1) or (2) holds*:

(1) *There exists a flat metric $\overline{g} \in C$.*

(2) *There exists a smooth family $\{g_t\}_{0 \leq t \leq 1}$ of locally conformally flat metrics on M such that $g_0 \in C$ and $Y(M, [g_1]) > 0$ (possibly $Y(M, [g_t]) < 0$ for some $t \in (0, 1)$).*

In the case (1), the universal covering $(\widetilde{M}, \widetilde{C})$ of (M, C) is conformal to $(S^3 - \{p_N\}, C_0)$ where $p_N := (1, 0, 0, 0) \in S^3$, and hence (M, C) is Kleinian. In the case (2), Theorem 1.1 implies that $(M, [g_1])$ is Kleinian. The argument in the proof of

Theorem 1.1 also implies that there exists a torsion free subgroup Γ of finite index in $\pi_1(M)$ such that Γ is either a trivial group or a nontrivial finitely generated free group. Then, the *virtual cohomological dimension* vcd $\pi_1(M)$ of $\pi_1(M)$ is either 0 or 1 (see [6]). Therefore, $(M, [g_1])$ is a closed Kleinian 3-manifold with vcd $\pi_1(M) < 3$. The quasiconformal stability of Kleinian groups [12, Theorem 2] implies that any flat conformal structure on M which is a smooth deformation of $[g_1]$ is also Kleinian; in particular C is too.

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Department of Mathematics, University of Tsukuba, Tsukuba 305-8571, Japan
E-mail address: aiyama@math.tsukuba.ac.jp

Division of Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan
E-mail address: akutagawa@math.is.tohoku.ac.jp