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A NOTE ON SOME CLASSICAL RESULTS OF GROMOV-LAWSON

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Abstract. In this paper we show how the higher index theory can be used to prove results concerning the non-existence of a complete Riemannian metric with uniformly positive scalar curvature at infinity. By improving some classical results due to M. Gromov and B. Lawson we show the efficiency of these methods to prove such non-existence theorems.

1. Introduction

Let (M, g) be an oriented complete non-compact manifold partitioned by a compact hypersurface N into two parts M_+ and M_- with $M_+ \cap M_- = \emptyset$ and $\overline{M}_+ \cap \overline{M}_- = N$. We assume that the positive unit normal to N points out from M_- to M_+. Let W be a Clifford bundle on M which is at the same time a Hilbert A-module bundle, where A is a C^*-algebra. We assume that this bundle is equipped with an A-linear connection which is compatible with the Clifford action of TM and denote the corresponding A-linear Dirac type operator by D. As an example let M be a spin manifold with spin bundle S and V be a Hilbert A-module bundle on M with a Hermitian connection. Then the spin Dirac operator twisted by V is an example of such an operator which acts on the smooth sections of $W = S \otimes V$.

Let $U = (D - i)(D + i)^{-1}$ be the Cayley transform of D which is an A-linear bounded operator on $H = L^2(M, W)$. Let ϕ_+ be a smooth function on M which coincides with the characteristic function of M_+ outside a compact set and put $\phi_- := 1 - \phi_+$. It turns out that the operator $U_+ = \phi_- + \phi_+ U$ is A-Fredholm in the sense of Fomenko-Mischenko. The Fomenko-Mischenko index $\mathrm{ind}(U_+)$ does not depend on ϕ_+ but on the cobordism class of the partitioning manifold N. This index is denoted by $\mathrm{ind}(D, N)$. A basic property of this index is the following vanishing theorem, which we have proved in [10, Theorem 2.4]:

If M is spin and $W = S \otimes V$, where V is a flat Hilbert A-module bundle, then $\mathrm{ind}(D, N) = 0 \in K_0(A)$ *provided that the scalar curvature of g is uniformly positive.*

The Clifford action of $i\vec{n}$ provides a $\mathbb{Z}_2$ grading for $W_{|N}$ and makes $W_{|N}$ a graded Clifford bundle on N. Let D_N denote the associated Dirac type operator on N which acts on smooth sections of $W_{|N}$. It is an A-linear elliptic operator and has

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the Fomenko-Mischenko index $\operatorname{ind} D_N \in K_0(A)$. The following equality generalizing a result due to J. Roe and N. Higson [8, 5] is proved in [10]:

$$\operatorname{ind} D_N = \operatorname{ind}(D, N)\ . \tag{1.1}$$

This equality has been used in [10] to prove the following:

If a complete spin manifold (M, g) is partitioned by an enlargeable hypersurface N and if there is a smooth map $\phi : M \to N$ whose restriction to N is of non-zero degree, then the scalar curvature of g cannot be uniformly positive.

In this paper we show that under the above conditions $\operatorname{ind}(D, N) \neq 0$ and conclude that the scalar curvature of g cannot be uniformly positive *even outside a compact subset* of M. Moreover we show that this conclusion is true if the map

$$j_* : \pi_1(N) \to \pi_1(M)$$

which is induced by the inclusion is injective. We will use these stronger results to improve some classical results due to M. Gromov and B. Lawson. Section 2 deals with some analytical aspect of regular operators on Hilbert Modules. Here we construct the wave operator for the spin Dirac operators twisted by flat Hilbert module bundles and prove its unit speed property. In section 3 we use these results to prove the vanishing Theorem 3.1 and apply it to improve some classical results due to M. Gromov and B. Lawson.

Here we give the definition of the enlargeability as it is introduced by Gromov and Lawson in [2].

Definition. Let N be a closed oriented manifold of dimension n with a fixed Riemannian metric g. The manifold N is enlargeable if for each real number $\epsilon > 0$ there is a Riemannian spin cover $(\tilde{N}, \tilde{g})$, with lifted metric, and a smooth map $f : \tilde{N} \to S^n$ such that: the function f is constant outside a compact subset K of $\tilde{N}$, the degree of f is non-zero, and the map $f : (\tilde{N},\ \tilde{g}) \to (S^n, g_0)$ is ϵ-contracting, where g_0 is the standard metric on S^n. Being ϵ-contracting means that $\|T_x f\| \leq \epsilon$ for each $x \in \tilde{N}$, where $T_x f \colon T_x\tilde{N} \to T_{f(x)}S^n$. The manifold N is said to be area-enlargeable if the function f is ϵ-area contracting. This means that $\|\Lambda^2 T_x f\| \leq \epsilon$ for each $x \in \tilde{N}$, where $\Lambda^2 T_x f \colon \Lambda^2 T_x\tilde{N} \to \Lambda^2 T_{f(x)}S^n$.

An enlargeable manifold does not admit Riemannian metrics with positive scalar curvature. The relevance of this theorem will be clear by noticing that enlargeability depends only on the homotopy type of M.

This paper deals mainly with non-compact manifolds. However we give an application to the compact case at the end of the paper by proving the following result, which is proved by M. Gromov and B. Lawson for $\dim M = 4$.

Let M be a closed manifold and N be a closed enlargeable submanifold of codimension 2. If the natural map $\pi_1(N) \to \pi_1(M)$ is injective and $|\pi_1(N)/\pi_1(M)| = \infty$, then M does not admit a metric with everywhere positive scalar curvature.

2. Functional calculus of the Dirac operator

Let $\mathbb{H}$ be a Hilbert A-module with A a C^*-algebra and let T be an A-linear map which is defined on a dense subspace $\operatorname{Dom}(T)$ of $\mathbb{H}$. The graph of T is the following subset of $\mathbb{H} \oplus \mathbb{H}$:

$$graph(T) := \{(u, T(u)) | u \in \operatorname{Dom}(T)\}.$$

The closure of this graph with respect to the norm topology in $\mathbb{H}\oplus\mathbb{H}$ is the graph of an operator $\bar{T}$ which is called the closure of T. The domain $\operatorname{Dom}(\bar{T})$ of this closure is a closed A-subspace of $\mathbb{H}$. The adjoint of T is the closed operator T^* such that $\langle Tu, v\rangle = \langle u, T^*v\rangle$ for $u \in \mathrm{Dom}(T)$ and $v \in \mathrm{Dom}(T^*)$. Let $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$. Following [6], T is selfadjoint if $T^* = T$ and it is normal if

$$\langle Tu, Tv\rangle = \langle T^*u, T^*v\rangle, \text{ for } u, v \in \mathrm{Dom}(T).$$

The operator T is called regular if there is a bounded adjointable operator $P \in \mathcal{L}_A(\mathbb{H} \oplus \mathbb{H}, \mathbb{H})$ with $\mathrm{Im}\, P = \mathrm{Dom}\, \bar{T}$ (cf. Proposition 5 of [6]). The fact that T is regular and selfadjoint makes it possible to associate to each continuous (not necessarily bounded) function f on $\mathrm{spec}(T)$ a closed A-linear operator $f(T)$ on $\mathbb{H}$. This correspondence defines a continuous functional calculus for T. This construction has been worked out in [1]. Here we follow the more geometric approach of [6]. The key observation of [6] is that the transformation $T \to Q(T) = T(1 + T^*T)^{-1/2}$ provides a $*$-preserving bijection between the set of all normal regular A-linear operators $\mathcal{R}_A(\mathbb{H})$ and the following set:

$$\mathcal{V}(\mathbb{H}) = \{Q \in \mathrm{End}_A(\mathbb{H}) | \|Q\| \leq 1 \text{ and } \mathrm{Im}(1 - Q^*Q) \text{ and } \mathrm{Im}(1 - QQ^*) \text{ are dense}\}.$$

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$. Each function $g \in C_0(\mathbb{C})$ determines a function $\bar{g} \in C_0(\mathbb{D})$ by the following relation:

$$\bar{g}(z(1 + |z|^2)^{-1/2}) = g(z)(1 + |g(z)|^2)^{-1/2}.$$

Clearly $\|\bar{g}\| \leq 1$, so the bounded operator $\bar{g}(Q(T))$ belongs again to $\mathcal{V}_A(\mathbb{H})$ and corresponds to a unique operator in $\mathcal{R}_A(\mathbb{H})$ which is defined to be $g(T)$. From this construction it is clear that if $g = g'$ on a closed subset containing the spectrum $\mathrm{spec}(T)$, then $g(T) = g'(T)$. It is clear also that the corresponding $g \to g(T)$ provides a C^*-representation $\phi : C_0(X) \to \mathrm{End}_A(\mathbb{H})$.

The regularity of T implies that $C_0(X)(T)\mathbb{H} = \mathbb{H}$ (see Corollary 14 and Theorem 15 of [6]), so given any $u \in \mathbb{H}$ there is $g \in C_0(X)$ and $v \in \mathbb{H}$ with $u = g(T)v$. If h is a bounded continuous function on X, then $hg \in C_0(X)$ and one defines $h(T)u := (hg)(T)v$. It is easy to verify that $h(T)$ is well defined and that it is bounded with $\|h(T)\| \leq \|h\|$, where $\|h\| = \sup\{h(\lambda) | \lambda \in \mathrm{spec}(T)\}$.

For an unbounded continuous function h, the set

$$\mathcal{C}(h) := \{g \in C_0(X) | hg \in C_0(X)\}$$

is a $*$-subalgebra of $C_0(X)$. The above argument can be used to define the unbounded operator $h(T)$ with domain $\phi(\mathcal{C}(h))(\mathbb{H})$. It is clear that $\mathrm{Dom}\, f(T) \subset \mathrm{Dom}\, g(T)$ if $|g| \leq |f|$ on $\mathrm{spec}(T)$. As an example, if $g(z) = z$, then it turns out that $g(T) = T$. The following theorem summarizes some properties of this functional calculus which are relevant to our purposes. These properties are straightforward consequences of the above discussion.

Theorem 2.1. *Let T be a densely defined regular normal A-linear operator on a Hilbert A-module $\mathbb{H}$ and let $X \subset \mathbb{C}$ be a closed subset containing the spectrum* $\mathrm{spec}(T)$. *To each continuous function $f \in C(X)$ one can correspond the regular normal operator $f(T)$ on $\mathbb{H}$ satisfying $f(T)^* = \overline{f(T^*)}$ such that:*

1. *For f, g in $C(X)$ the operator $(f + g)(T)$ is the closure of $f(T) + g(T)$, $(fg)(T)$ is the closure of $f(T)g(T)$ and $f \circ g(T) = f(g(T))$.*
2. *If T is bounded, then $f \to f(T)$ coincides with the functional calculus in the C^*-algebra of bounded operators on $\mathbb{H}$.*

(3) *Let $\{f_k\}_k$ be a sequence of continuous functions on X which is dominated be a continuous function F, i.e. $|f_k(z)| \leq |F(z)|$ for all $z \in X$. If $f_k \to f$ uniformly on compact subsets of X, then $f_k(T)(u) \to f(T)(u)$ for $u \in \mathrm{Dom}(F)$. If the f_k are uniformly bounded and the convergence to f is uniform, then $f_k(T) \to f(T)$ in the norm topology.*

As a special case of this theorem, if T is selfadjoint, then $f(t,T) = e^{itT}$, for $t \in \mathbb{R}$, is a one-parameter family of unitary operators on $\mathbb{H}$ satisfying $e^{i(t+s)T} = e^{itT}e^{isT}$. Moreover $Te^{itT} = e^{itT}T$, which implies that e^{itT} provides a unitary bijection between $\mathrm{Dom}\, T$ and $\mathrm{Im}\, T$. By the third part of the previous theorem, for $u \in \mathrm{Dom}(T)$ we have

$$\left(\frac{d}{dt}\right)_{t=0} e^{itT}u = \lim_{t\to 0}\left(\frac{e^{itx}-1}{t}(T)\right)(u) = iT(u).$$

So, for a selfadjoint regular operator T, the t-parameterized family of unitary operators e^{itT} satisfies the wave equation and the initial condition

$$(\frac{d}{dt} - iT)e^{itT}(u) = 0 \quad \text{for } u \in \mathrm{Dom}(T), \tag{2.1}$$

$$\lim_{t\to 0} e^{itT}u = u. \tag{2.2}$$

The wave equation (2.1) implies the following vanishing result for $u \in \mathrm{Dom}(T)$:

$$\begin{aligned}\frac{d}{dt}\langle e^{itT}(u), e^{itT}(u)\rangle &= \langle iTe^{itT}(u), e^{itT}(u)\rangle + \langle e^{itT}(u), iTe^{itT}(u)\rangle \\ &= 0.\end{aligned}$$

This conservation law and initial condition (2.2) prove the uniqueness of the wave operator with properties (2.1) and (2.2). If f is a smooth function in the Schwartz space $\mathcal{S}(\mathbb{R})$, then the Fourier transform $\hat{f}$ is in $\mathcal{S}(\mathbb{R})$ and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{isx}\, ds.$$

By applying the above theorem we get the following formula:

$$f(T)(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{isT}(u)\, ds\ , \quad \text{for } u \in \mathrm{Dom}(T). \tag{2.3}$$

The Hilbert module that we shall study in the sequel is the space of the L^2-sections of a Hilbert A-module bundle over a complete manifold M. Let (M, g) be a complete Riemannian manifold and let W be a Clifford Hilbert A-modules bundle over M, where A is a complex C^*-algebra. For σ and η two compactly supported smooth sections of W put

$$\langle \sigma, \eta\rangle = \int_M \langle \sigma(x), \eta(x)\rangle\, d\mu_g(x) \in A.$$

It is easy to show that $|\sigma| = \|\langle\sigma,\sigma\rangle\|^{1/2}$ is a norm on $C_c(M, W)$. The completion of $C_c^\infty(M, W)$ with respect to this norm is the Hilbert A-module $\mathbb{H} = L^2(M, W)$. Let D be an A-linear Dirac type operator acting on the compactly supported smooth sections of W which form a dense subspace of $\mathbb{H}$. We recall that D is formally selfadjoint, i.e. $\langle D\sigma, \eta\rangle = \langle\sigma, D\eta\rangle$ for σ and η as in the above. Moreover D is a regular operator (see e.g. [10, Lemma 2.1]), so one can apply Theorem 2.1 to D and define the bounded operator $f(D)$ on $L^2(M, W)$ for each bounded continuous function f on $\mathbb{R}$. In particular we can define the wave operator e^{itD}. In the following lemma we describe a context in which the wave operator has finite propagation speed.

Lemma 2.2. *Let $W = S \otimes V$ be the spin bundle S twisted by the flat Hilbert A-module bundle V. The wave operator e^{itD} has unit propagation speed.*

Proof. To prove the assertion we give another construction for the wave operator which satisfies the unit propagation speed. Then the uniqueness of the wave operator implies the desired assertion. In this proof we denote by V_0 the fiber of V which is a Hilbert A-module. Let $\{U_\alpha, \phi^S_\alpha \otimes \phi^V_\alpha\}_\alpha$ be a trivializing atlas for M such that $W_{|U_\alpha} \simeq U_\alpha \times S \otimes_{\mathbb{C}} V_0$. Since V is flat, we can assume that the transition functions $\phi^V_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{End}_A(V_0)$ are locally constant. Let D' denote the spin Dirac operator. Then $D = D' \otimes \mathrm{Id}_{V_0}$ on smooth sections $\xi = \sum_\alpha \xi^\alpha$ of W where ξ_α is supported in U_α and $\xi^\alpha = s(x) \otimes v$ for a fixed v in V_0. From the unit propagation formula for D', for $|t|$ sufficiently small the section $\xi^\alpha_t(x) := e^{itD'} s(x) \otimes v$ is supported in U_α too. Moreover it is the unique solution of the following wave equation with the given initial condition ξ:

$$(\frac{d}{dt} - iD' \otimes \mathrm{Id}_{V_0})\xi^\alpha_t(x) = 0.$$

Since $\phi^V_{\alpha\beta}$ is constant, the transition of the solution ξ^α_t to another chart is the solution of the wave equation in that chart with an appropriate initial condition. Therefore these local solutions of local wave equations actually paste together to define a global solution ξ_t of the wave equation for D for sufficiently small values of t. We can use ξ_t as the initial condition and repeat the above procedure to define the solution of the wave equation beyond t. This is the way we get a solution which is defined for $t \in \mathbb{R}$. We define $e^{itD}\xi_0$ to be ξ_t. From this construction it is clear that the wave operator e^{itD} has unit propagation speed. $\square$

For what follows rewrite the relation (2.3) with the Dirac operator D:

$$f(D)(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) e^{isD}(u)\, ds\ , \quad \text{for } u \in \mathrm{Dom}(D). \tag{2.4}$$

3. Vanishing theorem and its implications

As mentioned previously, the higher index $\mathrm{ind}(D, N)$ vanishes if the scalar curvature of the underlying Riemannian metric g is uniformly positive on M. In fact this index vanishes even if the scalar curvature is uniformly positive outside a compact subset of M. This is proved in the following theorem.

Theorem 3.1. *With the above notation, if the scalar curvature of g is uniformly positive at infinity and if $W = S \otimes V$, where V is a flat Hilbert A-module bundle, then* $\mathrm{ind}(D, N) = 0 \in K_0(A)$.

Proof. Let U_0 and U_1 be disjoint open subsets of M such that the closure of U_0 is compact and $M = \bar{U}_0 \cup U_1$. Moreover we assume that the scalar curvature κ of g is uniformly positive in $U_1(2r)$, e.g. $\kappa > 4\kappa_0$. Here $U_1(2r)$ consists of all points suited within distance $2r \geq 0$ from U_1 and r is a sufficiently big number that will be determined below. By multiplying the metric g with a sufficiently small positive number we can and will assume that the constant κ_0 is arbitrarily large. Let ϕ_0, ϕ_1 and ϕ_r be respectively the characteristic functions of $\bar{U}_0$, U_1 and $U_1(r)$ in M.

If ϕ is a function on M which is locally constant outside a compact subset, then $[(D+i)^{-1}, \phi]$ is compact; cf. [10, Lemma 2.2]. Therefore

$$\begin{aligned} U_+ &= \operatorname{Id} - 2i\phi_+ \sum_{i,j=0}^{1} \phi_i (D+i)^{-1} \phi_j \\ &\sim \operatorname{Id} - 2i\phi_+ \phi_1 (D+i)^{-1} \phi_r \end{aligned}$$

Here the symbol $\sim$ means "equal up to a compact operator". The function $(x+1)^{-1}$ can be uniformly approximated by compactly supported smooth functions, and hence by smooth functions with compactly supported Fourier transform. Let h be such a function whose Fourier transform $\hat{h}$ is supported in $[-r, r]$ (the r at the beginning of the proof is determined here). If h is sufficiently close to $(x+i)^{-1}$ in sup-norm, then $\operatorname{Id} - 2i\phi_+\phi_1 h(D)\phi_r$, being close to $\operatorname{Id} - 2i\phi_+\phi_1(D+i)^{-1}\phi_r$ in operator norm (cf. Theorem 2.1), is an A-Fredholm operator with the same index. Therefore we need to prove the vanishing of the index of the following operator:

$$\operatorname{Id} - 2i\phi_+\phi_1 h(D)\phi_r \ . \tag{3.1}$$

Let σ be a smooth section of $W = S \otimes V$ supported in $U_1(r)$. Since V is flat, the following generalized Lichnerowicz formula holds with respect to the A-valued L^2-inner product (cf. [9, page 199]):

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4},$$

which implies

$$\begin{aligned} \langle D\sigma, D\sigma \rangle &= \langle D^2\sigma, \sigma \rangle \\ &= \langle \nabla\sigma, \nabla\sigma \rangle + \langle \frac{\kappa}{4}\sigma, \sigma \rangle \\ &\geq \kappa_0 \|\sigma\|^2. \end{aligned}$$

In the last inequality we have used the fact that for a and b in a C^*-algebra, $a+b \geq b$ and $\|a+b\| \geq \|b\|$ provided that a and b are positive and selfadjoint. Consider the restriction of the Dirac operator D to the Hilbert A-module $\mathcal{H} := L^2(U_1(2r), W)$ and denote it by D_{2r}. This is an unbounded operator acting on smooth sections compactly supported in $U_1(2r)$. This operator is symmetric and satisfies the above positivity condition. In fact it has a selfadjoint regular extension to $\mathcal{H}$, as we are going to show. Here we use the notation of the proof of Lemma 2.2. We assume the trivializing charts $(\mathcal{U}_\alpha, \phi_\alpha^S)$ and $(U_\alpha, \phi_\alpha^V)$ for the vector bundles S and V over M such that the transition functions $\phi_{\alpha\beta} = \phi_\alpha^V \circ (\phi_\beta^V)^{-1}$ from $U_\alpha \cap U_\beta$ into $\operatorname{End}_A(V_0)$ are locally constant. Since the twisting bundle V is flat, the Hilbert A-module $\mathcal{H}$ is generated by elements $s \otimes v$, where s is a smooth section of the spin bundle $S \to U_1(2r)$ supported in one of the U_α's and v is a constant element of V_0 (i.e. a flat section of $V_{|U_\alpha}$). On these sections the operator D_{2r} takes the form $D' \otimes \operatorname{Id}$, where D' denotes the spin Dirac operator acting on the smooth sections of S which are compactly supported in $U_1(2r)$. With this domain, D' is a symmetric operator on $L^2(U_1(2r), S)$ satisfying the following positivity relation in $\mathbb{R}$:

$$\langle D's, D's \rangle \geq \kappa_0 \|s\|^2 \ .$$

The Friedrichs' extension theorem provides a selfadjoint extension $\bar{D}'$ of D' to $L^2(U_1(2r), S)$ satisfying still the above positivity condition. We recall two facts from the construction of the Friedrichs' extension. If s is compactly supported in

U_α, then $\bar{D}'s$ is compactly supported in U_α too. Moreover if s belongs to $\mathrm{Dom}(\bar{D}')$, then $\phi\, s \in \mathrm{Dom}(\bar{D}')$ for a smooth compactly supported function ϕ. Now define the operator $\bar{D}_{2r}$ as follows. Its domain consists of a sum of sections $s \otimes v$, where s is supported in U_α and belongs to $\mathrm{Dom}(\bar{D}')$ and v is an element of V. On these sections we define $\bar{D}_{2r}(s \otimes v) = \bar{D}'s \otimes v$. This is a selfadjoint operator on $\mathcal{H}$ satisfying the following relation:

$$\langle \bar{D}_{2r}\sigma, \bar{D}_{2r}\sigma\rangle \geq \kappa_0\|\sigma\|^2 \quad \text{for } \sigma \in \mathrm{Dom}(\bar{D}_{2r}) \subset \mathcal{H}.$$

Since $\bar{D}'$ is selfadjoint, $\mathrm{Im}(\bar{D}' + i) = L^2(U_1(2r), S)$; cf. [7, page 257]. So the above definition shows that $\mathrm{Im}(\bar{D}_{2r}+i) = \mathcal{H}$. Therefore D_{2r}, as an operator on H is selfadjoint and regular. Consequently we can apply the functional calculus of the previous section to define the bounded operator $h(D_{2r})$ and $e^{itD_{2r}}$ on $L^2(U_1(2r), W)$. The point is that the spectrum of D_{2r} is outside of the interval $(-\kappa_0, \kappa_0)$. Since h goes to zero at infinity, if κ_0 is sufficiently large, then $\|h(D_{2r})\|$ is arbitrarily small. Let σ be a smooth section of W supported in U_r. The smooth sections $e^{itD}\sigma$ and $e^{itD_{2r}}\sigma$ both satisfy the same wave equation with the same initial condition provided t is smaller than r. Here we have used the unit speed propagation property of Theorem 2.2. The uniqueness of the wave operator implies their equality for $0 \leq t \leq r$. Now using the relation (2.4) we conclude the equality $\phi_1 h(D)\phi_r = \phi_1 h(D_{2r})\phi_r$, which implies the invertibility of the operator (3.1) and the vanishing of its index in $K_0(A)$. □

The following theorem gives an application to the previous one.

Theorem 3.2. *Let M be a spin manifold and N be an enlargeable partitioning hypersurface of M dividing M into two non-compact parts. If any one of the following conditions is satisfied. Then M does not admit a complete Riemannian metric with positive scalar curvature at infinity:*

1. *There is a smooth map $\phi : M \to N$ such that its restriction to N is of non-zero degree.*
2. *The injection $j : N \hookrightarrow M$ induces an isomorphism $j_* : \pi_1(N) \xrightarrow{\sim} \pi_1(M)$.*

Proof. Fix a complete Riemannian metric on M. Using the previous theorem it suffices to find an appropriate flat Hilbert A-module bundle V over M such that $\mathrm{ind}(D, N) \neq 0 \in K_0(A)$, where D denotes the spin Dirac operator twisted by V. By enlargeability of N there is a flat Hilbert A-module bundle V' on N with the following properties (see [3, 4] and [10, Theorem 3.1]):

(a) the index of the spin Dirac operator of N twisted by V' is a non-zero element of $K_0(A)$;

(b) the index of the spin Dirac operator of N twisted by $\phi^*_{|N} V$ is equal to the index of the spin Dirac operator multiplied by $\deg \phi_{|N}$.

Under the condition (1) the bundle $V := \phi^* V'$ is a flat Hilbert A-module bundle over M. By applying the index formula (1.1) to the Clifford bundle $W := S \otimes \phi^* V$ and using the statements (a) and (b) in the above we conclude the non-vanishing of $\mathrm{ind}(D, N)$.

Now assume the condition (2). The flat bundle V' arises from a representation ζ of the fundamental group $\pi_1(N)$ in $\mathrm{End}_A(V'_0)$. Here V'_0 denotes a typical fiber of V' and is a finitely generated projective Hilbert A-module. This representation is of course a representation of $\pi_1(M)$ and, therefore, gives rise to a flat bundle V over M whose restriction to N is V'. Applying the index formula (1.1) to the

Clifford bundle $W := S \otimes V$ and using the statements (a) we get the non-vanishing of $\operatorname{ind}(D, N)$ in $K_0(A)$. $\square$

The following corollary is a direct consequence of this theorem.

Theorem 3.3. *Let (M, g) be a non-compact orientable complete spin n-manifold. Let N be a closed $(n-1)$-dimensional submanifold of M which is area-enlargeable. Let $M - N = M_+ \sqcup M_-$ and, say, M_+ is not compact. If any one of the following conditions is satisfied, then the scalar curvature of g cannot be uniformly positive outside a compact subset of M_+.*

(1) *There is a smooth map $\phi : M_+ \to N$ such that its restriction to N is of non-zero degree.*
(2) *The injection $j' : N \hookrightarrow M_+$ induces an isomorphism $j'_* : \pi_1(N) \xrightarrow{\sim} \pi_1(M_+)$.*

Proof. By deforming the Riemannian metric g in a compact collar neighborhood $N \times [0, 1)$ in $\overline{M}_+$ we can and will assume that g takes the product form $g_N + dt^2$, where g_N is a Riemannian metric on N. Let M_+^- denote the (non-complete) riemannian manifold M_+ with reversed orientation. The Riemannian metric $g_{|M_+}$ extends naturally (by reflection) to a complete Riemannian metric g' on the manifold $M_+ \sqcup M_+^-$. The scalar curvature of g' is uniformly positive at infinity provided that the scalar curvature of $g_{|M_+}$ is uniformly positive at infinity. Let the first condition be satisfied. We can deform ϕ in the above collar neighborhood and assume that its restriction to $N \times [0, 1)$ is independent of t. So the map ϕ extends (by reflection) to a smooth map $\tilde{\phi} := \phi \sqcup \phi$ from $M_+ \sqcup M_+^-$ onto N with $\deg \tilde{\phi}_{|N} \neq 0$. Under the condition (2) the injection $j : N \hookrightarrow M_+ \sqcup M_+^-$ induces an isomorphism $j_* : \pi_1(N) \xrightarrow{\sim} \pi_1(M_+ \sqcup M_+^-)$. Now we can apply the above theorem to deduce that the scalar curvature of g cannot be uniformly positive outside a compact set. $\square$

The set M_+ satisfying the condition (1) is called a *bad end* for M. The above theorem shows that the scalar curvature of a complete Riemannian metric cannot be uniformly positive at infinity of a bad end. In [2, Theorem 7.46] M. Gromov and B. Lawson proved this fact under the additional condition that the Ricci curvature of g be bounded from below on M_+. Moreover, the above theorem improves the following theorem of Gromov-Lawson; cf. [2, Theorem 7.44] (for if N_0 is area-enlargeable, then $N_0 \times S^1$ is area-enlargeable too).

Let (M, g) be a connected complete Riemannian manifold which contains a compact hypersurface N such that: N is diffeomorphic to $N_0 \times S^1$, where N_0 is enlargeable, $\pi_1(N_0) \to \pi_1(M)$ is injective and there is a non-compact component M_+ of $M - N$ and a map $\overline{M}_+ \to N$ such that its restriction to N has non-zero degree. If one of the following two conditions is satisfied, then the scalar curvature of g cannot be uniformly positive on M:

(1) *The map $\overline{M}_+ \to N$ is bounded.*
(2) *N_0 has no transversal of finite area.*

Here a transversal to N_0 is a properly embedded 2-dimensional submanifold of M which is transverse to N_0 with non-zero intersection number. By comparing this theorem with Theorem 3.3 it is clear that we have relaxed the strong condition on the topology of N (it is not to be of the product form $N_0 \times S^1$) and we need no condition on the fundamental groups in our theorem. Moreover we have relaxed the boundedness condition of the map $\overline{M}_+ \to N$. In addition we have the stronger result that the scalar curvature cannot be positive even at infinity. Of course we have paid a price for all of these: we have assumed M to be a spin manifold, while in the Gromov-Lawson theorem the spin condition is implicit in the enlargeability condition on N.

So far we have dealt with non-compact manifolds. The following theorem is an application of the above theorems to the compact case.

Theorem 3.4. *Let $N \subset M$ be an enlargeable submanifold of the closed spin manifold M with codimension 2 and let $j_* : \pi_1(N) \to \pi_1(M)$ denote the inclusion map. If j_* is injective and $|\pi_1(M)/j_*(\pi_1(N))| = \infty$, then M does not carry a Riemannian metric with everywhere positive scalar curvature metric.*

Proof. Let $\tilde{M}$ denote the (non-compact) covering of M corresponding to $j_*(\pi_1(N))$. Then the submanifold N can be lifted to $N \subset \tilde{M}$. Let $\tilde{N}$ denote the boundary of a tubular neighborhood of $N \subset \tilde{M}$. Being a circle bundle over an enlargeable manifold, $\tilde{N}$ is an enlargeable hypersurface of $\tilde{M}$. By passing, if necessary, to a 2-sheet covering we may assume that $\tilde{N}$ is orientable and so it partitions $\tilde{M}$ into a non-compact part $\tilde{M}_+$ and a compact one $\tilde{M}_-$. Using the Van Kampen theorem it is easy to verify the isomorphism $\pi_1(\tilde{N}) \simeq \pi_1(\tilde{M}_+)$. Now we can apply Theorem 3.3 to deduce that no complete metric on $\tilde{M}$ can have uniformly positive scalar curvature. On the other hand the lifting of any metric on M, with everywhere positive scalar curvature, to $\tilde{M}$ has uniformly positive scalar curvature. This contradiction proves the theorem. □

This theorem has been proved in [2, Theorem 9.4] for $\dim M = 4$. As a direct consequence of this theorem one can prove the following one.

Theorem 3.5. *Let M be a fiberation over a surface of non-positive Euler character. If the fibers are enlargeable, then M does not admit a metric with positive scalar curvature.*

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