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LINEAR DILATATION AND DIFFERENTIABILITY OF HOMEOMORPHISMS OF $\mathbb{R}^n$

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In memory of Juha Heinonen

Abstract. According to a classical result, if Ω is a domain in $\mathbb{R}^d$, where $d > 1$, $f : \Omega \to \mathbb{R}^d$ is a homeomorphism and the lim-sup dilatation H_f of f is finite almost everywhere on Ω, then f is differentiable almost everywhere on Ω. We show that this theorem fails if H_f is replaced by the lim-inf dilatation h_f. Our example demonstrates the sharpness of recent results of Kallunki and Koskela concerning the h_f function and also of Balogh and Csörnyei involving the lower-scaled oscillation of continuous functions $f : \Omega \to \mathbb{R}$.

1. Statement of main result

Throughout this section we assume that Ω is a domain in $\mathbb{R}^d$, where $d \geq 2$ and $f : \Omega \to \Omega' \subset \mathbb{R}^d$ is a homeomorphism. For any x in Ω the linear lim sup and lim inf dilatations of f at x are defined by

$$H_f(x) = \limsup_{r\to 0} H_f(x,r) = \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)}, \tag{1}$$

$$h_f(x) = \liminf_{r\to 0} H_f(x,r) = \liminf_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)}, \tag{2}$$

where

$$L_f(x,r) = \max\{|f(x) - f(y)| : |x-y| \leq r\}, \tag{3}$$

$$l_f(x,r) = \min\{|f(x) - f(y)| : |x-y| \geq r\}. \tag{4}$$

The H_f function is intimately connected with the theory of quasiconformal mappings as illustrated by the following well-known classical result (see, for example, [V], Theorem 34.1):

Theorem 1.1. *If there exists $K < \infty$ such that $H_f(x) \leq K$ for all $x \in \Omega$, then f is quasiconformal on Ω.*

In [G] Gehring showed that the hypotheses on H_f can be weakened a bit and still give the same conclusion as in Theorem 1.1:

Theorem 1.2. *Suppose that S is a subset of Ω with σ-finite $(d-1)$-dimensional measure, $H_f(x) < \infty$ for all $x \in \Omega \setminus S$ and there is a $K < \infty$ such that $H_f(x) \leq K$ a.e. on Ω. Then f is quasiconformal on Ω.*

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Another way to weaken the hypotheses in Theorem 1.1 is by replacing H_f with h_f. In [HK] Heinonen and Koskela showed that H_f can be replaced with h_f in Theorem 1.1:

Theorem 1.3. *Suppose there exists $K < \infty$ such that*

$$h_f(x) \leq K \tag{5}$$

for all $x \in \Omega$. Then f is quasiconformal on Ω.

More recently Koskela and Kallunki [KK] showed that Theorem 1.2 is also true with H_f replaced by h_f:

Theorem 1.4. *Suppose that S is a subset of Ω with σ-finite $(d-1)$-dimensional measure, $h_f(x) < \infty$ for all $x \in \Omega \backslash S$ and there is a $K < \infty$ such that $h_f(x) \leq K$ a.e. on Ω. Then f is quasiconformal on Ω.*

Since quasiconformal mappings are almost everywhere differentiable, it follows that the hypotheses of Theorem 1.2 imply that f is differentiable a.e. on Ω. In fact, one can significantly weaken the hypotheses on H_f and still get the a.e. differentiability of f:

Theorem 1.5. *Suppose that $H_f(x) < \infty$ a.e. on Ω. Then f is differentiable a.e. on Ω.*

Theorem 1.5 can be found, for example, in [KM] and is an easy consequence of the following versions of two classical theorems:

Theorem 1.6 (Rademacher-Stepanov)**.** *Define*

$$L_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{r}. \tag{6}$$

If $L_f(x) < \infty$ a.e. on Ω, then f is a.e. differentiable on Ω.

Theorem 1.7 (Lebesgue Differentiation)**.** *For almost every $x \in \Omega$ the following limit exists:*

$$\lim_{r \to 0} \frac{|f(B(x,r))|}{|B(x,r)|}.$$

Note that for each of these last two results we are still assuming that f is a homeomorphism from Ω into $\mathbb{R}^d$, a much stronger hypothesis than is required in either case. A natural question to ask at this point is whether we can replace H_f with h_f in Theorem 1.5. The answer to this question is an emphatic no, as our following main result shows.

Theorem 1.8. *Suppose that $d \geq 2$. Let $h : [0,\infty) \to [0,\infty)$ be a homeomorphism satisfying*

$$\frac{h(t)}{t^{d-1}} \to 0 \text{ as } t \to 0^+. \tag{7}$$

Then there exists a homeomorphism $f : [0,1]^d \to f([0,1]^d) \subset \mathbb{R}^d$ and a set $S \subset [0,1]^d$ such that

$$\mathcal{H}^h(S) = 0, \tag{8}$$

$$h_f(x) = 1 \quad \text{for all } x \in [0,1]^d \backslash S, \tag{9}$$

$$H_f(x) = \infty \text{ for all } x \in [0,1]^d, \tag{10}$$

$$f \text{ is nowhere differentiable on } [0,1]^d. \tag{11}$$

The notation denotes the Hausdorff measure with respect to h and will be defined precisely in section three. Note that if $h(t) = \frac{t^{d-1}}{-\log t}$ for small t, then h satisfies (7) and in this case S has Hausdorff dimension no larger than $d-1$. This demonstrates that Theorem 1.4 is quite sharp even regarding differentiability.

2. Scaled oscillation

In this section we again assume that Ω is a domain in $\mathbb{R}^d$, but we only require that $d \geq 1$ and we assume that $f : \Omega \to \mathbb{R}$ is continuous. We note that Theorem 1.6 remains true in this setting. In [BC] Balogh and Csörnyei consider analogues of this theorem and show that it is false if one replaces the upper scaled oscillation function L_f (defined in (6)) with the lower scaled oscillation function l_f, defined by:

$$l_f(x) = \liminf_{r \to 0} \frac{L_f(x,r)}{r}. \tag{12}$$

In fact, they show that

Theorem 2.1. *For every $d \geq 1$ there exists a nowhere differentiable, continuous function $f : [0,1]^d \to \mathbb{R}$ such that $l_f(x) = 0$ for a.e. $x \in [0,1]^d$.*

Note that to obtain Theorem 2.1 it suffices to prove that there exists a nowhere differentiable, continuous function $g : [0,1] \to \mathbb{R}$ such that $l_g(x) = 0$ for almost every $x \in [0,1]$. The higher-dimensional case then follows by considering the function $f : [0,1]^d \to \mathbb{R}$ defined by $f(x_1, x_2, ..., x_d) = g(x_1)$. In the positive direction, by strengthening the hypotheses on l_f, Balogh and Csörnyei obtain the following:

Theorem 2.2. *Assume that $l_f(x) < \infty$ on $\Omega \backslash E$ where E has σ-finite $(d-1)$-dimensional Hausdorff measure. Assume also that $l_f \in L^p_{loc}(\Omega)$ where $p > d$ if $d > 1$ and $p = 1$ if $d = 1$. Then f is differentiable a.e. on Ω.*

Theorem 2.1 shows that one cannot weaken the hypotheses in Theorem 2.2 by requiring only that the exceptional set E have d-dimensional Lebesgue measure 0. Our next result strengthens Theorem 2.1, showing that, in fact, E can have Hausdorff dimension at most $d-1$ in Theorem 2.2.

Theorem 2.3. *Suppose that $d \geq 1$. Let $h : [0,\infty) \to [0,\infty)$ be a homeomorphism satisfying (7). Then there exist a continuous function $f : [0,1]^d \to \mathbb{R}$ and a set $S \subset [0,1]^d$ such that (15) and (11) hold and*

$$l_f(x) = 0 \quad \text{for all } x \in [0,1]^d \backslash S, \tag{13}$$

$$L_f(x) = \infty \text{ for all } x \in [0,1]^d. \tag{14}$$

3. Hausdorff measure

Let $h : [0, \infty) \to [0, \infty)$ be a homeomorphism. For each $\delta > 0$ and $A \subset \mathbb{R}^d$ define

$$\mathcal{H}^h_\delta(A) = \inf \sum_i h(\mathrm{diam} E_i),$$

where the infimum is taken over all sequences (E_i) of subsets of $\mathbb{R}^d$ such that $A \subset \bigcup_i E_i$ and each E_i has diameter $\leq \delta$. Note that $\mathcal{H}^h_\delta$ is a decreasing function of δ, so we can define

$$\mathcal{H}^h(A) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(A) = \sup_{\delta > 0} \mathcal{H}^h_\delta(A).$$

Note that in the case that $h(t) = t^d$, where $d > 0$, then $\mathcal{H}^h(A)$ is just the standard d-dimensional Hausdorff measure of A.

We are now ready to prove Theorems 1.8 and 2.3.

4. Proofs of Theorems 1.8 and 2.3

We first note that the proofs of Theorems 1.8 and 2.3 follow rather quickly once we have established that Theorem 2.3 holds in the case $d = 1$. To see this assume for the moment that we have proved Theorem 2.3 for this case. Fix $d > 1$ and assume that h is a homeomorphism of $[0, \infty)$ onto itself satisfying (7). For $0 < t \leq 1$ define

$$h^*(t) = \sup_{0 < s \leq t} \frac{h(s)}{s^{d-1}}$$

and for $t > 1$ define $h^*(t) = th^*(1)$. It follows that h^* is an increasing homeomorphism of $(0, \infty)$ onto itself, and thus by our assumption we can find a continuous function $g : [0, 1] \to \mathbb{R}$ and a set $E \subset [0, 1]$ such that

$$\mathcal{H}^{h^*}(E) = 0, \tag{15}$$

$$l_g(x) = 0 \quad \text{for all } x \in [0, 1] \backslash E, \tag{16}$$

$$L_g(x) = \infty \text{ for all } x \in [0, 1], \tag{17}$$

$$g \text{ is nowhere differentiable on } [0, 1]. \tag{18}$$

To prove Theorem 1.8 define $f(x_1, x_2, ..., x_d) = (x_1, x_2, ..., x_{d-1}, x_d + g(x_1))$ and $S = E \times [0, 1]^{d-1}$. The continuity of g guarantees that f is a homeomorphism. Moreover, it is straighforward to show that (8) follows from (15) and also that (9), (10) and (11) follow from (16), (17) and (18).

For the proof of Theorem 2.3 we define S as in the preceding paragraph, but now define $f(x_1, x_2, ..., x_d) = g(x_1)$. Then (11) again follows from the nowhere differentiability of g and (13) and (14) follow easily from (16) and (17).

It remains to show that we can find a function g and a set $E \subset [0, 1]$ such that (15)-(18) all hold. The construction of g is a variation of a standard construction of a nowhere differentiable, continuous function. To begin we need a bit of notation and a simple lemma.

For any nonnegative integer n define $\mathcal{I}_n = \{I_{s,n} : s = 0, 1, ..., 2^n - 1\}$, where $I_{s,n} = [s2^{-n}, (s + 1)2^{-n}]$. Given nonnegative integers j and k, we say that f is (j, k)-*dyadic* if f is continuous on $[0, 1]$ and for each $I \in \mathcal{I}_j$, f is linear on I with $f(I) \in \mathcal{I}_k$.

Definition 4.1. Let j and k be nonnegative integers and f be (j,k)-dyadic. Suppose that l and m are integers satisfying

$$m \geq k+2 \text{ and } l-m > j-k. \tag{19}$$

Then the (l,m)-successor of f is the unique (l,m)-dyadic function g that satisfies the following three conditions:

$$g(s2^{-j}) = f(s2^{-j}) \text{ for } s = 0,1,2,...,2^j. \tag{20}$$

For each $I = I_{s,j} \in \mathcal{I}_j$, g is linear on the intervals $[s2^{-j}, s2^{-j}+2^{m-l-k-1}]$ and $[(s+1)2^{-j}-2^{m-l-k-1}, (s+1)2^{-j}]$ with slope the same sign as the sign of the slope of f on I.

For each $I = I_{s,j} \in \mathcal{I}_j$, the sign of the slope of g is $(-1)^t$ on each interval $I_{t,l} \subset I' = [s2^{-j}+2^{m-l-k-1}, (s+1)2^{-j}-2^{m-l-k-1}]$.

We then have the following lemma, which requires no proof:

Lemma 4.2. *Suppose that j,k,l,m satisfy (19), f is (j,k)-dyadic and g is the (l,m)-successor of f.*

Then

$$|f(x)-g(x)| \leq 2^{-k} \textit{ for all } x \in [0,1], \tag{21}$$

$$\forall I \in \mathcal{I}_j \ |g(x)-g(y)| \leq 2^{-m} \textit{ for all } x,y \in I'. \tag{22}$$

We now begin the construction of g. We first choose (j_n) to be an increasing sequence of nonnegative integers divisible by 3 and satisfying the following:

$$j_0 = 0, \tag{23}$$

$$j_{n+1} \geq 3j_n + 3 \ \forall n = 0,1,2,..., \tag{24}$$

$$h^*(2^{\frac{-j_{n+1}}{3}}) \leq 2^{-j_n-n-1}. \tag{25}$$

For each $n = 0,1,2,...$ define $k_n = \frac{2j_n}{3}$.

Let $g_0(x) = x$. Note that for all $n = 0,1,2,...$ we have $k_{n+1} \geq k_n + 2$ and $j_{n+1} - k_{n+1} > j_n - k_n$. Thus, we can use Lemma 4.2 to inductively define a sequence of functions (g_n) defined on $[0,1]$ such that for each $n = 0,1,2,...$, g_{n+1} is the (j_{n+1}, k_{n+1})-successor of g_n. Since (k_n) is strictly increasing and $\lim_{n\to\infty} k_n = \infty$, it follows from (21) that (g_n) converges uniformly on $[0,1]$ to a continuous function g.

We next show that (17) holds. Let $x \in [0,1]$. Given a positive integer i, choose s_i such that $x \in I_{s_i,j_i} = [a_i, b_i]$. By construction of g we have

$$|\frac{g(b_i)-g(a_i)}{b_i-a_i}| = \frac{2^{-k_i}}{2^{-j_i}} = 2^{\frac{1}{2}k_i}.$$

It follows that

$$\max\{|\frac{g(b_i)-g(x)}{b_i-x}|, |\frac{g(x)-g(a_i)}{x-a_i}|\} \geq 2^{\frac{1}{2}k_i}.$$

Since $2^{\frac{1}{2}k_i} \to \infty$ and $b_i - a_i \to 0$, we see that $L_g(x) = \infty$. Thus, (17) is established and (18) clearly follows from (17).

We now move on to the proof of (15) and (16). For each $s = 0,1,2,...,2^{j_n}-1$ define

$$I''_{s,j_n} = [s2^{-j_n} + 2^{-\frac{1}{2}k_{n+1}-k_n}, (s+1)2^{-j_n} - 2^{-\frac{1}{2}k_{n+1}-k_n}].$$

Note that if $x \in I''_{s,j_n}$ and $|x-y| \le 2^{-\frac{1}{2}k_{n+1}-k_n-1}$, then $x, y \in I'_{s,j_n} = [s2^{-j_n} + 2^{-\frac{1}{2}k_{n+1}-k_n-1}, (s+1)2^{-j_n} - 2^{-\frac{1}{2}k_{n+1}-k_n-1}]$, and hence by (22) we have

$$|g_{n+1}(x) - g_{n+1}(y)| \le 2^{-k_{n+1}}.$$

Moreover, from the construction of g it is easy to see that we can replace g_{n+1} with g in the inequality above. Summing up, we have:

(26) If $x \in I''_{s,j_n}$ and $|x-y| \le 2^{-\frac{1}{2}k_{n+1}-k_n-1}$, then $|g(x)-g(y)| \le 2^{-k_{n+1}}$.

Now for each positive integer n, define

$$F_n = \bigcup_{s=0}^{2^{j_n}-1} I''_{s,j_n},$$

$$F = \bigcap_{n=1}^{\infty} (\bigcup_{k=n}^{\infty} F_k)$$

and let $E = [0,1] \backslash F$. Fix $x_0 \in F = [0,1] \backslash E$. Then there is an increasing sequence of integers (n_i) such that $x_0 \in F_{n_i}$ for $i = 1, 2, \ldots.$ Fix i and choose s such that $x_0 \in I''_{s,n_i}$. Let $r = 2^{-\frac{1}{2}k_{n_i+1}-k_{n_i}-1}$ and assume that $|x_0 - x| \le r$. Using (26), we get $|g(x_0) - g(x)| \le 2^{-k_{n_i+1}}$, and a simple calculation using (24) shows that

$$\frac{|g(x_0) - g(x)|}{r} \le 2^{-\frac{j_{n_i}}{3}}.$$

Since $2^{-\frac{j_{n_i}}{3}} \to 0$, we have (16), as desired.

To complete the proof we need to establish (15). To that end define $E_n = [0,1] \backslash F_n$ so that

$$E = \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} E_k),$$

where

$$E_n = \bigcup_{s=0}^{2^{j_n}-1} (I_{s,j_n} \backslash I''_{s,j_n}).$$

Note that each E_n is the union of 2^{j_n+1} intervals of length $2^{-\frac{1}{2}k_{n+1}-k_n}$. Since $k_n \ge 1$ for $n \ge 1$, the diameter of each of these intervals is less than $2^{\frac{-j_{n+1}}{3}}$, and thus by (25) we have

$$\mathcal{H}^{h^*}_{\delta_n}(E_n) \le 2^{j_n+1} h^*(2^{-\frac{j_{n+1}}{3}}) \le 2^{-n},$$

where $\delta_n = 2^{\frac{-j_{n+1}}{3}}$. This confirms that (15) holds, and we are done with the proof.

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References

[BC] Balogh, Z., and M. Csörnyei, *Scaled oscillation and regularity*, Proc. Amer. Math. Soc. **134** 9 (2006), 2667-2675. MR2213746 (2007b:46052)

[G] Gehring, F.W., *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353-393. MR0139735 (25:3166)

[HK] Heinonen, J., and P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), 61-79. MR1323982 (96e:30051)

[KK] Kallunki, S., and P. Koskela, *Exceptional sets for the definition of quasiconformality*, American Journal of Mathematics **122** (2000), 735-743. MR1771571 (2001h:37095)

[KM] Kallunki, S., and O. Martio, *ACL homeomorphisms and linear dilatation*, Proc. Amer. Math. Soc. **130** (2002), 1073-1078. MR1873781 (2002i:30022)

[V] Väisälä, J., *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Mathematics, No. 229, Springer-Verlag, New York, 1971. MR0454009 (56:12260)

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