

PLURISUBHARMONIC FUNCTIONS AND NEF CLASSES ON COMPLEX MANIFOLDS

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ABSTRACT. We prove the existence of plurisubharmonic functions with prescribed logarithmic singularities on complex 3-folds equipped with a nef class of positive volume. We prove the same result for rational classes on Moishezon n -folds.

1. INTRODUCTION

We recall that a function $f : \Omega \rightarrow [-\infty, \infty)$ for a domain $\Omega \subset \mathbb{C}^n$ is plurisubharmonic if it is upper semi-continuous and for every $a, b \in \mathbb{C}^n$, the map

$$z \in \mathbb{C} \mapsto f(a + bz) \in [-\infty, \infty)$$

is subharmonic where it is defined.

Now let (X, ω) be a compact Kähler manifold of complex dimension n . There are complex coordinate charts B_i on which we can write $\omega = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g_i$ for smooth potential functions g_i (unique up to adding pluriharmonic functions). We define a function $\varphi : X \rightarrow [-\infty, +\infty)$ to be ω -plurisubharmonic if φ is upper semi-continuous, not identically $-\infty$, and each $g_i + \varphi$ is plurisubharmonic (clearly, this does not depend on the choice of potentials g_i). A basic fact is that an L^1 function φ on X satisfying

$$(1.1) \quad \omega + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi \geq 0,$$

in the sense of currents, agrees with a unique ω -plurisubharmonic function almost everywhere. Conversely, every ω -plurisubharmonic φ is in L^1 and satisfies (1.1).

The space of ω -plurisubharmonic φ has been the focus of considerable study in the last few decades. Note that every closed positive real $(1, 1)$ -current cohomologous to $[\omega] \in H^{1,1}(X, \mathbb{R})$ can be written as $\omega + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi$ for some such φ .

It was shown in [18, 26] (see also [27]) that there exists a constant $\alpha > 0$ depending only on $[\omega]$ such that

$$\int_X e^{-\alpha \varphi} \omega^n \leq C,$$

for all ω -plurisubharmonic φ . This shows in particular that singularities of φ can be at most logarithmic.

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An interesting and well-known problem is: can we find a φ with prescribed logarithmic singularities at given points on X ? In the case when $[\omega]$ is the Chern class of a holomorphic line bundle L , this is equivalent to prescribing singular Hermitian metrics on L . This can be naturally extended to line bundles L which are only nef. Deep results in [6, 7, 25] and others used the construction of singular φ to prove effective results in algebraic geometry.

In this short paper we investigate this problem on general complex (non-Kähler) manifolds. Our motivation is to try to understand whether techniques from Kähler geometry can be extended to non-Kähler complex geometry, at least when natural analogues exist.

Suppose now that X is only a compact complex manifold. Let β be a closed real $(1,1)$ -form on X , consider the (finite-dimensional) real Bott-Chern cohomology group

$$H_{\text{BC}}^{1,1}(X, \mathbb{R}) = \frac{\{\text{closed real } (1,1)\text{-forms}\}}{\{\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi, \psi \in C^\infty(X, \mathbb{R})\}},$$

and call $[\beta]$ the class of β in $H_{\text{BC}}^{1,1}(X, \mathbb{R})$.

Since a positive β only exists if X is Kähler, we consider instead the case when the class $[\beta]$ is nef (as defined in [9]), which means that for any $\varepsilon > 0$ there exists a representative $\beta + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi_\varepsilon$ so that $\beta + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi_\varepsilon > -\varepsilon \omega$, where ω is some fixed Hermitian metric on X .

The closed form β admits local potential functions, and thus we can define the notion of β -plurisubharmonic in the same way as described above. Under the cohomological assumption that $\int_X \beta^n > 0$, we look for β -plurisubharmonic functions φ with prescribed logarithmic singularities.

Our main result is the following:

Main Theorem. *Let X be a compact complex manifold of dimension n . Suppose there exists a class $[\beta] \in H_{\text{BC}}^{1,1}(X, \mathbb{R})$ which is nef and satisfies $\int_X \beta^n > 0$. Assume either*

(i) $n = 2$ or $n = 3$

or

(ii) X is Moishezon and $[\beta] \in H_{\text{BC}}^{1,1}(X, \mathbb{Q}) := H_{\text{BC}}^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q})$.

Fix points $x_1, \dots, x_N \in X$ and choose positive real numbers τ_1, \dots, τ_N so that

$$(1.2) \quad \sum_j \tau_j^n < \int_X \beta^n.$$

Then there exists a β -plurisubharmonic φ with logarithmic poles at x_1, \dots, x_N :

$$(1.3) \quad \varphi(z) \leq \tau_j \log |z| + O(1),$$

in a coordinate neighborhood (z_1, \dots, z_n) centered at x_j , where $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. In particular, the Lelong number of φ at each point x_j is at least τ_j .

Recall that a Moishezon manifold is a compact complex manifold which is bimeromorphic to a projective manifold. An equivalent definition is that a Moishezon manifold is a compact complex manifold admitting a big line bundle L (meaning $\dim H^0(X, L^k) > ck^n$ for k large, for some fixed $c > 0$).

If X is Kähler (without imposing (i) or (ii)), this result is due to Demailly [6], whose proof made use of Yau's solution of the complex Monge-Ampère equation on Kähler manifolds [30].

In the case of one point ($N = 1$), inequality (1.2) is sharp in general: taking $X = \mathbb{C}P^n$ and $[\beta]$ to be the (ample) anticanonical class (so β is $n + 1$ times the Fubini-Study metric), equation (1.2) says that $\tau < n + 1$, and it is well known (see e.g. [4, Proposition 2.1]) that $n + 1$ is indeed the maximum order of a logarithmic pole of any β -plurisubharmonic function.

We remark that our result in case (i) is only really new in the case $n = 3$. The reason is that in dimension 2, any surface X as in the Main Theorem is necessarily Kähler. In fact, the existence of a closed real $(1, 1)$ -form β with $\int_X \beta^2 > 0$ implies that the intersection form on $H^{1,1}(X, \mathbb{R})$ is not negative definite, and thanks to a classical theorem of Kodaira [20] this implies that $b_1(X)$ is even. A theorem of Miyaoka and Siu [23, 24] (see also [2, 21]) then implies that X is Kähler.

On the other hand there are certainly many non-Kähler 3-folds satisfying the hypotheses of the Main Theorem (see for example [17] and the description in [16, Example 3.4.1, p.443]). Indeed, such manifolds were discussed by Demailly and Păun in [9], where it was conjectured that a compact complex n -fold X with a nef class $[\beta]$ of type $(1, 1)$ with $\int_X \beta^n > 0$ is bimeromorphic to a Kähler manifold.

Demailly [6] proved the Kähler version of the Main Theorem using Yau’s existence result for solutions to the complex Monge-Ampère equation on Kähler manifolds [30]. In this paper we follow along the same lines of argument as Demailly, but now apply the recent extension of Yau’s Theorem to general complex manifolds [28] (see also [3, 14, 29, 11, 1, 13]). However, a difficulty arises here in the non-Kähler case due to the fact that for a $(1, 1)$ -form Ω and a function f , the equality

$$\int_X \left(\Omega + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} f \right)^n = \int_X \Omega^n$$

does not hold in general if Ω is not closed. We can overcome this obstacle in dimensions 2 and 3 by making use of Gauduchon metrics.

We conjecture that the Main Theorem holds for any dimension without the Moishezon and rationality assumptions in (ii).

2. PROOF OF THE MAIN THEOREM

Let ω be a Gauduchon metric on X , which means that $\partial \bar{\partial}(\omega^{n-1}) = 0$ (such a metric always exists [12]). Following Demailly [6], we choose coordinates z^1, \dots, z^n in a neighborhood centered at x_j and put

$$(2.4) \quad \gamma_{j,\varepsilon} = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \left(\chi \left(\log \frac{|z|}{\varepsilon} \right) \right),$$

where $|z|^2 = |z^1|^2 + \dots + |z^n|^2$, and $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, convex, increasing and satisfies $\chi(t) = t$ for $t \geq 0$ and $\chi(t) = -1/2$ for $t \leq -1$. Observe that $\gamma_{j,\varepsilon}$ is a nonnegative real $(1, 1)$ -form defined only on this coordinate chart, which equals $\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |z|$ outside the ball $\{|z| \leq \varepsilon\}$. We therefore have that $\gamma_{j,\varepsilon}^n = 0$ if $|z| > \varepsilon$, so we can extend it to zero on the whole of X . This way, $\gamma_{j,\varepsilon}^n$ is a smooth nonnegative (n, n) -form on X that satisfies

$$\int_X \gamma_{j,\varepsilon}^n = \int_{|z| \leq \varepsilon} \gamma_{j,\varepsilon}^n = \int_{|z| \leq \varepsilon} \left(\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |z| \right)^n = 1,$$

using Stokes’ theorem, and $\gamma_{j,\varepsilon}^n \rightarrow \delta_{x_j}$ as $\varepsilon \rightarrow 0$. Since β is nef, for any $\varepsilon > 0$ there exists a smooth function ψ_ε such that $\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi_\varepsilon$ is Hermitian.

Now recall that the Hermitian version of Yau’s theorem [28] states that given a Hermitian metric $\hat{\omega}$ and a smooth function F there exists a unique smooth f and a unique constant $K > 0$ solving

$$\left(\hat{\omega} + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}f\right)^n = Ke^F \hat{\omega}^n, \quad \hat{\omega} + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}f > 0, \quad \sup_X f = 0.$$

Note that, in the case of Yau’s theorem where $\hat{\omega}$ is Kähler, integration by parts shows that $K = \int_X \hat{\omega}^n / \int_X e^F \hat{\omega}^n$. In the non-Kähler case, no such formula holds in general and this is the source of the difficulty.

Applying this with reference metric $\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\psi_\varepsilon$, we obtain a smooth φ_ε with

$$\left(\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)\right)^n = C_\varepsilon \left(\sum_j \tau_j^n \gamma_{j,\varepsilon}^n + \delta\omega^n\right)$$

and

$$\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon) > 0, \quad \sup_X(\psi_\varepsilon + \varphi_\varepsilon) = 0,$$

where $\delta > 0$ is fixed and C_ε is a (uniquely determined) positive constant. The key fact that we need now is the lower bound $C_\varepsilon \geq 1$ for ε and δ sufficiently small. We remark that there are some formal similarities between the argument given here for this lower bound and the proofs of Proposition 3.8 in [10] and Theorem 1.2 in [19].

We consider first the case (i) when $n = 2$. Then $\partial\bar{\partial}\omega = 0$ and, using the fact that β is closed, we have

$$\begin{aligned} C_\varepsilon &= \frac{\int_X (\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon))^2}{\int_X \sum_j \tau_j^2 \gamma_{j,\varepsilon}^2 + \delta\omega^2} = \frac{\int_X (\beta + \varepsilon\omega)^2}{\sum_j \tau_j^2 + \delta \int_X \omega^2} \\ &\geq \frac{\int_X \beta^2 - \varepsilon^2 \int_X \omega^2}{\sum_j \tau_j^2 + \delta \int_X \omega^2}, \end{aligned}$$

since

$$\int_X (\beta + \varepsilon\omega)^2 = \int_X \beta^2 + 2\varepsilon \int_X \left(\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\psi_\varepsilon\right) \wedge \omega - \varepsilon^2 \int_X \omega^2.$$

Choosing δ sufficiently small and using (1.2), we get for any $\varepsilon > 0$ sufficiently small,

$$C_\varepsilon \geq 1.$$

We now consider the case when $n = 3$. For simplicity call $\beta_\varepsilon = \beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)$ and note that

$$\beta + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon) = \beta_\varepsilon - \varepsilon\omega,$$

and so

$$\int_X (\beta_\varepsilon - \varepsilon\omega)^3 = \int_X \left(\beta + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)\right)^3 = \int_X \beta^3 > 0.$$

Using the Gauduchon condition $\partial\bar{\partial}(\omega^2) = 0$, we see that

$$(2.5) \quad \int_X \beta_\varepsilon \wedge \omega^2 = \int_X (\beta + \varepsilon\omega) \wedge \omega^2 \leq C \int_X \omega^3.$$

On the other hand, using (2.5), we have

$$\begin{aligned} \int_X \beta_\varepsilon^3 &= \int_X (\beta_\varepsilon - \varepsilon\omega + \varepsilon\omega)^3 \\ &= \int_X (\beta_\varepsilon - \varepsilon\omega)^3 + 3\varepsilon \int_X \beta_\varepsilon^2 \wedge \omega - 3\varepsilon^2 \int_X \beta_\varepsilon \wedge \omega^2 + \varepsilon^3 \int_X \omega^3 \\ &\geq \int_X \beta^3 - C'\varepsilon^2, \end{aligned}$$

and so

$$C_\varepsilon = \frac{\int_X \beta_\varepsilon^3}{\int_X \sum_j \tau_j^3 \gamma_{j,\varepsilon}^3 + \delta\omega^3} \geq \frac{\int_X \beta^3 - C'\varepsilon^2}{\sum_j \tau_j^3 + \delta \int_X \omega^3},$$

and hence choosing δ sufficiently small and using (1.2), we get for $\varepsilon > 0$ sufficiently small

$$C_\varepsilon \geq 1.$$

Finally we prove $C_\varepsilon \geq 1$ in the case (ii). Because $[\beta]$ is rational we have $\ell[\beta] = c_1(L)$ for some line bundle L over X and some integer $\ell \geq 1$. In this case there exists a Hermitian metric h_ε on L such that $\frac{1}{\ell}c_1(L, h_\varepsilon) = \beta + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)$, where $c_1(L, h_\varepsilon)$ is the curvature form of the Hermitian metric h_ε . Denote by $X(0)$ the set of $x \in X$ such that $c_1(L, h_\varepsilon)$ has 0 negative eigenvalues. We now apply Demailly’s holomorphic Morse inequalities [5] to see that for k large we have

$$\begin{aligned} \dim H^0(X, L^k) &\leq \frac{k^n}{n!} \int_{X(0)} c_1(L, h_\varepsilon)^n + o(k^n) \\ &\leq \frac{k^n}{n!} \int_{X(0)} (c_1(L, h_\varepsilon) + \varepsilon\omega)^n + o(k^n) \\ &= \frac{C_\varepsilon \ell^n k^n}{n!} \int_{X(0)} \left(\sum_j \tau_j^n \gamma_{j,\varepsilon}^n + \delta\omega^n \right) + o(k^n) \\ (2.6) \quad &\leq \frac{C_\varepsilon \ell^n k^n}{n!} \left(\sum_j \tau_j^n + \delta \int_X \omega^n \right) + o(k^n). \end{aligned}$$

We now estimate the number of sections of L^k using the Riemann-Roch theorem. Since the manifold X is Moishezon there exists a modification $\mu : \tilde{X} \rightarrow X$ with \tilde{X} a projective manifold. The pullback μ^*L is then a nef line bundle on \tilde{X} with $\int_{\tilde{X}} c_1(\mu^*L)^n = \ell^n \int_X \beta^n > 0$. Because μ^*L is nef, its higher cohomology groups satisfy $\dim H^q(\tilde{X}, \mu^*L^k) = O(k^{n-1})$, for $q > 0$ (see Example 1.2.36 in [22]), and a standard Leray spectral sequence argument (see (2.1) in [8]) shows that $\dim H^q(X, L^k) = O(k^{n-1})$, for $q > 0$. By the Riemann-Roch theorem we now have

$$(2.7) \quad \dim H^0(X, L^k) = \frac{\ell^n k^n}{n!} \int_X \beta^n + o(k^n).$$

Combining (2.6) and (2.7) and taking k large we get

$$\int_X \beta^n \leq C_\varepsilon \left(\sum_j \tau_j^n + \delta \int_X \omega^n \right).$$

Choosing $\delta > 0$ sufficiently small we obtain $C_\varepsilon \geq 1$.

We claim that $\psi_\varepsilon + \varphi_\varepsilon$ is uniformly bounded in L^1 . This is because we can choose a large constant A so that $A\omega \geq \beta + \varepsilon\omega$ for all $0 < \varepsilon \leq 1$, and then the function $\psi_\varepsilon + \varphi_\varepsilon$ satisfies $A\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon) > 0$, and then Proposition 2.1 in [11] (for example) gives a uniform L^1 bound that depends only on X, A, ω .

This implies that there is a sequence $\varepsilon_k \rightarrow 0$ such that $\psi_{\varepsilon_k} + \varphi_{\varepsilon_k}$ converges in L^1 to a β -plurisubharmonic function φ . Indeed, we can recover this from the local statement about compactness of plurisubharmonic functions in a domain in \mathbb{C}^n which are uniformly bounded in L^1 , in the following way: we cover X with finitely many coordinate charts B_i so that on each B_i there is a smooth function ρ_i with $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\rho_i > \beta + \omega$. Then on each B_i the functions $\rho_i + \psi_\varepsilon + \varphi_\varepsilon$ are plurisubharmonic and uniformly bounded in L^1 (independent of ε), so the local statement applies.

We now show that φ has the desired logarithmic singularities.

Take Ω to be a neighborhood of x_j (which we can assume contains the set $\{|z| < 1\}$) and consider the smooth plurisubharmonic function on Ω ,

$$u = C_\varepsilon^{1/n}\tau_j(\chi(\log(|z|/\varepsilon)) + \log \varepsilon) + C_1$$

for C_1 a large constant, where χ is the same cutoff function as in (2.4). Let h be a smooth function on $\bar{\Omega}$ with

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}h \geq \beta + \omega$$

and put $v = h + \psi_\varepsilon + \varphi_\varepsilon$, so v is a smooth plurisubharmonic function on Ω which is bounded from above by C_0 , say.

Then if ε is sufficiently small and C_1 is sufficiently large we have

$$u|_{\partial\Omega} = C_\varepsilon^{1/n}\tau_j \log |z| + C_1 \geq C_0 \geq v|_{\partial\Omega}.$$

But in addition we have on Ω

$$\begin{aligned} \left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}v\right)^n &\geq \left(\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)\right)^n \\ &\geq C_\varepsilon\tau_j^n\gamma_{j,\varepsilon}^n = \left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}u\right)^n. \end{aligned}$$

Then by the Bedford-Taylor comparison principle for Monge-Ampère (e.g. Lemma 6.7 in [6]),

$$u \geq v \quad \text{on } \Omega,$$

and hence when $|z| < 1/2$ and ε is small we have

$$\begin{aligned} \psi_\varepsilon + \varphi_\varepsilon &\leq C_\varepsilon^{1/n}\tau_j(\chi(\log(|z|/\varepsilon)) + \log \varepsilon) + C_2 \leq C_\varepsilon^{1/n}\tau_j \log(|z| + \varepsilon) + C_2 \\ &\leq \tau_j \log(|z| + \varepsilon) + C_2, \end{aligned}$$

where we are using the fact that $C_\varepsilon \geq 1$. Since $\psi_\varepsilon + \varphi_\varepsilon$ converges to φ in L^1 , Hartogs' Lemma (see for example Proposition 2.6 (2) in [15]) implies (1.3). Q.E.D.

Let us remark here that, in any dimension, the constant C_ε is always bounded above (independent of ε but depending on δ) when ε is small. Indeed, at the point

on X where the function $\psi_\varepsilon + \varphi_\varepsilon$ achieves its maximum, we have that $\beta + \varepsilon\omega$ is positive definite and moreover $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon) \leq 0$, and so at that point we have

$$\begin{aligned} C\omega^n &\geq (\beta + \varepsilon\omega)^n \geq \left(\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)\right)^n \\ &= C_\varepsilon \left(\sum_j \tau_j^n \gamma_{j,\varepsilon}^n + \delta\omega^n\right) \geq C_\varepsilon \delta\omega^n, \end{aligned}$$

giving $C_\varepsilon \leq C/\delta$.

A similar argument only gives a lower bound of the form $C_\varepsilon \geq c\varepsilon^{3n}$, in any dimension. In fact, a direct calculation shows that on X we have

$$\gamma_{j,\varepsilon}^n \leq \frac{C}{\varepsilon^{2n}}\omega^n,$$

for a constant C independent of ε . We then compute

$$\left(\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)\right)^n = C_\varepsilon \left(\sum_j \tau_j^n \gamma_{j,\varepsilon}^n + \delta\omega^n\right) \leq \frac{C}{\varepsilon^{2n}}C_\varepsilon\omega^n.$$

Recall that since $[\beta]$ is nef there is a smooth function $\psi_{\varepsilon/2}$ so that $\beta + \frac{\varepsilon}{2}\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\psi_{\varepsilon/2} > 0$. We then write

$$\begin{aligned} \beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon) &= \left(\beta + \frac{\varepsilon}{2}\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\psi_{\varepsilon/2}\right) \\ &\quad + \left(\frac{\varepsilon}{2}\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon - \psi_{\varepsilon/2})\right). \end{aligned}$$

At the point on X where $\psi_\varepsilon + \varphi_\varepsilon - \psi_{\varepsilon/2}$ achieves its minimum we have $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon - \psi_{\varepsilon/2}) \geq 0$, and so at that point

$$\frac{\varepsilon^n}{2^n}\omega^n \leq \left(\beta + \varepsilon\omega + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon)\right)^n \leq \frac{C}{\varepsilon^{2n}}C_\varepsilon\omega^n,$$

which gives $C_\varepsilon \geq c\varepsilon^{3n}$.

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