

ON LOCAL TRIGONAL FIBRATIONS

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ABSTRACT. We show that certain degenerating hyperelliptic curves, called horizontal types, of genus g appear as the central fiber of a local family of trigonal curves of genus g by constructing such a family explicitly.

INTRODUCTION

Let $\varphi : \mathcal{C} = \{\mathcal{C}_t\}_{t \in \Delta_\epsilon} \rightarrow \Delta_\epsilon$ be a local family of curves over a small disc $\Delta_\epsilon := \{t \in \mathbb{C} \mid |t| < \epsilon\}$, where ϵ is a small positive real number, which means that \mathcal{C} is a smooth surface, φ is proper and surjective, all fibers of φ are connected and $\varphi^{-1}(t)$ is smooth for $t \neq 0$. We call $\varphi : \mathcal{C} \rightarrow \Delta_\epsilon$ a local hyperelliptic (resp. trigonal) fibration of genus g if \mathcal{C}_t ($t \neq 0$) is a hyperelliptic (resp. trigonal) curve of genus g .

In [4], Chen and Tan gave two examples of local trigonal fibrations of genus 3 such that their central fibers, \mathcal{C}_0 , are smooth hyperelliptic curves of genus 3. A purpose of this paper is to show that any hyperelliptic curve of genus g appears as the central fiber of a local trigonal fibration.

Let us explain our setting and question precisely. Let B_1 be a reduced divisor on $\mathbb{P}^1 \times \Delta_\epsilon$ such that B_1 meets $\mathbb{P}^1 \times \{t\}$ ($t \neq 0$) at $2g + 2$ distinct points transversely. In particular, all singularities of B_1 are in $\mathbb{P}^1 \times \{0\}$. Let $f_o : W \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$ be the double cover with $\Delta_{f_o} = B_1$, where Δ_{f_o} denotes the branch locus of f_o , and let $\mu_o : \widetilde{W} \rightarrow W$ be the canonical resolution (see [7] for the canonical resolution). We consider a local hyperelliptic fibration of genus g given by putting $\mathcal{C} = \widetilde{W}$ and $\varphi_o := \text{pr}_2 \circ f_o \circ \mu_o$.

We say that

(i) $\varphi_o : \mathcal{C} \rightarrow \Delta_\epsilon$ is of *horizontal type* if B_1 does not contain $\mathbb{P}^1 \times \{0\}$ as its irreducible component, and

(ii) $\varphi_o : \mathcal{C} \rightarrow \Delta_\epsilon$ is of *non-horizontal type* if B_1 contains $\mathbb{P}^1 \times \{0\}$ as its irreducible component.

Note that we may assume that φ_o is either horizontal or non-horizontal by taking ϵ (> 0) small enough. Also any local hyperelliptic fibration is obtained as the relatively minimal model of $\varphi_o : \mathcal{C} \rightarrow \Delta_\epsilon$. We now formulate our question as follows:

Question 0.1. For an arbitrary local hyperelliptic fibration $\varphi_o : \mathcal{C} \rightarrow \Delta_\epsilon$, does there exist a local trigonal fibration $\varphi : \widetilde{\mathcal{C}} \rightarrow \Delta_\epsilon$ such that $\mathcal{C}_0 = \widetilde{\mathcal{C}}_0$? In other words,

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can the central fiber of any local hyperelliptic fibration appear as that of a certain local trigonal fibration?

In this paper, we give an answer to Question 0.1 in the case when φ_o is of horizontal type.

Theorem 0.1. *Question 0.1 is true for a local hyperelliptic fibration of horizontal type.*

Since any hyperelliptic curve of genus g appears as the central fiber of a local hyperelliptic fibration of horizontal type, i.e., a trivial fibration, we have

Corollary 0.1. *Let D be any hyperelliptic curve of genus g . There exists a local trigonal fibration $\tilde{\varphi} : \tilde{C} \rightarrow \Delta_\epsilon$ such that $\tilde{C}_0 = D$.*

1. COVERS AND THE FUNDAMENTAL GROUP

Let X and M be a normal variety and a complex manifold, respectively. We call X a (branched) cover of M if there exists a finite surjective morphism $\pi : X \rightarrow M$. When needed, the covering morphism will be specified as a cover $\pi : X \rightarrow Y$. Let B be a reduced divisor on M . The following facts are well-known:

- Choose a point $* \in M \setminus B$. Then the inclusion morphism $\iota : M \setminus B \rightarrow M$ induces an epimorphism $\iota_* : \pi_1(M \setminus B, *) \rightarrow \pi_1(M, *)$.
- Let H be a subgroup of $\pi_1(M \setminus B, *)$. Then there exists an unramified cover $\pi_H : X_o \rightarrow M \setminus B$ over $M \setminus B$ with $\pi_H(X_o, \hat{*}) \cong H$, $\pi_H(\hat{*}) = *$, such that X_o can be extended over M uniquely. We also denote the extended cover of $\pi_H : X_o \rightarrow M \setminus B$ by $\pi_H : X \rightarrow M$. Note that the branch locus Δ_{π_H} of π_H is a subset of B and $\deg \pi_H = [\pi_1(M \setminus B, *) : H]$. Conversely, if there exists a branched cover $\pi : X \rightarrow M$ with $\deg \pi = n$, then there exists a subgroup H_π of $\pi_1(M \setminus \Delta_\pi, *)$ of index n .
- Let H be a normal subgroup of $\pi_1(M \setminus B, *)$. Then there exists a Galois cover $\pi_H : X \rightarrow M$ with $\text{Aut}_M(X) \cong \pi_1(M \setminus B, *)/H$.

For the first statement, see [9], for example, and for the last two statements, see [6], EXPOSE XII, for example.

Let G be a finite group. We call a Galois cover with $\text{Aut}_M(X) \cong G$ a G -cover.

The following lemma is fundamental throughout this article. Note that Tan proved a similar theorem ([11], Corollary 2.3) using algebro-geometric methods.

Lemma 1.1. *Let M and B be as above. Let D_6 be the dihedral group of order 6, which we describe by $\langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1 \rangle$. If there exists a non-Galois triple cover $\pi : X \rightarrow M$ with $\Delta_\pi = B$, then there exists a D_6 -cover $\hat{\pi} : \hat{X} \rightarrow M$, with $\Delta_{\hat{\pi}} = B$. Conversely, if there exists a D_6 -cover $\hat{\pi} : \hat{X} \rightarrow M$ with $\Delta_{\hat{\pi}} = B$, then the quotient surface $X := \hat{X}/\langle \sigma \rangle$ by σ admits a non-Galois triple cover $\pi : X \rightarrow M$ with $\Delta_\pi = B$.*

Proof. Let $\pi : X \rightarrow M$ be a given non-Galois triple cover of M as above. Then there exists a corresponding subgroup H of $\pi_1(M \setminus B, *)$ with $\pi = \pi_H$ and $[\pi_1(M \setminus B, *) : H] = 3$. By considering the permutation action over the set of all left cosets of H , we have a homomorphism from $\pi_1(M \setminus B, *)$ to the symmetric group \mathfrak{S}_3 of three letters and its image is a transitive subgroup of $\mathfrak{S}_3 \cong D_6$ (see [10, Chapter I, §7]). Since π is not Galois, H is not normal. This implies that there exists an

epimorphism $\pi_1(M \setminus B, *) \rightarrow D_6$ such that $K := \ker(\pi_1(M \setminus B, *) \rightarrow D_6) \subset H$. The D_6 -cover corresponding to K satisfies the desired property. Conversely, suppose that there exists a D_6 -cover $\varpi : \mathcal{X} \rightarrow M$ with $\Delta_\varpi = B$. As we have an epimorphism $\pi_1(M \setminus B, *) \rightarrow D_6$, there exists a subgroup, H , of $\pi_1(M \setminus B, *)$ such that $[\pi_1(M \setminus B, *) : H] = 3$ and H contains $\ker(\pi_1(M \setminus B, *) \rightarrow D_6)$. The triple cover corresponding to H is the desired one. As for the equality $\Delta_\varpi = \Delta_{\pi_H}$, see [12]. \square

Remark 1.1. Let $\pi : X \rightarrow M$ be as in Lemma 1.1. There exists a double cover corresponding to the preimage of the subgroup of order 3 in D_6 . We denote it by $\beta_1(\pi) : D(X/M) \rightarrow M$. Note that $\Delta_{\beta_1(\pi)} \subseteq \Delta_\pi$.

2. PROOF OF THEOREM 0.1

2.1. **Settings.** Throughout this section, we always assume that

(*) a local hyperelliptic fibration is of horizontal type.

Let B_1 be the branch locus of $f_o : W \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$ as in the Introduction. Let B_o be a double section of $\text{pr}_2 : \mathbb{P}^1 \times \Delta_\epsilon \rightarrow \Delta_\epsilon$ such that

- B_o is smooth,
- $\text{pr}_2|_{B_o} : B_o \rightarrow \Delta_\epsilon$ has a unique ramification point over $0 \in \Delta_\epsilon$, and
- $B_o \cap B_1 = \emptyset$.

Put $B := B_o + B_1$ and $G_B := \pi_1(\mathbb{P}^1 \times \Delta_\epsilon \setminus B, *)$. In order to study non-Galois triple covers with branch locus B , by Lemma 1.1, we need to know the description of G_B and its normal subgroup K with $G_B/K \cong D_6$.

2.2. **A description of G_B .** We describe G_B via generators and their relations. The method used here is well-known in computing the fundamental group of the complements of plane curves via the so-called ‘‘Zariski-van Kampen method’’. We refer to [1, 2, 5, 9] and use results there freely.

Choose a point t_o in $\Delta_\epsilon^* := \Delta_\epsilon \setminus \{0\}$. Put $\mathbb{P}_{t_o}^1 := \mathbb{P}^1 \times \{t_o\}$ and $\mathbf{Y} := \mathbb{P}_{t_o}^1 \cap B$. Let η be a loop given by $\{|t_o| \exp(2\pi i\theta) \mid 0 \leq \theta \leq 1\}$. Note that $\pi_1(\Delta_\epsilon^*, t_o) \cong \langle \eta \rangle \cong \mathbb{Z}$. Choose a point $b = (\hat{t}_o, t_o) \in (\mathbb{P}^1 \times \Delta_\epsilon^*) \setminus B$ with $(\{\hat{t}_o\} \times \Delta_\epsilon) \cap B = \emptyset$ and a geometric basis $\gamma_1, \gamma_2, \dots, \gamma_{2g+4}$ of $\pi_1(\mathbb{P}_{t_o}^1 \setminus \mathbf{Y}, t_o)$ (see [1, Definition 1.13] for the definition of a geometric basis). One can define a right action of $\pi_1(\Delta_\epsilon^*, t_o)$ on $\pi_1(\mathbb{P}_{t_o}^1 \setminus \mathbf{Y}, \hat{t}_o)$, and G_B is described through this action as follows:

$$G_B \cong \langle \gamma_1, \dots, \gamma_{2g+4} \mid \gamma_i^\eta = \gamma_i \ (i = 1, \dots, 2g + 4), \ \gamma_1 \dots \gamma_{2g+4} = 1 \rangle.$$

We may assume that

$\gamma_1, \dots, \gamma_{2g+2}$ are meridians for the points in $\mathbb{P}_{t_o}^1 \cap B_1$,
and

γ_{2g+3} and γ_{2g+4} are meridians for $\mathbb{P}_{t_o}^1 \cap B_o$.

Under these circumstances, we have

Lemma 2.1.

$$G_B \cong \langle \gamma_1, \dots, \gamma_{2g+4} \mid \gamma_i^\eta = \gamma_i \ (i = 1, \dots, 2g + 2), \ \gamma_{2g+3} = \gamma_{2g+4}, \ \gamma_1 \dots \gamma_{2g+4} = 1 \rangle.$$

Let H be the subgroup of G_B generated by

$$\gamma_i^2 \ (i = 1, \dots, 2g + 4) \text{ and } \gamma_j \gamma_{j+1} \ (j = 1, \dots, 2g + 3).$$

Let $f : S \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$ be a double cover with $\Delta_f = B$.

Lemma 2.2. *H is the subgroup of G_B corresponding to the double cover*

$$f' := f|_{S \setminus f^{-1}(B)} : S \setminus f^{-1}(B) \rightarrow (\mathbb{P}^1 \times \Delta_\epsilon) \setminus B.$$

Proof. Put $\bar{\varphi} := \text{pr}_2 \circ f$ and $S_{t_o} := \bar{\varphi}^{-1}(t_o)$. Let $f_{t_o} : S_{t_o} \rightarrow \mathbb{P}_{t_o}^1$ be the restriction of f to the fiber over t_o and put $f'_{t_o} := f_{t_o}|_{S_{t_o} \setminus f_{t_o}^{-1}(\mathbf{Y})}$. We have a commutative diagram:

$$\begin{array}{ccc} S_{t_o} \setminus f_{t_o}^{-1}(\mathbf{Y}) & \longrightarrow & S \setminus f^{-1}(B) \\ f'_{t_o} \downarrow & & \downarrow f' \\ \mathbb{P}_{t_o}^1 \setminus \mathbf{Y} & \longrightarrow & (\mathbb{P}^1 \times \Delta_\epsilon) \setminus B. \end{array}$$

From the above diagram, we have a commutative diagram of groups:

$$\begin{array}{ccc} \tilde{N} & \longrightarrow & N \\ (f'_{t_o})_\# \downarrow & & \downarrow (f')_\# \\ \tilde{G} := \pi_1(\mathbb{P}_{t_o}^1 \setminus \mathbf{Y}, \hat{t}_o) & \longrightarrow & G_B, \end{array}$$

where \tilde{N} and N are the subgroups of index 2 corresponding to the double covers f'_{t_o} and f' ; i.e., \tilde{N} and N can be considered as the fundamental groups of $S_{t_o} \setminus f_{t_o}^{-1}(\mathbf{Y})$ and $S \setminus f^{-1}(B)$, respectively. Since $S \setminus f^{-1}(B \cup \mathbb{P}^1 \times \{0\}) \rightarrow \Delta_\epsilon \setminus \{0\}$ is a submersion, $S \setminus f^{-1}(B \cup \mathbb{P}^1 \times \{0\})$ is a fiber bundle over $\Delta_\epsilon \setminus \{0\}$ whose fiber is $S_{t_o} \setminus f_{t_o}^{-1}(\mathbf{Y})$. Hence we have an exact sequence

$$\pi_1(S_{t_o} \setminus f_{t_o}^{-1}(\mathbf{Y})) \rightarrow \pi_1(S \setminus f^{-1}(B \cup (\mathbb{P}^1 \times \{0\}))) \rightarrow \pi_1(\Delta_\epsilon \setminus \{0\}) \rightarrow 1,$$

where the base points of each fundamental group are appropriately chosen.

Now we can apply the argument of the proof of [9, Lemma 1.5, C]. We have

$$\pi_1(S_{t_o} \setminus f_{t_o}^{-1}(\mathbf{Y})) \rightarrow \pi_1(S \setminus f^{-1}(B)) \rightarrow \pi_1(\Delta_\epsilon) \rightarrow 1.$$

Hence $\tilde{N} \rightarrow N$ is surjective. As $[G_B : N] = 2$, our statement follows from the claim below:

Claim. Let \tilde{H} be the subgroup of \tilde{G} generated by

$$\gamma_i^2 \ (i = 1, \dots, 2g + 4) \text{ and } \gamma_j \gamma_{j+1} \ (j = 1, \dots, 2g + 3).$$

Then $\tilde{H} = \tilde{N}$.

Proof of Claim. Since the branch locus of $f_{t_o} : S_{t_o} \rightarrow \mathbb{P}_{t_o}^1$ is \mathbf{Y} , $\tilde{H} \subset \tilde{N}$. It is enough to show that $\tilde{G} = \tilde{H} \cup \gamma_1^{-1} \tilde{H}$, i.e., $[\tilde{G} : \tilde{H}] = 2$.

Step 1. $\gamma_i^\pm \in \gamma_1^{-1} \tilde{H}$.

We first note that $\gamma_1 = \gamma_1^{-1} \gamma_1^2 \in \gamma_1^{-1} \tilde{H}$. Suppose that $\gamma_i \in \gamma_1^{-1} \tilde{H}$ and put $\gamma_i = \gamma_1^{-1} h_i, h_i \in \tilde{H}$. Then

$$\gamma_{i+1} = \gamma_i^{-1} \gamma_i \gamma_{i+1} = \gamma_i \gamma_i^{-2} (\gamma_i \gamma_{i+1}) = \gamma_1^{-1} h_i \gamma_i^{-2} (\gamma_i \gamma_{i+1}) \in \gamma_1^{-1} \tilde{H}.$$

Thus $\gamma_i \in \gamma_1^{-1} \tilde{H}$ for $i = 1, \dots, 2g + 4$. Also $\gamma_i^{-1} = \gamma_i \gamma_i^{-2} \in \gamma_1^{-1} \tilde{H}$.

Step 2. Any element on $w \in \tilde{G}$ is in either \tilde{H} or $\gamma_1^{-1} \tilde{H}$.

Since

$$\tilde{G} = \langle \gamma_1, \dots, \gamma_{2g+4} \mid \gamma_1 \gamma_2 \cdots \gamma_{2g+4} = 1 \rangle,$$

we may assume that any $w \in \tilde{G}$ is of the form

$$w = w_1 \cdots w_M,$$

where $w_k = \gamma_i$ or γ_i^{-1} for some i .

Suppose that M is even. As $w = (w_1 w_2) \cdots (w_i w_{i+1}) \cdots (w_{M-1} w_M)$, it is enough to show that

$$\gamma_i^\pm \gamma_j^\pm, \gamma_i^\pm \gamma_j^\mp \in \tilde{H}$$

for any i, j . Since

$$\gamma_i^{-1} \gamma_j^{-1} = (\gamma_j \gamma_i)^{-1}, \quad \gamma_i^{-1} \gamma_j = \gamma_i^{-2} \gamma_i \gamma_j, \quad \gamma_i \gamma_j^{-1} = \gamma_i \gamma_j \gamma_j^{-2},$$

we only need to show that $\gamma_i \gamma_j \in \tilde{H}$ for any i, j . There are three possibilities: namely, (i) $i = j$, (ii) $i < j$ and (iii) $i > j$, and we check each case separately.

The case (i). $\gamma_i \gamma_j = \gamma_i^2 \in \tilde{H}$.

The case (ii). Since

$$\gamma_i \gamma_j = (\gamma_i \gamma_{i+1}) \gamma_{i+1}^{-2} (\gamma_{i+1} \gamma_{i+2}) \cdots \gamma_{j-1}^{-2} \gamma_j^{-1} \gamma_j,$$

$\gamma_i \gamma_j \in \tilde{H}$.

The case (iii). Since

$$\gamma_i \gamma_j = (\gamma_i \gamma_{i-1}) \gamma_{i-1}^{-2} (\gamma_{i-1} \gamma_{i-2}) \cdots \gamma_{j+1}^{-2} \gamma_{j+1} \gamma_j$$

and

$$\gamma_{k+1} \gamma_k = \gamma_{k+1}^2 (\gamma_{k+1}^{-1} \gamma_k^{-1}) \gamma_k^2 = \gamma_{k+1}^2 (\gamma_k \gamma_{k+1})^{-1} \gamma_k^2,$$

we infer that $\gamma_i \gamma_j \in \tilde{H}$. Thus $w \in \tilde{H}$ when M is even.

Suppose that M is odd. In this case, w is of the form $\gamma_i \tilde{w}$ or $\gamma_i^{-1} \tilde{w}$ for some i and $\tilde{w} \in \tilde{H}$ by Case 1. Since $\gamma_i^\pm \in \gamma_1^{-1} \tilde{H}$, we infer that $w \in \gamma_1^{-1} \tilde{H}$.

From Step 1 and Step 2, we have the Claim and Lemma 2.2 follows. □

2.3. Existence of a D_6 -cover. In this section, we show the existence of a normal subgroup K of G_B such that (i) K is a subgroup of H , (ii) $G_B/K \cong D_6$ and (iii) the D_6 -cover corresponding to K is branched along B with ramification index 2.

Let $\mu : \tilde{S} \rightarrow S$ be the canonical resolution of the double cover $f : S \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$. Since singularities of S are on those of B , we have $S \setminus f^{-1}(B) \cong \tilde{S} \setminus (f \circ \mu)^{-1}(B)$. Hence we have an epimorphism $\bar{\delta} : \pi_1(S \setminus f^{-1}(B), b_+) \rightarrow \pi_1(\tilde{S}, b_+)$, where b_+ is a point in $f^{-1}(b)$. In particular, we have an epimorphism $\delta : H \rightarrow H_1(\tilde{S}, \mathbb{Z})$.

Lemma 2.3. *For any $h \in H$, we have*

$$\delta(\gamma_1^{-1} h \gamma_1) = -\delta(h) = \delta(h^{-1}).$$

Proof. We first note that $\gamma_i^2 \in \ker \delta$ for $i = 1, \dots, 2g + 4$. Since

$$\gamma_1^{-1} (\gamma_i \gamma_{i+1}) \gamma_1 = (\gamma_i \gamma_1)^{-1} \gamma_i^2 \gamma_{i+1}^2 (\gamma_i \gamma_{i+1})^{-1} (\gamma_i \gamma_1),$$

we have

$$\delta(\gamma_1^{-1} (\gamma_i \gamma_{i+1}) \gamma_1) = -\delta(\gamma_i \gamma_{i+1}).$$

Let h be an arbitrary element in H and suppose that h is of the form $u_1 \cdots u_n$, where u_i ($i = 1, \dots, n$) are either $(\gamma_j^2)^\pm$ or $(\gamma_l \gamma_{l+1})^\pm$ for some j, l . Then

$$\begin{aligned} \delta(\gamma_1^{-1} h \gamma_1) &= \sum_{i=1}^n \delta(\gamma_1^{-1} u_i \gamma_1) \\ &= -\sum_{i=1}^n \delta(u_i) \\ &= -\delta(h). \end{aligned} \quad \square$$

Let A be a subgroup of $H_1(\tilde{S}, \mathbb{Z})$ and put $K_A := \delta^{-1}(A)$. We have

Lemma 2.4. K_A is a normal subgroup of G_B .

Proof. Let k be an arbitrary element of K_A . By Lemma 2.3, $\delta(\gamma_1^{-1} k \gamma_1) = -\delta(k) \in A$. Hence $\gamma_1^{-1} k \gamma_1 \in K_A$. As $H \triangleright K_A$ and $G_B = H \cup \gamma_1^{-1} H = H \cup \gamma_1 H$, we have $G_B \triangleright K_A$. \square

Corollary 2.1. Let A be a subgroup of $H_1(\tilde{S}, \mathbb{Z})$ of index 3. Then K_A is a normal subgroup of G_B such that $G_B/K_A \cong D_6$.

Proof. Choose $h \in H \setminus K_A$ such that $H = K_A \cup hK_A \cup h^{-1}K_A$. By Lemma 2.3, we have $\delta(\gamma_1^{-1} h \gamma_1) = \delta(h^{-1})$. Hence we have $\gamma_1^{-1} h \gamma_1 \in h^{-1}K_A$. This implies that G_B/K_A is non-abelian. \square

2.4. Proof of Theorem 0.1. We keep our notation as before. Put $\varphi = \text{pr}_2 \circ f \circ \mu$. We first note that $H_1(\tilde{S}, \mathbb{Z}) \cong H_1(|\varphi^{-1}(0)|, \mathbb{Z})$, where $|\bullet|$ denotes the underlying topological space of \bullet by [3, Theorem 8.8, Chapter I] and [8]. We call the irreducible component of $\varphi^{-1}(0)$ coming from $\mathbb{P}^1 \times \{0\}$ the main component. Our first observation is as follows:

Lemma 2.5. Let $\overline{\gamma_{2g+2}\gamma_{2g+3}}$ be the class of $\gamma_{2g+2}\gamma_{2g+3}$ in $H_1(\varphi^{-1}(0), \mathbb{Z})$. Then

$$H_1(|\varphi_o^{-1}(0)|, \mathbb{Z}) \oplus \mathbb{Z}\overline{\gamma_{2g+2}\gamma_{2g+3}} \cong H_1(|\varphi^{-1}(0)|, \mathbb{Z}).$$

Proof. We first note that the closure of $\varphi^{-1}(\mathbb{P}_0^1 \setminus (B \cap \mathbb{P}_0^1))$ in \tilde{S} gives either an irreducible curve Γ_0 or two irreducible curves Γ_0^+ and Γ_0^- . We call them the *essential component(s)* of $\varphi^{-1}(0)$. Note that $\Gamma_0^\pm \cong \mathbb{P}^1$ since f_o is a double cover.

By our construction, $|\varphi^{-1}(0)|$ is obtained by attaching a “pinched” handle as in Figure 1.

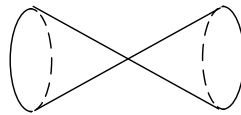


FIGURE 1. A pinched handle

Thus $|\varphi^{-1}(0)| \setminus \text{a pinched handle}$ is homotopic to U_1 , U_1 is either $(\Gamma_0^+ \setminus \text{a point}) \cup (\Gamma_0^- \setminus \text{a point}) \cup \text{other remaining components of } \varphi^{-1}(0)$ or $\Gamma_0 \setminus \text{two points} \cup \text{other remaining components}$.

Now let U_2 be the pinched handle body as in Figure 1. By applying the Mayer-Vietoris exact sequence to U_1 and U_2 , we have:

$$\begin{aligned} H_2(U_1 \cap U_2, \mathbb{Z}) &\rightarrow H_2(U_1, \mathbb{Z}) \oplus H_2(U_2, \mathbb{Z}) \rightarrow H_2(|\varphi^{-1}(0)|, \mathbb{Z}) \rightarrow \\ H_1(U_1 \cap U_2, \mathbb{Z}) &\rightarrow H_1(U_1, \mathbb{Z}) \oplus H_1(U_2, \mathbb{Z}) \rightarrow H_1(|\varphi^{-1}(0)|, \mathbb{Z}) \rightarrow \\ H_0(U_1 \cap U_2, \mathbb{Z}) &\rightarrow H_0(U_1, \mathbb{Z}) \oplus H_0(U_2, \mathbb{Z}) \rightarrow H_0(|\varphi^{-1}(0)|, \mathbb{Z}) \rightarrow 0. \end{aligned}$$

Also by replacing U_2 by two disjoint discs, we have

$$\begin{aligned} H_2(U_1 \cap U_2, \mathbb{Z}) &\rightarrow H_2(U_1, \mathbb{Z}) \oplus H_2(U_2, \mathbb{Z}) \rightarrow H_2(|\varphi_o^{-1}(0)|, \mathbb{Z}) \rightarrow \\ H_1(U_1 \cap U_2, \mathbb{Z}) &\rightarrow H_1(U_1, \mathbb{Z}) \oplus H_1(U_2, \mathbb{Z}) \rightarrow H_1(|\varphi_o^{-1}(0)|, \mathbb{Z}) \rightarrow \\ H_0(U_1 \cap U_2, \mathbb{Z}) &\rightarrow H_0(U_1, \mathbb{Z}) \oplus H_0(U_2, \mathbb{Z}) \rightarrow H_0(|\varphi_o^{-1}(0)|, \mathbb{Z}) \rightarrow 0. \end{aligned}$$

For both exact sequences, we have that $U_1 \cap U_2$ is homotopic to a disjoint union of two disjoint circles S^1 . Hence we infer that

$$H_1(|\varphi_o^{-1}(0)|, \mathbb{Z}) \cong \text{Coker}(H_1(U_1 \cap U_2, \mathbb{Z}) \rightarrow H_1(U_1, \mathbb{Z})).$$

Hence we have

$$H_1(|\varphi_o^{-1}(0)|, \mathbb{Z}) \oplus \mathbb{Z} \cong H_1(|\varphi^{-1}(0)|, \mathbb{Z}).$$

Since $\overline{\gamma_i \gamma_{i+1}}$ ($i = 1, \dots, \gamma_{2g+1}$) generates $H_1(|\varphi_o^{-1}(0)|, \mathbb{Z})$ and $\overline{\gamma_{2g+3} \gamma_{2g+4}}$ is 0 in $H_1(|\varphi^{-1}(0)|, \mathbb{Z})$, we have our statement. \square

Put $A = H_1(|\varphi_o^{-1}(0)|, \mathbb{Z}) \oplus \langle \overline{(\gamma_{2g+2} \gamma_{2g+3})^3} \rangle$, let $q : \widehat{X} \rightarrow \widetilde{S}$ be the cyclic triple cover corresponding to K_A in H , and let $\nu : \widehat{X} \rightarrow \widehat{S}$ be the Stein factorization of $\widehat{X} \rightarrow \widetilde{S} \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$. By Corollary 2.1, \widehat{S} is a D_6 -cover, and we denote its covering morphism by $\widehat{\pi} : \widehat{S} \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$. By our construction, $\widehat{\pi}^{-1}(\mathbb{P}^1 \times \{0\})$ satisfies the following conditions:

- $\widehat{\pi}^{-1}(\mathbb{P}^1 \times \{0\})$ consists of three irreducible curves F_1, F_2 and F_3 .
- We may assume that $\sigma^*F_1 = F_1, \sigma^*F_2 = F_3$ and $\tau^*F_1 = F_2, \tau^*F_2 = F_3$, where σ and τ are the elements of D_6 as in §1.
- For each $i, \nu^{-1}(F_i)$ is isomorphic to $\varphi_o^{-1}(0)$. The involution induced by σ acts on $\nu^{-1}(F_i)$ in the same way as the involution on \widetilde{W} induced by the covering transformation of $f_o : W \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$ acts on $f_o^{-1}(\mathbb{P}^1 \times \{0\})$.

Let X be the quotient surface of \widehat{S} by σ . By Lemma 1.1, X is a non-Galois triple cover of $\mathbb{P}^1 \times \Delta_\epsilon$ with branch locus B , and we denote its covering morphism by $\pi : X \rightarrow \mathbb{P}^1 \times \Delta_\epsilon$. Let $\varphi_X : X \rightarrow \Delta_\epsilon$ be the induced fibration. Then we have

- $\varphi_X^{-1}(t)$ is a smooth curve of genus g for $t \neq 0$ such that $\pi|_{\varphi_X^{-1}(t)} : \varphi_X^{-1}(t) \rightarrow \mathbb{P}^1 \times \{t\}$ is a 3-to-1 morphism.
- $\varphi_X^{-1}(0)$ consists of two reduced components G_1 and G_2 , where G_1 is the image of F_1 and G_2 is the image of both F_2 and F_3 . Since $G_1 \cong \mathbb{P}^1$ and $G_1 G_2 = 1$, G_1 is an exceptional curve of the first kind. Also G_2 is isomorphic to $f_o^{-1}(\mathbb{P}^1 \times \{0\})$.

Let x_o be a singular point of X . $y_o = \pi(x_o)$ is a singular point of B_1 , and there exists a small neighborhood U_{y_o} and V_{x_o} of y_o and x_o , respectively, such that $\pi(V_{x_o}) = U_{y_o}$ and $\pi|_{V_{x_o}} : V_{x_o} \rightarrow U_{y_o}$ is a double cover. Since σ just exchanges F_2 and F_3 , $\pi|_{V_{x_o}}$ is locally isomorphic to f_o over U_{y_o} . We now blow down G_1 and take the canonical resolution \widetilde{X} of all singularities of X . Since G_1 is a vertical divisor, φ_X induces another fibration $\varphi_{\widetilde{X}} : \widetilde{X} \rightarrow \Delta_\epsilon$, which gives the desired local trigonal fibration in Theorem 0.1.

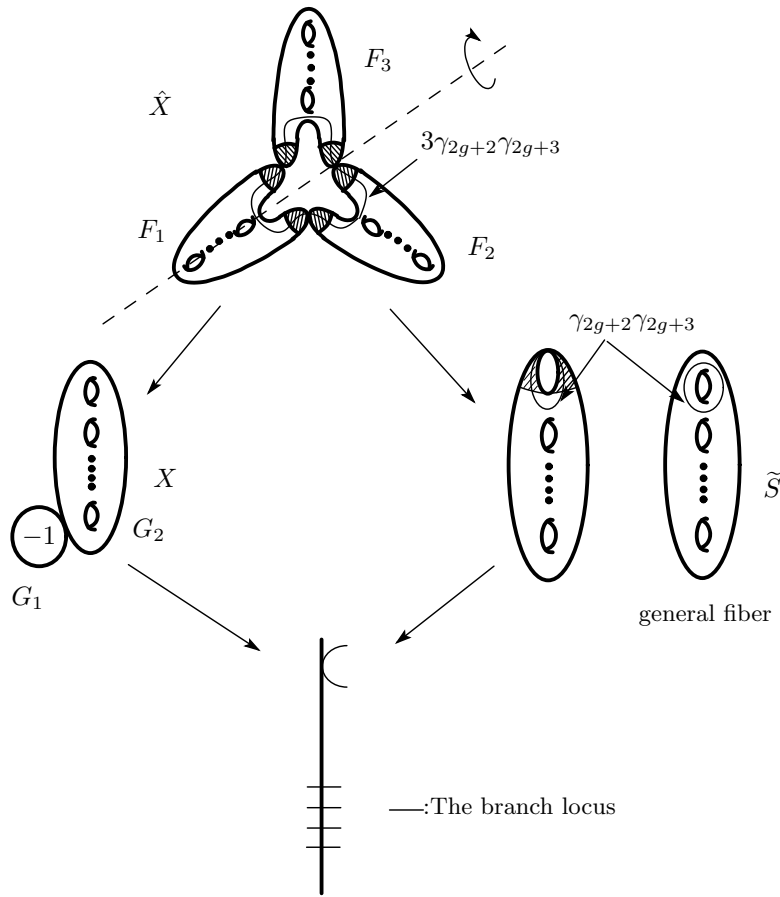


FIGURE 2

In the case when B_1 is smooth, the above observation can be explained in Figure 2.

3. AN EXAMPLE FOR THE NON-HORIZONTAL CASE

In this section, we give a local trigonal fibration of genus 3 such that the central fiber is given as the one for a local hyperelliptic fibration of non-horizontal type.

Note that some of the local trigonal fibrations of non-horizontal type can be reduced to those of horizontal type by considering elementary transformations at $\mathbb{P}^1 \times \{0\}$. Our example, given below, is the one which cannot be reduced to non-horizontal type.

Example 3.1. Let B_1 be a reduced divisor on $\mathbb{P}^1 \times \Delta_\epsilon$ given by

$$B_1 : t(x^4 - t)(x^4 + t) = 0,$$

where $x = X_1/X_0$, $[X_0, X_1]$ being a homogeneous coordinate of \mathbb{P}^1 and t is a coordinate of Δ_ϵ . Let $\varphi_o : \mathcal{C} \rightarrow \Delta_\epsilon$ be the local hyperelliptic fibration obtained as in the Introduction. φ_o is not relatively minimal and let $\bar{\varphi}_o : \bar{\mathcal{C}} \rightarrow \Delta_\epsilon$ be its

relatively minimal model. We denote its central fiber by F_0 . The configuration of F_0 is

$$F_0 = 4E + 2C_1 + 2C_2,$$

where E is a curve of genus 1 and $C_i, i = 1, 2$, are smooth rational curves with $E^2 = -1, C_i^2 = -2, EC_i = 1$ and $C_1C_2 = 0$.

We first note that $\overline{\varphi}_o : \overline{C} \rightarrow \Delta_\epsilon$ in Example 3.1 is never obtained as the relatively minimal model of a local hyperelliptic fibration of horizontal type. In fact, suppose that there exists a local hyperelliptic fibration of horizontal type such that the relatively minimal model of $\varphi_1 : C' \rightarrow \Delta_\epsilon$ is $\overline{\varphi}_o$. This means that C' is obtained from \overline{C} by successive blowing-ups. Since F_0 has no reduced component, we infer that the central fiber of φ_1 also has no reduced component. On the other hand, the irreducible component of the central fiber of φ_1 arising from $\mathbb{P}^1 \times \{0\}$ is reduced by taking a local section into account. This leads us to a contradiction.

Therefore we cannot apply Theorem 0.1 to obtain a local trigonal fibration with central fiber F_0 in Example 3.1. Nevertheless, there exists a local trigonal fibration with central fiber F_0 . We end this section by constructing such an example explicitly.

Example 3.2. Let us start with a family of plane quartic curves as follows:

Let $[X_0, X_1, X_2]$ be the homogeneous coordinates of \mathbb{P}^2 . Consider the surface H_0 of $\mathbb{P}^2 \times \Delta_\epsilon$ defined by

$$(X_1X_2 - X_0^2)^2 + t^8(X_1^4 - X_2^4) = 0.$$

Let $\pi_0 : H_0 \rightarrow \Delta_\epsilon$ be the morphism induced from the second projection $\mathbb{P}^2 \times \Delta_\epsilon \rightarrow \Delta_\epsilon$. We see that the fiber $\pi_0^{-1}(0)$ is the singular locus of H_0 . Since the fibers $\pi_0^{-1}(t)$ ($t \neq 0$) are non-singular plane curves of degree four, they are non-hyperelliptic, i.e., trigonal, curves of genus 3. We blow up $\mathbb{P}^2 \times \Delta_\epsilon$ along the ideal generated by $X_1X_2 - X_0^2$ and t^4 . Setting $Z_0t^4 = X_1X_2 - X_0^2$, we obtain the defining equation of the proper transformation H_1 of H_0 in the exceptional set as

$$Z_0^2 + (X_1^4 - X_2^4) = 0.$$

The family $\pi_1 : H_1 \rightarrow \Delta_\epsilon$ is non-singular. Moreover, since the restriction $h : \pi_1^{-1}(0) \rightarrow \pi_0^{-1}(0)$ of the morphism $H_1 \rightarrow H_0$ is the double cover branched at eight points $X_1X_2 - X_0^2 = X_1^4 - X_2^4 = t = 0$, the fiber $\pi_1^{-1}(0)$ is a hyperelliptic curve.

We now define the automorphism of $G : H_0 \rightarrow H_0$ induced from the automorphism of $\mathbb{P}^2 \times \Delta_\epsilon$ as

$$X_0 \mapsto iX_0, \quad X_1 \mapsto -X_1, \quad X_2 \mapsto X_2, \quad t \mapsto it,$$

where $i = \sqrt{-1}$. The fixed points of G are $P_1 = [0, 0, 1]$ and $P_2 = [0, 1, 0]$ on $\mathbb{P}^2 \times \{0\}$. We can naturally define the automorphism $G' : H_1 \rightarrow H_1$ of H_1 induced from G . Note that G' acts on the coordinate Z_0 as $Z_0 \mapsto -Z_0$. Consider the quotient S of H_1 by the cyclic group $\langle G' \rangle$ generated by G' . We see that $(G')^4$ is the identity and $(G')^2$ has four fixed points on $\pi_1^{-1}(0)$ which are the inverse image of P_1 and P_2 by h . Since G' acts on the coordinate Z_0 as $Z_0 \mapsto -Z_0$ and G' interchanges the two points of the inverse image of P_i ($i = 1, 2$) by h , we see that the restriction $G'|_{\pi_1^{-1}(0)}$ of G' is the automorphism of order four and the quotient $\pi_1^{-1}(0)/\langle G'|_{\pi_1^{-1}(0)} \rangle$ is an elliptic curve.

Thus, we see that the singular fiber $\phi^{-1}(0)$ of $\phi: S \rightarrow \Delta_\varepsilon$ is the elliptic curve with multiplicity four and S has two rational double points of type A_1 on its singular fiber. Then, the resolution $\psi: \tilde{S} \rightarrow \Delta_\varepsilon$ of the family $\phi: S \rightarrow \Delta_\varepsilon$ has the singular fiber F_0 and its general fibers are non-hyperelliptic curves.

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