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NICHOLS ALGEBRAS ASSOCIATED TO THE TRANSPOSITIONS OF THE SYMMETRIC GROUP ARE TWIST-EQUIVALENT

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Abstract. Using the theory of covering groups of Schur we prove that the two Nichols algebras associated to the conjugacy class of transpositions in $\mathbb{S}_n$ are equivalent by twist and hence they have the same Hilbert series. These algebras appear in the classification of pointed Hopf algebras and in the study of the quantum cohomology ring of flag manifolds.

1. Introduction

Nichols algebras play a fundamental role in the classification of finite-dimensional pointed Hopf algebras over $\mathbb{C}$. They are graded Hopf algebras

$$\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V) = \mathbb{C} \oplus V \oplus \mathfrak{B}^2(V) \oplus \cdots$$

in the category of Yetter-Drinfeld modules over a Hopf algebra H, and they are uniquely determined by V, the homogeneous component of $\mathfrak{B}(V)$ of degree one.

Let H be the group algebra of a finite group G. In the study of Nichols algebras a basic question is to describe those Yetter-Drinfeld modules V over H for which $\mathfrak{B}(V)$ is finite-dimensional. Whereas deep results were found for the case where G is abelian, [5, 16, 17], the situation is widely unknown for non-abelian groups G.

The first examples of finite-dimensional pointed Hopf algebras with non-abelian coradical appeared in [21], as bosonizations of Nichols algebras related to the transpositions in $\mathbb{S}_3$ and $\mathbb{S}_4$. The analogous Nichols algebra over $\mathbb{S}_5$ was computed by Graña; see [14]. These Nichols algebras are computed from the conjugacy class of transpositions and a 2-cocycle (cocycle for short) associated to this conjugacy class. The cocycles arise from a cohomology theory defined for racks (see for example [3, 9, 13]). In [2], Theorem 1.1, it is proved that for all $n \in \mathbb{N}$, $n \geq 4$, there are precisely two rack 2-cocycles associated to the conjugacy class of transpositions in $\mathbb{S}_n$ that might have finite-dimensional Nichols algebras. Explicitly, one of these cocycles is the constant cocycle -1. The other one is the cocycle given by

$$\chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{if } \sigma(i) > \sigma(j) \end{cases} \tag{1}$$

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for all transpositions σ and $\tau = (i\ j)$ with $i < j$. For all $n \in \{4,5\}$ the Nichols algebra associated to the conjugacy class of transpositions in $\mathbb{S}_n$ and any of the two cocycles -1, χ is finite-dimensional. Moreover, both of these algebras have the same Hilbert series. It is not known whether these algebras are finite-dimensional for $n > 5$. The main result of this work is to connect these two algebras by *twisting the cocycle*. More precisely, we prove that the constant cocycle -1 and χ are equivalent by a twist. This gives an affirmative answer to a question due to Andruskiewitsch; see [1], Question 7. However, the problem arose already earlier in the literature. For example, in the last paragraph of [19], Majid discusses the relationship between these two algebras and the related quadratic algebras. To reach our main result, we use the existence of projective representations of $\mathbb{S}_n$. Projective representations of $\mathbb{S}_n$ were originally studied by Schur in 1911; see [22] for an English translation of his fundamental paper about this subject. As a corollary of our result we obtain that for all $n \geq 4$, both Nichols algebras associated to the conjugacy class of transpositions of $\mathbb{S}_n$ have the same Hilbert series.

We recall briefly another application for Nichols algebras which may have connections with the main result of this work. In [8], Borel identified the cohomology ring of a flag manifold with S_W, the algebra of coinvariants of the associated Coxeter group W. This admits certain divided-difference operators which create classes of Schubert manifolds. In [11], Fomin and Kirillov introduced a new model for the Schubert calculus of a flag manifold, realizing S_W as a commutative subalgebra of a non-commutative quadratic algebra $\mathcal{E}_W$, when W is a symmetric group. In [6], Bazlov proved that Nichols algebras provide the correct setting for this model for Schubert calculus on a flag manifold. It is an open problem whether the Nichols algebra associated to χ coincides with the quadratic algebra $\mathcal{E}_W$ [19, 21].

2. Preliminaries

2.1. **Racks and cohomology.** We briefly recall basic facts about racks; see [3] for more information and references.

A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is a function, such that the map $x \mapsto i \triangleright x$ is bijective for all $i \in X$, and $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. A *subrack* of X is a non-empty subset $Y \subseteq X$ such that $(Y, \triangleright)$ is also a rack.

Example 2.1. A group G is a rack with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If a subset $X \subseteq G$ is stable under conjugation by G, then it is a subrack of G. In particular, the conjugacy class of transpositions in $\mathbb{S}_n$ is a rack; it will be denoted by X_n.

In this work we are interested only in racks which can be realized as a finite conjugacy class of a group. Let X be such a rack. A map $q : X \times X \to \mathbb{C}^\times$ is a 2-*cocycle* if and only if

$$q_{x, y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z}$$

for all $x, y, z \in X$. We write $Z^2_R(X, \mathbb{C}^\times)$ for the set of all rack 2-cocycles. Let $q, q' \in Z^2_R(X, \mathbb{C}^\times)$. We write $q \sim q'$ if there exists $\gamma : X \to \mathbb{C}^\times$ such that

$$q_{x,y} = \gamma^{-1}_{x \triangleright y} q'_{x,y} \gamma_y$$

for all $x, y \in X$. Since $\sim$ is an equivalence relation and $Z^2_R(X, \mathbb{C}^\times)$ is stable under $\sim$ it is possible to define the *second rack cohomology group* as

$$H^2_R(X, \mathbb{C}^\times) = Z^2_R(X, \mathbb{C}^\times)/\sim .$$

All these notions are based on the abelian cohomology theory of racks proposed independently in [9], [13]. For more details about cohomology theories of racks, see [3], §4.

2.2. **Nichols algebras.** We refer to [4] for an introduction to Yetter-Drinfeld modules and Nichols algebras.

Let $n \in \mathbb{N}$. We recall the well-known presentation of the braid group $\mathbb{B}_n$ by generators and relations. The group $\mathbb{B}_n$ has generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\begin{aligned} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j &\quad \text{for } |i-j| = 1; \\ \sigma_i\sigma_j = \sigma_j\sigma_i &\quad \text{for } |i-j| > 1. \end{aligned}$$

There exists a canonical projection $\mathbb{B}_n \to \mathbb{S}_n$ that admits the so-called Matsumoto section $\mu : \mathbb{S}_n \to \mathbb{B}_n$ such that $\mu\left((i\ i+1)\right) = \sigma_i$. This section satisfies the following: $\mu(xy) = \mu(x)\mu(y)$ for any $x, y \in \mathbb{S}_n$ such that $l(xy) = l(x) + l(y)$, where l is the length function on $\mathbb{S}_n$.

Let V be a vector space over $\mathbb{C}$ and let $c \in \mathbf{GL}(V \otimes V)$. The map c is a solution of the *braid equation* if and only if

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

If c is a solution of the braid equation, we say that (V, c) is a *braided vector space.* A solution of the braid equation induces a representation

$$\rho_n : \mathbb{B}_n \to \mathbf{GL}(V^{\otimes n})$$

defined by

$$\rho_n(\sigma_i) = \mathrm{id}^{\otimes(i-1)} \otimes c \otimes \mathrm{id}^{\otimes(n-i-1)}$$

for $1 \leq i \leq n-1$. Let

$$Q_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n(\mu(\sigma)) : V^{\otimes n} \to V^{\otimes n}$$

for $n \geq 2$, $Q_0 = \mathrm{id}_{\mathbb{C}}$ and $Q_1 = \mathrm{id}_V$. The *Nichols algebra* associated to the braided vector space V is

$$\mathfrak{B}(V) = T(V) / \bigoplus_{n \geq 2} \ker Q_n \simeq \bigoplus_{n \geq 0} \operatorname{im} Q_n.$$

By [3], Theorem 4.14, Yetter-Drinfeld modules over group algebras can also be studied in terms of racks and rack 2-cocycles. Therefore we are interested in Nichols algebras of braided vector spaces arising from racks and 2-cocycles.

Let $(X, \triangleright)$ be a rack and let $q \in Z^2_R(X, \mathbb{C}^\times)$. We consider $V = \mathbb{C}X$, the vector space with basis $x \in X$, and define $c : V \otimes V \to V \otimes V$ as

$$c(x \otimes y) = q_{x,y} x \triangleright y \otimes x.$$

Then c is a solution of the braid equation. The Nichols algebra associated to the pair (X, q) is the Nichols algebra of the braided vector space (V, c). This algebra will be denoted by $\mathfrak{B}(X, q)$.

Recall that X_n is defined as the rack associated to the conjugacy class of transpositions in $\mathbb{S}_n$. In [2], Theorem 1.1, it is proved that there are two rack 2-cocycles associated to X_n that might have a finite-dimensional Nichols algebra. One is the constant 2-cocycle -1. The other is the 2-cocycle χ given by Equation (1).

Remark 2.2. It can be checked directly that the 2-cocycles -1 and χ associated to the rack X_3 are cohomologous. Then the Nichols algebras $\mathfrak{B}(X_3, \chi)$ and $\mathfrak{B}(X_3, -1)$ are isomorphic and hence they have the same Hilbert series.

For all $m \in \mathbb{N}$ let $(m)_t = 1 + t + \cdots + t^{m-1} \in \mathbb{Z}[t]$.

Example 2.3. The Nichols algebras $\mathfrak{B}(X_4, -1)$ and $\mathfrak{B}(X_4, \chi)$ both have dimension 576. In both cases the Hilbert series is

$$\mathcal{H}_4(t) = (2)_t^2 (3)_t^2 (4)_t^2.$$

These algebras appeared first in [11, 21]. For more information about these algebras, see [12], Theorem 2.4 and Proposition 2.5, or [15], Proposition 5.11.

Example 2.4. The Nichols algebras $\mathfrak{B}(X_5, -1)$ and $\mathfrak{B}(X_5, \chi)$ both have dimension 8294400. In both cases the Hilbert series is

$$\mathcal{H}_5(t) = (4)_t^4 (5)_t^2 (6)_t^4.$$

These algebras were first computed by Graña [14]. For more information about these algebras, see [12], Theorem 2.4 and Proposition 2.5, or [15], Proposition 5.13.

2.3. **Twisting.** In [1], Section 3.4 it is shown how to relate two rack 2-cocycles by a twisting in such a way that some properties of the corresponding Nichols algebras are preserved. This method is based on the twisting method of [10] and its relationship with the bosonization given in [20].

Let X be a subrack of a conjugacy class of a group G. Let q be a rack 2-cocycle on X and let ϕ be a group 2-cocycle on G. Define $q^\phi : X \times X \to \mathbb{C}^\times$ by

$$q^\phi_{x,y} = \phi(x,y)\phi(x \triangleright y, x)^{-1}\, q_{x,y} \tag{2}$$

for $x, y \in X$.

Remark 2.5. Let X be a rack and $q \in H^2_R(X, \mathbb{C}^\times)$. For a map $\phi : X \times X \to \mathbb{C}^\times$, define q^ϕ by Equation (2). Then q^ϕ is a rack 2-cocycle if and only if

$$\begin{aligned} &\phi(x,z)\phi(x \triangleright y, x \triangleright z)\phi(x \triangleright (y \triangleright z), x)\phi(y \triangleright z, y) \\ &\qquad = \phi(y,z)\phi(x, y \triangleright z)\phi(x \triangleright (y \triangleright z), x \triangleright y)\phi(x \triangleright z, x) \end{aligned} \tag{3}$$

for any $x, y, z \in X$. Thus, if X is a subrack of a group G and ϕ is a group 2-cocycle, $\phi \in Z^2(G, \mathbb{C}^\times)$, then $\phi|_{X \times X}$ satisfies equation (3). See [1], Remark 3.10, or [9], Theorem 7.1, for a similar result.

Lemma 2.6. *The Hilbert series of $\mathfrak{B}(X, q)$ and $\mathfrak{B}(X, q^\phi)$ are equal.*

Proof. See [1], Section 3.4. □

Definition 2.7. The 2-cocycles q and q' on X are *equivalent by twist* if there exists $\phi : X \times X \to \mathbb{C}^\times$ such that $q' = q^\phi$ as in (2).

3. The Schur cover of $\mathbb{S}_n$

3.1. **Projective representations and covering groups.** We review some aspects of Schur's theory of projective representations and construct the Schur cover of $\mathbb{S}_n$. See [7, 18, 22] for details.

A *projective representation* of a finite group G is a group homomorphism $G \to \mathbf{PGL}(V)$. Equivalently, such a representation may be viewed as a map $f : G \to \mathbf{GL}(V)$ such that

$$f(x)f(y) = \phi(x,y)f(xy)$$

for all $x, y \in G$ and suitable scalars $\phi(x,y) \in \mathbb{C}^\times$. The map $G \times G \to \mathbb{C}^\times$, $(x,y) \mapsto \phi(x,y)$, is called a *factor set*. The associativity of the group $\mathbf{GL}(V)$ implies the 2-*cocycle condition* of the factor set ϕ:

$$\phi(x,y)\phi(xy,z) = \phi(x,yz)\phi(y,z) \tag{4}$$

for all $x, y, z \in G$.

Two projective representations $\rho_1 : G \to \mathbf{GL}(V_1)$ and $\rho_2 : G \to \mathbf{GL}(V_2)$ are *equivalent* if there exists a $\mathbb{C}$-linear isomorphism $S : V_1 \to V_2$ and a map $b : G \to \mathbb{C}^\times$ such that

$$b(x)S\rho_1(x)S^{-1} = \rho_2(x)$$

for all $x \in G$. Two factor sets ϕ and ϕ' are *equivalent* if they differ only by a factor $b_x b_y / b_{xy}$ for some $b : G \to \mathbb{C}^\times$. The *Schur multiplier* of G is the abelian group of factor sets modulo equivalence. It is isomorphic to the second cohomology group $H^2(G, \mathbb{C}^\times)$.

Recall that a central extension of G is a pair (E, p), where $p : E \to G$ is a surjective group homomorphism such that $\ker p$ is contained in the center of the group E. Schur proved that every finite group G has a central extension (E, p) with the property that every projective representation ρ of G lifts to an ordinary representation $\bar{\rho}$ of E such that the diagram

$$\begin{tikzcd} E \arrow[r, "\bar{\rho}"] \arrow[d, "p"'] & \mathbf{GL}(V) \arrow[d] \\ G \arrow[r, "\rho"] & \mathbf{PGL}(V) \end{tikzcd}$$

commutes.

There exist extensions with $\ker p \simeq H^2(G, \mathbb{C}^\times)$. Moreover, $H^2(G, \mathbb{C}^\times)$ is the unique minimal possibility for $\ker p$. These minimal central extensions of G are called *Schur covering groups* of G.

We recall that $\mathbb{S}_n$ has a Coxeter group presentation

$$\mathbb{S}_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = (s_j s_{j+1})^3 = (s_k s_l)^2 = 1 \rangle \qquad (1 \le i \le n-1,\ 1 \le j \le n-2,\ k \le l-2),$$

where $s_1, \ldots, s_{n-1}$ denote the adjacent transpositions of $\mathbb{S}_n$ defined by

$$s_1 = (1\,2),\ s_2 = (2\,3), \ldots, s_{n-1} = (n-1\,n).$$

Theorem 3.1. *Given $n \ge 4$, define the group T_n as follows:*

$$T_n = \langle z, t_1, \ldots, t_{n-1} \mid t_i^2 = (t_j t_{j+1})^3 = 1,\ (t_k t_l)^2 = z,\ z^2 = [z, t_i] = 1 \rangle \qquad (1 \le i \le n-1,\ 1 \le j \le n-2,\ k \le l-2).$$

Then T_n is a Schur covering group of $\mathbb{S}_n$. Therefore, there exists a central extension

$$0 \to A \xrightarrow{i} T_n \xrightarrow{p} \mathbb{S}_n \to 1,$$

where $A = \langle z \rangle$.

Proof. See [18], Theorem 2.12.3. $\square$

3.2. **Transpositions in T_n.** We want to study the lifting of the conjugacy class of transpositions in $\mathbb{S}_n$. See [23], Section 3 for a detailed description about permutations in T_n.

Remark 3.2. Let $t \in T_n$. For any $\sigma \in \mathbb{S}_n$ we have that $p^{-1}(\sigma) = \{\bar\sigma, \bar\sigma z\}$. Since the involution z is a central element of T_n, the group $\mathbb{S}_n$ acts on T_n by conjugation:

$$\sigma \triangleright t = \bar\sigma t (\bar\sigma)^{-1} = (\bar\sigma z) t (\bar\sigma z)^{-1}.$$

Therefore it is possible to write the conjugation in T_n as $\sigma \triangleright t = \sigma t \sigma^{-1}$, where $t \in T_n$ and $\sigma \in \mathbb{S}_n$.

Definition 3.3. For $i, j \in \mathbb{N}$ such that $1 \leq i, j \leq n$, $i \neq j$, let $[i\ j]$ be an element of T_n defined inductively as

$$\begin{aligned} [i\ i+1] &= t_i, \\ [i\ j] &= t_i \triangleright [i+1\ j] z, \quad \text{for } i+1 < j, \\ [j\ i] &= [i\ j] z. \end{aligned}$$

Lemma 3.4. *Let $i, j, k \in \{1, ..., n\}$. Then $s_k \triangleright [i\ j] = [s_k(i)\ s_k(j)] z$.*

Proof. Multiplying both sides by z if needed, we may assume that $i < j$. If $\{k, k+1\} \cap \{i, j\} = \emptyset$, then the claim follows from [22], Paragraph 6, III. If $k = i-1$, then the claim follows from Definition 3.3. The case $k = i$ follows from the case $k = i-1$ by applying s_{i-1}. Since $s_j \triangleright t_{j-1} = s_{j-1} \triangleright t_j$, a straightforward computation settles the case $k = j$. Finally, the case $k = j-1$ follows from the case $k = j$ by applying s_j. □

Proposition 3.5. *Let $l \in \mathbb{N}$, $\sigma = s_{i_1} s_{i_2} \cdots s_{i_l} \in \mathbb{S}_n$ and $i, j \in \{1, ..., n\}$. Then*

$$\sigma \triangleright [i\ j] = [\sigma(i)\ \sigma(j)] z^l.$$

Proof. This follows from Lemma 3.4 by induction on l. □

3.3. **Nichols algebras over symmetric groups.** Recall that X_n is the rack of transpositions in $\mathbb{S}_n$. There exist two rack 2-cocycles that we want to consider. One of these rack 2-cocycles is the constant cocycle -1. The other one is the 2-cocycle given by equation (1).

Lemma 3.6. *Let A be an abelian group and let $\psi : A \to \mathbb{C}^\times$ be a group homomorphism. For $\phi \in Z^2(\mathbb{S}_n, A)$, define $\phi_\psi(x, y) = \psi(\phi(x, y))$. Then $\phi_\psi \in Z^2(\mathbb{S}_n, \mathbb{C}^\times)$.*

Proof. This follows from [7], Chapter 6, §3. □

Lemma 3.7. *There exists a section $s : \mathbb{S}_n \to T_n$ such that if $\tau = (i\ j)$, $i < j$, then*

$$s(\sigma) \triangleright s(\tau) = \begin{cases} s(\sigma \triangleright \tau) z & \text{if } \sigma(i) < \sigma(j), \\ s(\sigma \triangleright \tau) & \text{if } \sigma(i) > \sigma(j) \end{cases} \tag{5}$$

for all σ.

Proof. By Theorem 3.1 there exists a central extension

$$0 \to A \xrightarrow{i} T_n \xrightarrow{p} \mathbb{S}_n \to 1,$$

where $A = \langle z \rangle$. Take any set-theoretical section $\bar{s} : \mathbb{S}_n \to T_n$ such that $\bar{s}(\mathrm{id}) = 1$ and define a map $s : \mathbb{S}_n \to T_n$ by

$$s(\pi) = \begin{cases} \bar{s}(\pi) & \text{if } \pi \notin X_n, \\ [i\ j] & \text{if } \pi = (i\ j) \in X_n, \text{ with } i < j. \end{cases} \tag{6}$$

Then $ps = \mathrm{id}$ and $s(\mathrm{id}) = 1$. Since $\sigma \in X_n$, the length of σ is 1. Remark 3.2 and Proposition 3.5 imply that

$$s(\sigma) \triangleright s(\tau) = \sigma \triangleright s(\tau) = \sigma[i\ j]\sigma^{-1} = [\sigma(i)\ \sigma(j)]z.$$

Hence the claim follows. $\square$

Theorem 3.8. *Let $n \geq 4$. The rack 2-cocycles χ and -1 associated to X_n are twist equivalent. Hence the Hilbert series of $\mathfrak{B}(X_n, -1)$ and $\mathfrak{B}(X_n, \chi)$ are equal.*

Proof. By Theorem 3.1 there exists a central extension

$$0 \to A \xrightarrow{i} T_n \xrightarrow{p} \mathbb{S}_n \to 1,$$

where $A = \langle z \rangle$. Let $s : \mathbb{S}_n \to T_n$ be the section of Lemma 3.7 and let $\phi(x, y) \in A$ be defined by the equation

$$s(x)s(y) = i(\phi(x, y))s(xy).$$

Then $\phi \in Z^2(\mathbb{S}_n, A)$. For any group homomorphism $\psi : A \to \mathbb{C}^\times$ apply Lemma 3.6 to get a group 2-cocycle $\phi_\psi \in Z^2(\mathbb{S}_n, \mathbb{C}^\times)$. Take $\psi = \mathrm{sgn}$, where $\psi(z) = -1$. We claim that

$$-1 = \phi_\psi(\sigma, \tau)\phi_\psi(\sigma \triangleright \tau, \sigma)^{-1}\chi(\sigma, \tau)$$

for all $\sigma, \tau \in X_n$. In fact,

$$\begin{aligned} &\phi_\psi(\sigma, \tau)\phi_\psi(\sigma \triangleright \tau, \sigma)^{-1} \\ &\qquad = \psi\left(s(\sigma)s(\tau)s(\sigma\tau)^{-1}\right)\psi\left(s(\sigma\tau\sigma^{-1})s(\sigma)s(\sigma\tau)^{-1})\right)^{-1} \\ &\qquad = \psi\left(s(\sigma)s(\tau)s(\sigma\tau)^{-1}(s(\sigma\tau\sigma^{-1})s(\sigma)s(\sigma\tau)^{-1})^{-1}\right) \\ &\qquad = \psi\left(s(\sigma)s(\tau)s(\sigma\tau)^{-1}s(\sigma\tau)s(\sigma)^{-1}s(\sigma\tau\sigma^{-1})^{-1}\right) \\ &\qquad = \psi\left(s(\sigma)s(\tau)s(\sigma)^{-1}s(\sigma\tau\sigma^{-1})^{-1}\right). \end{aligned}$$

Therefore the first claim follows from Lemma 3.7. For the second claim, use Lemma 2.6. $\square$

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