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TWISTED COHOMOLOGY AND HOMOLOGY GROUPS ASSOCIATED TO THE RIEMANN-WIRTINGER INTEGRAL

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Dedicated to Professor Keizo Yamaguchi on his sixtieth birthday

Abstract. We study twisted cohomology and homology groups on a one-dimensional complex torus minus n distinct points with coefficients in a certain local system of rank one. This local system comes from the integrand of the Riemann-Wirtinger integral introduced by Mano. We construct bases of non-vanishing cohomology and homology groups, give an interpretation as a pairing of a cohomology class and a homology class to the Riemann-Wirtinger integral, and finally describe briefly the Gauss-Manin connection on the cohomology groups.

1. Introduction

The twisted de Rham theory developed by Aomoto [1], [3], [6] has brought a unified treatment, and a systematic way of generalization, of various hypergeometric integrals which were invented and investigated by many authors. According to Aomoto, any such integral is interpreted as a pairing of a homology class and a cohomology class on a complex projective space $\mathbf{P}^n$ minus an effective divisor D with coefficients in a local system of rank one, which is defined by a multi-valued function on $\mathbf{P}^n$ ramified just along D. Moreover, knowing the structures of the corresponding homology and cohomology groups enables us not only to produce systematically a system of differential equations satisfied by such integrals but also to determine the connection formulae and the monodromy representations of such integrals [2], [4], [5].

The purpose of this paper is to study twisted cohomology and homology groups on a one-dimensional complex torus minus n distinct points. Let us formulate our problem in this paper. Let $\mathbf{C}$ be the additive group of complex numbers and Γ the lattice in $\mathbf{C}$ generated by 1 and τ with $\operatorname{Im}\tau > 0$. A one-dimensional complex torus is given by $E = E_\tau = \mathbf{C}/\Gamma$. Let $t_1, ..., t_n$ be n distinct points of E. We set $M = M(t_1, \ldots, t_n, \tau) = E \setminus \{t_1, ..., t_n\}$. Let $T(u)$ be a multi-valued function on M given by $T(u) = e^{2\pi\sqrt{-1}c_0 u}\theta(u-t_1)^{c_1}\cdots\theta(u-t_n)^{c_n}$, where c_i $(i = 0, ..., n)$ denote

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complex constants satisfying $c_1 + \cdots + c_n = 0$, and $\theta(u)$ denotes the theta function with $\theta(0) = 0$ (see *Notation for theta functions*). Let $\mathcal{L}$ be the local system on M defined by the multi-valuedness of $T(u)^{-1}$, and $\check{\mathcal{L}}$ be its dual: $\mathcal{L} = \mathbf{C}T(u)^{-1}$ and $\check{\mathcal{L}} = \mathbf{C}T(u)$. For a complex number λ, let R_λ be the local system of rank one on E determined by the following one-dimensional representation e_λ of the fundamental group of E (which is isomorphic to Γ): $e_\lambda(1) = 1,\ e_\lambda(\tau) = e^{2\pi\sqrt{-1}\lambda}$. We denote by $\check{R}_\lambda$ its dual. By the standard argument (e.g. [11], §9), we can identify the local system R_λ with a flat line bundle on E (which we denote by the same letter R_λ by abuse of notation) and assume that the parameter λ runs over the Jacobian of E. Besides, we can regard a multi-valued function $\mathfrak{s}(u;\lambda)$ defined in *Notation for theta functions* as a global meromorphic section of the line bundle R_λ. Then we can define the twisted cohomology and homology groups with coefficients in local systems: $H^i(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$ and $H_i(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$. The main purpose of this paper is to investigate their structures at length.

The study of these cohomology and homology groups gives us the foundation for analysis of integrals of the following type:

(1.1)

$$f_j = \int_\gamma e^{2\pi\sqrt{-1}c_0 u}\theta(u-t_1)^{c_1}\cdots\theta(u-t_n)^{c_n}\mathfrak{s}(u-t_j;\lambda)du,\ j = 1,\ldots,n,\ \gamma : \text{a cycle},$$

which we call the *Riemann-Wirtinger integral*. According to Mano [14], [15], the integral (1.1) appears as a particular solution of a system of partial differential equations of the integrability condition of monodromy-preserving deformation of a Fuchsian differential equation with n singularities $t_1, \ldots, t_n$ on a one-dimensional complex torus. Moreover, the integral (1.1) includes Gauss' hypergeometric function; namely, it is known [21] that, in the case where $n = 4$, $\lambda = 0$, $t_1 = 0$, $t_2 = 1/2$, $t_3 = \tau/2$, $t_4 = (1+\tau)/2$, it reduces essentially to Euler's integral representation for Gauss' hypergeometric function. Therefore the integral (1.1) in this case is called in [17] the *Wirtinger integral*. In [19], [20] Watanabe studied the monodromy representation and the differential equations for the Wirtinger integral from the viewpoint of the theory of theta functions. As a natural and interesting generalization of the Wirtinger integral, we can also consider the integral (1.1) with the following conditions: $\lambda = 0,\ n = N^2,\ N \geq 3$ and the set of singularities $\{t_1, \ldots, t_n\}$ coincides with that of N-torsion points on the torus (which we call the *generalized Wirtinger integral* of level N). For the generalized Wirtinger integrals, we are interested in solving the connection problem and the monodoromy representation and deriving differential equations satisfied by them. These topics will be discussed elsewhere. On the other hand the Riemann-Wirtinger integral is very similar to the integral representation constructed by Felder and Varchenko [9] which is an integral of a product of fractional powers of theta functions solving the elliptic Knizhnik-Zamolodchikov-Bernard (KZB) equation. We will compare these two integral representations in the present paper.

This paper is a natural generalization of the paper [18] for the Wirtinger integral, so several techniques used there are applied to the discussion in this paper. K. Ito [12] studied the twisted homology groups with coefficients in the local system defined by a multi-valued function of a special form on the complex torus minus four points. We can also regard this paper as a generalization of [12].

In Section 2, we study the twisted cohomology groups $H^i(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$. The main task here is to determine the structure of the non-vanishing cohomology group

$H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$. It changes according to whether $\lambda = 0$ or $\lambda \in P \setminus \{0\}$, where $P = \{a\tau + b \in \mathbf{C} \mid 0 \leq a, b < 1\}$ denotes a realization on the complex plane of the Jacobian of E (Theorem 2.7). In the case where $\lambda \in P \setminus \{0\}$, similarly to the case of projective spaces (i.e., integrals of a product of fractional powers of rational functions), we can take a basis consisting of only logarithmic 1-forms. In the case where $\lambda = 0$, however, we cannot find any basis consisting of only logarithmic 1-forms, but we need a 1-form having a pole of order more than one to give its basis (Corollary 2.9). Proposition 2.5 is crucial to prove Theorem 2.7. Although this proposition has already been proved algebro-geometrically by Deligne [8], we give here another proof based on the theory of complex analysis (see Remark 2.6). As we have already encountered in the case of the Wirtinger integral [18], the phenomenon in Corollary 2.9 is new in the sense that the non-vanishing cohomology groups for complex projective spaces are described exclusively by logarithmic forms but the cohomology group for the complex torus is not necessarily described in such a way.

In Section 3, we study the twisted homology groups $H_i(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$. By giving a cell decomposition for the space M, we choose twisted cycles as generators of this group and establish the linear relation satisfied by them explicitly (Theorem 3.1).

In Section 4, as an application of the results obtained in the preceding sections, we study the dependence of cohomology and homology groups with respect to the variables $t_1, \ldots, t_n, \tau$. Namely, we introduce a "base space" $\mathcal{T}$ parameterized by these data and take the union $\bigcup_{(t_1,\ldots,t_n,\tau)\in\mathcal{T}} H^1(M(t_1, \ldots, t_n, \tau), \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$. The tensor product of this union and the structure sheaf $\mathcal{O}_\mathcal{T}$ of $\mathcal{T}$ is, roughly speaking, equal to $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_\mathcal{T}$, where f denotes a natural projection of the total space $\mathcal{M} = \bigcup_{(t_1,\ldots,t_n,\tau)\in\mathcal{T}} M(t_1, \ldots, t_n, \tau)$ onto $\mathcal{T}$. To describe the Gauss-Manin connection on $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_\mathcal{T}$ ([13]), we establish the formulas of differentiation with respect to the variables $t_1, \ldots, t_n, \tau$ on $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_\mathcal{T}$ (Proposition 4.1).

In Section 5, we refer to a relation between the Riemann-Wirtinger integral and Felder-Varchenko's integral solution of the KZB equation. Solutions of the KZB equation are generally represented by multiple integrals over direct products of elliptic curves. The Riemann-Wirtinger integral is related to the so-called one integral variable $\mathfrak{sl}_2$-case of the KZB equation. In Proposition 5.1, we give a system of partial differential equations which is a generalization of the KZB equation. Then we see that the Riemann-Wirtinger integral is another specialization of this system.

Notation for theta functions. In this paper, we follow Chandrasekharan's notation for theta functions [7]:

$$\theta(u) = \theta(u, \tau) = -\sqrt{-1} \sum_{n=-\infty}^{+\infty} (-1)^n e^{\pi\sqrt{-1}(n+1/2)^2\tau} e^{2\pi\sqrt{-1}(n+1/2)u},$$

$$\theta_1(u) = \theta_1(u, \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi\sqrt{-1}(n+1/2)^2\tau} e^{2\pi\sqrt{-1}(n+1/2)u},$$

$$\theta_2(u) = \theta_2(u, \tau) = \sum_{n=-\infty}^{+\infty} (-1)^n e^{\pi\sqrt{-1}n^2\tau} e^{2\pi\sqrt{-1}nu},$$

$$\theta_3(u) = \theta_3(u, \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi\sqrt{-1}n^2\tau} e^{2\pi\sqrt{-1}nu}.$$

Then $\theta(u)$ is an odd function and $\theta_1(u)$, $\theta_2(u)$, $\theta_3(u)$ are even functions. These functions are related to each other as follows:

$$\theta_1(u) = \theta(u+\frac{1}{2}),$$
$$\theta_2(u) = -\sqrt{-1}e^{\pi\sqrt{-1}(\tau/4+u)}\theta(u+\frac{\tau}{2}),$$
$$\theta_3(u) = e^{\pi\sqrt{-1}(\tau/4+u)}\theta(u+\frac{\tau+1}{2}).$$

We also use the following symbols: $\theta' = \theta'(0,\tau)$, $\theta_i = \theta_i(0,\tau)$, $i = 1,2,3$. We introduce the following function:

$$\mathfrak{s}(u;\lambda) = \frac{\theta(u-\lambda)\theta'}{\theta(u)\theta(-\lambda)}.$$

Then $\mathfrak{s}(u;\lambda)$ has the quasi-periodicity

$$\mathfrak{s}(u+1;\lambda) = \mathfrak{s}(u;\lambda), \quad \mathfrak{s}(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}\mathfrak{s}(u;\lambda).$$

Moreover we use the symbol $\rho(u) = \theta'(u)/\theta(u)$.

2. Twisted cohomology groups

Let $\mathbf{H}$ be the upper half plane. For $\tau \in \mathbf{H}$, we set $\Gamma = \Gamma_\tau = \mathbf{Z}+\mathbf{Z}\tau$, a subgroup of the additive group $\mathbf{C}$ of complex numbers, where $\mathbf{Z}$ is the additive group of integers. The fundamental group of the torus $E = E_\tau = \mathbf{C}/\Gamma$ is isomorphic to the group Γ. Let λ be a complex number. We define a one-dimensional representation of the fundamental group, which we denote by $e_\lambda : \Gamma \ni \gamma \to e_\lambda(\gamma) \in \mathbf{C}^*$ ($\mathbf{C}^*$ denotes the multiplicative group of non-zero complex numbers), by the assignment of generators of Γ into $\mathbf{C}^*$: $e_\lambda(1) = 1$, $e_\lambda(\tau) = e^{2\pi\sqrt{-1}\lambda}$. Let R_λ be the local system of rank one on E determined by this representation e_λ. Let $\mathcal{O}_E$ be the sheaf of holomorphic functions on E. We set $\mathcal{O}_{E,\lambda} = \mathcal{O}_E \otimes_{\mathbf{C}} R_\lambda$, the tensor product of $\mathcal{O}_E$ and R_λ. Let L_λ be the line bundle on E whose local sections are generated by the sheaf $\mathcal{O}_{E,\lambda}$. The sheaf $\mathcal{O}_{E,\lambda}$ is also denoted by $\mathcal{O}_E(L_\lambda)$ in the literature. Then we have

Lemma 2.1. *Assume that $\lambda \notin \Gamma$. Then $H^0(E,\mathcal{O}_{E,\lambda}) = H^1(E,\mathcal{O}_{E,\lambda}) = 0$.*

Proof. Let f be in $H^0(E,\mathcal{O}_{E,\lambda})$. By definition, $f(u)$ is holomorphic on E, satisfying the relations $f(u+1) = f(u)$ and $f(u+\tau) = e^{2\pi\sqrt{-1}\lambda}f(u)$. By the definition of $\mathfrak{s}(u;-\lambda)$, the product $f(u)\mathfrak{s}(u;-\lambda)$ must be an elliptic function on E with at most a single simple pole at $u = 0$, from which we have $f = 0$. Therefore $H^0(E,\mathcal{O}_{E,\lambda}) = 0$. Now the Riemann-Roch theorem holds: $\dim H^0(E,\mathcal{O}_{E,\lambda}) - \dim H^1(E,\mathcal{O}_{E,\lambda}) - c(L_\lambda) = 1-g$, where $c(L_\lambda)$ is the first Chern class of the line bundle L_λ. Since we can take $\mathfrak{s}(u;\lambda)$ as a global meromorphic section of L_λ, we conclude that $c(L_\lambda) = 0$. Since $g = 1$ and $H^0(E,\mathcal{O}_{E,\lambda}) = 0$, we have $\dim H^1(E,\mathcal{O}_{E,\lambda}) = 0$ by the Riemann-Roch theorem, which completes the proof of Lemma 2.1. □

Let $t_1,\ldots,t_n$ be n distinct points of E, where $n \geq 2$. We set $D = \{t_1,\ldots,t_n\}$ and $M = M(t_1,\ldots,t_n,\tau) = E \setminus D$. Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be an open covering of E. Without loss of generality we may assume that $\mathcal{U}$ is a Leray open covering such that every open set $U_i \in \mathcal{U}$ contains at most one point t_k of D. Let us consider a collection $\mu = \{h_i\}_{i\in\Lambda}$, where h_i is a holomorphic section of L_λ on U_i if U_i contains no point of D, or it is a holomorphic section of L_λ on $U_i - \{t_k\}$ and may have an isolated singularity (i.e., a pole or an essential singularity) at t_k if U_i contains a point $t_k \in D$. Such a collection μ is called a *Mittag-Leffler distribution* if, $\rho_{U,V}$ denoting the restriction map of $H^0(U, \mathcal{O}_{E,\lambda})$ to $H^0(V, \mathcal{O}_{E,\lambda})$ for $U \supset V$, the difference $\rho_{U_i,U_i\cap U_j}(h_i) - \rho_{U_j,U_i\cap U_j}(h_j)$ is holomorphic on $U_i \cap U_j$. By a *solution* of μ we mean a global analytic section h of L_λ holomorphic on M such that the difference $\rho_{E,U_i}(h) - h_i$ is a holomorphic section on U_i for every $i \in \Lambda$. Here we note that our definition of a Mittag-Leffler distribution is a little wider than the one given in [10]. Then we have

Lemma 2.2. *If $\lambda \notin \Gamma$, every Mittag-Leffler distribution μ has a solution.*

Proof. We set $h_i|_{U_i\cap U_j} = \rho_{U_i,U_i\cap U_j}(h_i)$. The collection $\{h_i|_{U_i\cap U_j} - h_j|_{U_i\cap U_j}\}_{i\neq j}$ forms a 1-cocycle: $\{h_i|_{U_i\cap U_j} - h_j|_{U_i\cap U_j}\}_{i\neq j} \in Z^1(\mathcal{U}, \mathcal{O}_{E,\lambda})$. Since $H^1(E, \mathcal{O}_{E,\lambda}) = 0$ by Lemma 2.1 and $\mathcal{U}$ is a Leray covering, there exists a local holomorphic section g_i on every U_i such that $h_i|_{U_i\cap U_j} - h_j|_{U_i\cap U_j} = g_i|_{U_i\cap U_j} - g_j|_{U_i\cap U_j}$. Then we have $h_i|_{U_i\cap U_j} - g_i|_{U_i\cap U_j} = h_j|_{U_i\cap U_j} - g_j|_{U_i\cap U_j}$. Gluing the local sections $h_i - g_i$ over E, we have a global section on E, which is the desired solution of μ. □

When $\lambda \in \Gamma$, we may assume $\lambda = 0$ without loss of generality.

Lemma 2.3. *Assume that $\lambda = 0$. Let a_k $(1 \le k \le n)$ be the residue of the 1-form $h_i du$ at the singularity $u = t_k$ if $t_k \in U_i$. Then the Mittag-Leffler distribution μ has a solution if and only if $\sum_{k=1}^n a_k = 0$.*

For the proof, see [10], Chapter 2, §18.

Let $\mathcal{O}_M$ be the sheaf of holomorphic functions on M and let Ω^1_M be the sheaf of holomorphic 1-forms on M. We define the sheaves $\mathcal{O}_\lambda$ and Ω^1_λ on M by $\mathcal{O}_\lambda = \mathcal{O}_M \otimes_{\mathbf{C}} R_\lambda$ and $\Omega^1_\lambda = \Omega^1_M \otimes_{\mathbf{C}} R_\lambda$, where we denote the restriction of R_λ to M by the same symbol R_λ by abuse of notation. Since R_λ is locally constant and without torsion, we have the exact sequence of sheaves on M:

$$0 \to R_\lambda \to \mathcal{O}_\lambda \xrightarrow{d} \Omega^1_\lambda \to 0,$$

where d denotes the sheaf mapping induced by the differential $d : \mathcal{O}_M \to \Omega^1_M$. Let c_0 be an arbitrary complex number, and $c_1, \ldots, c_n$ $(n \ge 2)$ be non-integral complex numbers satisfying $c_1 + \cdots + c_n = 0$. We define a multi-valued function $T(u)$ on M by $T(u) = e^{2\pi\sqrt{-1}c_0 u}\theta(u - t_1)^{c_1} \cdots \theta(u - t_n)^{c_n}$. We set $\omega = d(\log T(u))$. We define a connection ∇ by $\nabla\varphi = d\varphi + \omega \wedge \varphi$. Then we have $\nabla\nabla = 0$ and $\nabla(1) = \omega$. The connection ∇ defines a sheaf morphism $\mathcal{O}_M \to \Omega^1_M$, and therefore $\mathcal{O}_\lambda \to \Omega^1_\lambda$. Let $\mathcal{L}$ and $\check{\mathcal{L}}$ be the local systems on M defined by the multi-valuedness of $T(u)^{-1}$ and $T(u)$, respectively: $\mathcal{L} = \mathbf{C}T(u)^{-1}$ and $\check{\mathcal{L}} = \mathbf{C}T(u)$. They are dual to each other. In the case where we emphasize the dependence on the parameters c_i contained in $\mathcal{L}$ and $\check{\mathcal{L}}$, we also write $\mathcal{L} = \mathcal{L}(c_0, \ldots, c_n)$ and $\check{\mathcal{L}} = \check{\mathcal{L}}(c_0, \ldots, c_n)$. Since the local system $\mathcal{L}$ is locally constant and without torsion, we have the following exact sequence on M:

$$0 \to \mathcal{L} \otimes_{\mathbf{C}} R_\lambda \to \mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_\lambda \xrightarrow{\mathrm{id}\otimes d} \mathcal{L} \otimes_{\mathbf{C}} \Omega^1_\lambda \to 0. \tag{2.1}$$

Let φ be a local section of $\mathcal{O}_\lambda$. Then the assignment $\varphi \mapsto T(u)\varphi$ defines a sheaf isomorphism $\mathcal{O}_\lambda \to \mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_\lambda$. Since $d(T(u)\varphi) = T(u)\nabla\varphi$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_\lambda & \xrightarrow{\nabla} & \Omega^1_\lambda \\ \downarrow & & \downarrow \\ \mathcal{L}\otimes_{\mathbf{C}}\mathcal{O}_\lambda & \xrightarrow[\mathrm{id}\otimes d]{} & \mathcal{L}\otimes_{\mathbf{C}}\Omega^1_\lambda, \end{array} \tag{2.2}$$

where the vertical arrows represent isomorphisms. Combining this commutative diagram and the exact sequence (2.1), we have the exact sequence

$$0\to\mathcal{L}\otimes_{\mathbf{C}}R_\lambda\to\mathcal{O}_\lambda\xrightarrow{\nabla}\Omega^1_\lambda\to 0, \tag{2.3}$$

from which we have the following long exact sequence of the cohomology groups:

$$\begin{aligned} 0 &\to H^0(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)\to H^0(M,\mathcal{O}_\lambda)\xrightarrow{\nabla}H^0(M,\Omega^1_\lambda)\to H^1(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)\\ &\to H^1(M,\mathcal{O}_\lambda)\xrightarrow{\nabla}H^1(M,\Omega^1_\lambda)\to H^2(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)\to H^2(M,\mathcal{O}_\lambda)\\ &\xrightarrow{\nabla}H^2(M,\Omega^1_\lambda)\to\cdots. \end{aligned}$$

Proposition 2.4. *We have* $H^i(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)=0$ *for* $i\neq 1$*, and*

$$H^1(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)\cong H^0(M,\Omega^1_\lambda)/\nabla(H^0(M,\mathcal{O}_\lambda)).$$

Proof. The local system $\mathcal{L}\otimes_{\mathbf{C}}R_\lambda$ has no global section on M but the zero section. So we have $H^0(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)=0$. Note that $\mathcal{O}_\lambda$ and Ω^1_λ are coherent $\mathcal{O}_M$-modules. Since the open Riemann surface M is Stein, we have $H^i(M,\mathcal{O}_\lambda)=H^i(M,\Omega^1_\lambda)=0$ $(i>0)$. Combining these results with (2.2), we have the short exact sequence

$$0\to H^0(M,\mathcal{O}_\lambda)\xrightarrow{\nabla}H^0(M,\Omega^1_\lambda)\to H^1(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)\to 0$$

and $H^i(M,\mathcal{L}\otimes_{\mathbf{C}}R_\lambda)=0$ $(i\geq 2)$, from which the proposition follows. □

Let $\tilde{\mathcal{K}^0}$ be the sheaf of meromorphic functions on E, and $\tilde{\mathcal{K}^1}$ be the sheaf of meromorphic 1-forms on E. Let $\mathcal{K}^0=H^0(E,\tilde{\mathcal{K}^0})$ be the field of global meromorphic functions on E, and $\mathcal{K}^1=H^0(E,\tilde{\mathcal{K}^1})$ be the $\mathcal{K}^0$-vector space of global meromorphic 1-forms on E. We define sheaves $\tilde{\mathcal{K}^p_\lambda}$ $(p=0,1)$ on E by $\tilde{\mathcal{K}^p_\lambda}=\tilde{\mathcal{K}^p}\otimes_{\mathbf{C}}R_\lambda$. Let $\mathcal{S}^p$ $(p=0,1)$ be the sheaf on M whose sections on an open set $U\subset M$ form the $\mathbf{C}$-vector space $\mathcal{S}^p(U)=\{s\in\mathcal{K}^p\mid s \text{ is holomorphic on } U\}$. By definition, $\mathcal{S}^0$ is a subsheaf of $\mathcal{O}_M$, and $\mathcal{S}^1$ a subsheaf of Ω^1_M. We define sheaves $\mathcal{S}^0_\lambda$ and $\mathcal{S}^1_\lambda$ on M by $\mathcal{S}^0_\lambda=\mathcal{S}^0\otimes_{\mathbf{C}}R_\lambda$ and $\mathcal{S}^1_\lambda=\mathcal{S}^1\otimes_{\mathbf{C}}R_\lambda$. Let us consider the subcomplex of sheaves

$$0\to\mathcal{S}^0_\lambda\xrightarrow{\nabla}\mathcal{S}^1_\lambda\to 0 \tag{2.4}$$

of the complex

$$0\to\mathcal{O}_\lambda\xrightarrow{\nabla}\Omega^1_\lambda\to 0. \tag{2.5}$$

The inclusion ι of (2.4) to (2.5) induces the natural homomorphism ι_* of the de Rham cohomologies:

$$\iota_*: H^0(M,\mathcal{S}^1_\lambda)/\nabla(H^0(M,\mathcal{S}^0_\lambda))\to H^0(M,\Omega^1_\lambda)/\nabla(H^0(M,\mathcal{O}_\lambda)). \tag{2.6}$$

Let $\mathcal{O}_E(*D)$ be the sheaf of functions meromorphic on E and holomorphic on M, and $\Omega^1_E(*D)$ be the sheaf of 1-forms meromorphic on E and holomorphic on M. We set $\mathcal{O}_\lambda(*D)=\mathcal{O}_E(*D)\otimes_{\mathbf{C}}R_\lambda$ and $\Omega^1_\lambda(*D)=\Omega^1_E(*D)\otimes_{\mathbf{C}}R_\lambda$. Since

$H^0(M, \mathcal{S}^1_\lambda) = H^0(E, \Omega^1_\lambda(*D))$ and $H^0(M, \mathcal{S}^0_\lambda) = H^0(E, \mathcal{O}_\lambda(*D))$, (2.6) is written by

(2.7) $\quad \iota_* : H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D))) \to H^0(M, \Omega^1_\lambda)/\nabla(H^0(M, \mathcal{O}_\lambda)).$

In fact we have

Proposition 2.5. *For any $\lambda \in \mathbf{C}$, ι_* is an isomorphism.*

Proof. Since we have

$$\mathcal{L}(c_0, c_1, \dots, c_n) \otimes_{\mathbf{C}} R_{\lambda+m\tau+n} \cong \mathcal{L}(c_0+m, c_1, \dots, c_n) \otimes_{\mathbf{C}} R_\lambda,$$

for $m, n \in \mathbf{Z}$, we may assume that $\lambda \in P = \{a\tau + b | 0 \le a, b < 1\}$ without loss of generality. For the proof we need the following claim:

$(*)$ For $\varphi \in H^0(M, \Omega^1_\lambda)$, there exist $\psi \in H^0(E, \Omega^1_\lambda(*D))$ and $\tilde{Q}_* \in H^0(M, \mathcal{O}_\lambda)$ such that $\varphi = \psi + \nabla \tilde{Q}_*$.

In fact, let φ be in $H^0(M, \Omega^1_\lambda)$. Since the canonical line bundle of E is trivial, we may set $\varphi = f(u)du$, where $f(u)$ is a section belonging to $H^0(M, \mathcal{O}_\lambda)$ and may have isolated essential singularities at $u = t_k$ $(k = 1, \dots, n)$. Let $P_k(u)$ be the principal part of the Laurent expansion of $f(u)$ at $u = t_k$. Let us find a function $Q_k(u)$ single-valued around $u = t_k$ satisfying the equation $P_k(u)du = \nabla Q_k$, that is,

$$P_k(u) = \frac{dQ_k}{du} + Q_k \frac{d}{du}(\log T(u)).$$

Here we may assume that $P_k(u) = \sum_{n \le -1} a^{(k)}_n (u - t_k)^n$. By quadrature we have a general solution of this equation: $Q_k = T(u)^{-1}[\int T(u)P_k(u)du + C]$ for some constant C. Since Q_k is single-valued at t_k, the condition $C = 0$ is necessary. Let us investigate the behaviour of the solution $Q_k = Q_k(u)$ with $C = 0$ around $u = t_k$. Since c_k is not an integer, the multi-valuedness of $T(u)$ around $u = t_k$ comes from the factor $(u - t_k)^{c_k}$. Then we can write $T(u) = (u - t_k)^{c_k} \times$ (single-valued holomorphic function) around $u = t_k$. Moreover, since we can write $T(u)P_k(u) = \sum_{n=-\infty}^{+\infty} e^{(k)}_n (u - t_k)^{c_k+n}$ around $u = t_k$, we have

$$\int T(u)P_k(u)du = \sum_{n=-\infty}^{+\infty} \frac{e^{(k)}_n}{c_k + n + 1}(u - t_k)^{c_k+n+1},$$

which is of the form $(u - t_k)^{c_k} \times$(single-valued analytic function which may have an isolated singularity at $u = t_k$) around $u = t_k$. Consequently, the function $Q_k(u) = T(u)^{-1} \int T(u)P_k(u)du$ is a single-valued analytic function around $u = t_k$ which may have an isolated singularity at $u = t_k$ and therefore can be expanded in a Laurent series at $u = t_k$. We set $Q_k(u) = \sum_{n=-\infty}^{+\infty} b^{(k)}_n (u - t_k)^n$, the Laurent expansion at $u = t_k$. Moreover we set $Q_{k-}(u) = \sum_{n \le 0} b^{(k)}_n (u - t_k)^n$ and $Q_{k+}(u) = \sum_{n \ge 1} b^{(k)}_n (u - t_k)^n$. Substituting $Q_k = Q_{k-} + Q_{k+}$ into the original equation above, we have $P_k = Q'_{k-} + Q'_{k+} + Q_{k-} \cdot (\log T(u))' + Q_{k+} \cdot (\log T(u))'$. Since $(\log T(u))'$ has a pole of order one at $u = t_k$ and Q_{k+} has a zero of order one at $u = t_k$, the product $Q_{k+} \cdot (\log T(u))'$ is holomorphic at $u = t_k$, and so is Q'_{k+}. Consequently, we see that in the right-hand side of the preceding relation the sum $Q'_{k-} + Q_{k-} \cdot (\log T(u))'$ contributes to the principal part P_k. Therefore, setting $\nabla Q_{k-} = g_k(u)du$, we see that the principal part of the Laurent expansion of $g_k(u)$ at $u = t_k$ is equal to P_k. Let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ be a Leray open covering of E such that every open set U_i

contains at most one point t_k of D. To conclude the proof of the claim $(*)$, we consider the two cases: (i) $\lambda \neq 0$ and (ii) $\lambda = 0$.

(i) Assume $\lambda \neq 0$. Let $\mu = \{Q_{*i}\}_{i\in\Lambda}$ be a Mittag-Leffler distribution subordinate to the open covering $\mathcal{U}$ consisting of local sections Q_{*i} of L_λ on U_i such that for every open set U_i containing $t_k \in D$ the section Q_{*i} coincides with the branch containing $Q_{k-} - b_0^{(k)}$ and for every open set U_j not containing any point of D the section Q_{*j} is holomorphic. Then, by Lemma 2.2, there exists a global analytic section Q_* of L_λ on E, holomorphic on M such that the difference $Q_{k-} - b_0^{(k)} - Q_*$ is holomorphic around $u = t_k$. We note that for every k the 1-form $P_k(u)du - \nabla b_0^{(k)} - \nabla Q_*$ is holomorphic at $u = t_k$ and that the 1-form $\nabla b_0^{(k)}$ locally defined around $u = t_k$ has a pole of order one at $u = t_k$ with residue $c_k b_0^{(k)}$. We set $\psi = \sum_{k=1}^n b_0^{(k)} c_k \mathfrak{s}(u - t_k, \lambda)du \in H^0(E, \Omega^1_\lambda(*D))$. Then the 1-form $f(u)du - \psi - \nabla Q_*$ is a global holomorphic section on E, and therefore by Lemma 2.1 we have $f(u)du = \psi + \nabla Q_*$, which proves the claim $(*)$.

(ii) Assume $\lambda = 0$. Then the claim $(*)$ follows if we apply Lemma 2.3 and the reasoning of the proof of Lemma 2.2 in [18]. We omit the details.

We are now in a position to prove the proposition with the aid of the claim $(*)$. Let us take $[\varphi] \in H^0(M, \Omega^1_\lambda)/\nabla(H^0(M, \mathcal{O}_\lambda))$ arbitrarily, where $\varphi \in H^0(M, \Omega^1_\lambda)$. If we form $[\psi] \in H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D)))$ from the element $\psi \in H^0(E, \Omega^1_\lambda(*D))$ whose existence is guaranteed by the claim $(*)$, then we have $\iota_*[\psi] = [\varphi]$, which proves the surjectivity of ι_*. The proof of the injectivity of ι_* is as follows. For $[\psi] \in H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D)))$, assume that $\iota_*[\psi] = 0$. We set $\psi = f(u)du$, where $f(u)$ is a section of $H^0(E, \mathcal{O}_\lambda(*D))$ that may have poles at points of D if $f(u)$ is not holomorphic there. The equation $\iota_*[\psi] = 0$ is translated into the assertion that there exists a section $g \in H^0(M, \mathcal{O}_\lambda)$ such that $f(u)du = \nabla g$. The equation is rewritten as

$$f(u) = \frac{dg}{du} + g(u)\frac{d}{du}(\log T(u)),$$

from which we have the solution $g(u) = T(u)^{-1}\int T(u)f(u)du$. By the same argument as when we constructed Q_k from P_k and investigated the behaviour of Q_k at $u = t_k$, we see that $g(u)$ is in $H^0(E, \mathcal{O}_\lambda(*D))$, and $[\psi] = 0$ as the equality in $H^0(E, \Omega^1_\lambda(*D))/\nabla(H^0(E, \mathcal{O}_\lambda(*D)))$, which proves the injectivity of ι_*. □

Remark 2.6. It is well known that this proposition is proved algebro-geometrically by Deligne's comparison theorem [8], II, Section 6. On the other hand, to study the structure of the quotient group $H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D))$ at length, we have to choose an effective divisor D such that the vector space $H^0(E, \Omega^1_\lambda(D))$ maps surjectively to the above group. In the analogous theory of integrals on complex projective spaces [6] one can choose for such a D the "reduced" one, that is, the effective divisor all of whose integral coefficients are equal to one. In our case, however, it is indispensable to choose a non-reduced divisor $D' = 2[t_1] + [t_2] + \cdots + [t_n]$ especially when $\lambda = 0$ (cf. Theorem 2.7). Our proof of this proposition tells us what divisor D is suitable to take for establishing the surjection of $H^0(E, \Omega^1_\lambda(D))$ onto $H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D))$.

Let D and D' be the divisors on E defined by $D = [t_1] + \cdots + [t_n]$ and $D' = 2[t_1] + [t_2] + \cdots + [t_n]$. Then we have the following natural homomorphisms:

$$I : H^0(E, \Omega^1_\lambda(D)) \to H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D)),$$
$$I' : H^0(E, \Omega^1_\lambda(D')) \to H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D)),$$

where, for a divisor F on E, we denote $H^0(E, \Omega^p_\lambda(F)) = \{f \in H^0(E, \tilde{\mathcal{K}^p_\lambda}) \mid (f)+F \geq 0\}$.

Theorem 2.7. *Under the assumptions $c_i \notin \mathbf{Z}$, $i = 1, \ldots, n$, the following assertions hold:*

(i) *For $\lambda \in P \setminus \{0\}$, I is an isomorphism.*
(ii) *For $\lambda \in P$, I' is surjective and* $\dim \ker I' = 1$.

Proof. First, we prove (i). We introduce a filtration to the vector spaces $H^0(E, \Omega^p_\lambda(*D))$ $(p = 0, 1)$ by the following:

$$F_k = F_k H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda(kD)), \quad k = 0, 1, 2, \ldots.$$

It is obvious that

$$F_k H^0(E, \Omega^p_\lambda(*D)) \subset F_{k+1} H^0(E, \Omega^p_\lambda(*D)),$$
$$\bigcup_{k=0}^{\infty} F_k H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda(*D)),$$

and

$$F_0 H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda) = 0.$$

For $k \geq 1$, it is easy to see that the set of n functions $\{\frac{\partial^{k-1}}{\partial u^{k-1}} \mathfrak{s}(u - t_i; \lambda)\}_{i=1}^n$ forms a basis of $\mathrm{Gr}^F_k H^0(E, \mathcal{O}_\lambda(*D)) = F_k/F_{k-1}$. From this fact and the assumptions $c_i \notin \mathbf{Z}$ $(i = 1, \ldots, n)$, we can check that the induced homomorphism

$$\mathrm{Gr}^F_k \nabla : \mathrm{Gr}^F_k H^0(E, \mathcal{O}_\lambda(*D)) \to \mathrm{Gr}^F_{k+1} H^0(E, \Omega^1_\lambda(*D))$$

is an isomorphism. Hence we have

$$H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D)) \cong F_1 H^0(E, \Omega^1_\lambda(*D)) = H^0(E, \Omega^1_\lambda(D)),$$

which concludes the assertion (i).

Next, we prove (ii). We introduce another filtration to $H^0(E, \Omega^p_\lambda(*D))$:

$$F'_k H^0(E, \Omega^p_\lambda(*D)) = H^0(E, \Omega^p_\lambda(D' + (k-1)D)), \ k = 0, 1, 2, \ldots.$$

For $\lambda \in P$, we define $n + 1$ functions $\varphi_j(u; \lambda)$ $(j = 0, \ldots, n)$ by

$$\varphi_0(u; \lambda) = -\lambda \mathfrak{s}(u - t_1; \lambda),$$
$$\varphi_1(u; \lambda) = \frac{\partial \mathfrak{s}}{\partial u}(u - t_1; \lambda),$$
$$\varphi_j(u; \lambda) = \mathfrak{s}(u - t_j; \lambda) - \mathfrak{s}(u - t_1; \lambda), \ j = 2, \ldots, n.$$

Here by $\varphi_j(u; 0)$ we understand $\varphi_j(u; 0) = \lim_{\lambda \to 0} \varphi_j(u; \lambda)$. This definition is valid because $\mathfrak{s}(u; \lambda)$ is expanded as follows to a Laurent series with respect to λ:

$$\mathfrak{s}(u; \lambda) = -\frac{1}{\lambda} + \rho(u) + \cdots,$$

where $\rho(u) = \theta'(u)/\theta(u)$. Then we see that $F'_0 H^0(E, \mathcal{O}_\lambda(*D)) = \mathbf{C}\varphi_0(u;\lambda)$, and that $\{\frac{\partial^{k-1}}{\partial u^{k-1}}\varphi_i(u;\lambda)\}_{i=1}^n$ forms a basis of $\mathrm{Gr}_k^{F'} H^0(E, \mathcal{O}_\lambda(*D))$. The induced homomorphism

$$\mathrm{Gr}_k^{F'}\nabla : \mathrm{Gr}_k^{F'} H^0(E, \mathcal{O}_\lambda(*D)) \to \mathrm{Gr}_{k+1}^{F'} H^0(E, \Omega^1_\lambda(*D))$$

is an isomorphism for $k \geq 1$. Hence we have

$$H^0(E, \Omega^1_\lambda(*D))/\nabla H^0(E, \mathcal{O}_\lambda(*D)) \cong F'_1 H^0(E, \Omega^1_\lambda(*D))/\nabla F'_0 H^0(E, \mathcal{O}_\lambda(*D)).$$

Noting that $\nabla\varphi_0(u;\lambda) \neq 0$, we can conclude the assertion (ii). □

Remark 2.8. In fact, we have

$$\begin{aligned}\nabla\varphi_0(u;\lambda) =& \Big(2\pi\sqrt{-1}c_0 - c_1\rho(\lambda) + \sum_{j=2}^n c_j \mathfrak{s}(t_j - t_1;\lambda)\Big)\varphi_0(u;\lambda)du \\ &+ (c_1 - 1)\lambda\varphi_1(u;\lambda)du - \lambda\sum_{j=2}^n \mathfrak{s}(t_j - t_1;\lambda)\varphi_j(u;\lambda)du.\end{aligned}$$

Corollary 2.9. *For any $\lambda \in P$, we have $\dim H^1(M, \mathcal{L}\otimes_{\mathbf{C}} R_\lambda) = n$. For $\lambda \in P\backslash\{0\}$, the set of n classes $\{[\mathfrak{s}(u - t_i;\lambda)du]\}_{i=1}^n$ forms a basis of $H^1(M, \mathcal{L}\otimes_{\mathbf{C}} R_\lambda)$. For $\lambda \in P$, the set of $n+1$ classes $\{[\varphi_i(u;\lambda)du]\}_{i=0}^n$ generates the $\mathbf{C}$-vector space $H^1(M, \mathcal{L}\otimes_{\mathbf{C}} R_\lambda)$, and these classes are subject to a single relation*

$$\begin{aligned}&\Big(2\pi\sqrt{-1}c_0 - c_1\rho(\lambda) + \sum_{j=2}^n c_j \mathfrak{s}(t_j - t_1;\lambda)\Big)[\varphi_0(u;\lambda)du] \\ &\qquad + (c_1 - 1)\lambda[\varphi_1(u;\lambda)du] - \lambda\sum_{j=2}^n \mathfrak{s}(t_j - t_1;\lambda)[\varphi_j(u;\lambda)du] = 0.\end{aligned}$$

In particular, for $\lambda = 0$, we have

$$\begin{aligned}&\varphi_0(u;0)du = du,\\ &\varphi_1(u;0)du = \rho'(u - t_1)du,\\ &\varphi_j(u;0)du = (\rho(u - t_j) - \rho(u - t_1))du, \quad j = 2, \dots, n,\end{aligned}$$

and the 1-form $\rho'(u - t_1)du$ has a pole at $u = t_1$ of order two.

3. Twisted homology groups

Let $\check{\mathcal{L}}$ and $\check{R}_\lambda$ be the local systems dual to $\mathcal{L}$ and R_λ respectively. The universal coefficient theorem implies that the natural pairings $H_i(M, \check{\mathcal{L}}\otimes_{\mathbf{C}} \check{R}_\lambda) \times H^i(M, \mathcal{L}\otimes_{\mathbf{C}} R_\lambda) \to \mathbf{C}$ $(i = 0, 1, 2)$ are non-degenerate. From the results of the last section, we have

$$H_0(M, \check{\mathcal{L}}\otimes_{\mathbf{C}} \check{R}_\lambda) = H_2(M, \check{\mathcal{L}}\otimes_{\mathbf{C}} \check{R}_\lambda) = 0,$$
$$\dim H_1(M, \check{\mathcal{L}}\otimes_{\mathbf{C}} \check{R}_\lambda) = n.$$

In this section, let us find generators of $H_1(M, \check{\mathcal{L}}\otimes_{\mathbf{C}} \check{R}_\lambda)$ and relations among them. Let $\tilde{M}$ be the manifold with circle boundaries which is homotopy equivalent to M. We give a cell decomposition of $\tilde{M}$ in Figure 1: A is a 2-cell, $l_0, l_2, \dots, l_n, l_\infty$, $s_2, \dots, s_n$, m_0, m_1, m_2, m_3 are 1-cells, and the dots $\bullet$'s stand for 0-cells. Then we

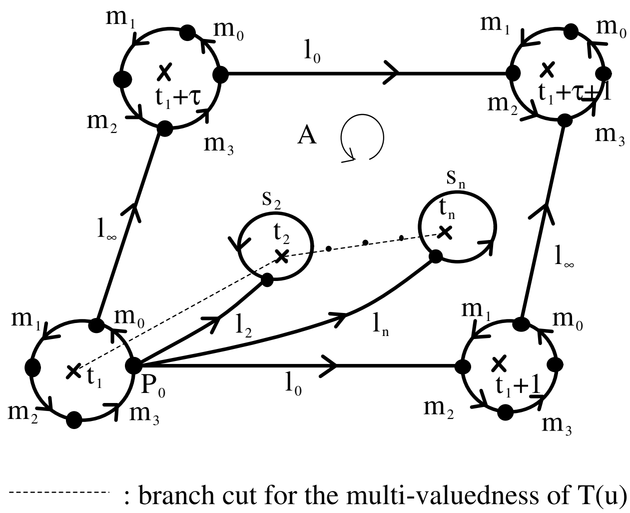

FIGURE 1. Cell decomposition of $\tilde{M}$.

see that the following $n+1$ elements generate the group $Z_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ of 1-cycles:

$$\begin{aligned}
\gamma_\infty &= \mathrm{reg}(t_1, t_1+\tau) = l_\infty + \frac{1-e^{-2\pi\sqrt{-1}c_\infty}}{e^{2\pi\sqrt{-1}c_1}-1}(m_0+m_3) \\
&\quad + \frac{1-e^{2\pi\sqrt{-1}(c_1-c_\infty)}}{e^{2\pi\sqrt{-1}c_1}-1}(m_1+m_2), \\
\gamma_0 &= \mathrm{reg}(t_1, t_1+1) = l_0 + \frac{1-e^{-2\pi\sqrt{-1}c_0}}{e^{2\pi\sqrt{-1}c_1}-1}(m_0+e^{2\pi\sqrt{-1}c_1}m_1) \\
&\quad + \frac{e^{2\pi\sqrt{-1}c_1}-e^{2\pi\sqrt{-1}c_0}}{e^{2\pi\sqrt{-1}c_1}-1}(m_2+m_3), \\
\gamma_j &= \mathrm{reg}(t_1, t_j) = l_j - \frac{s_j}{e^{2\pi\sqrt{-1}c_j}-1} \\
&\quad + \frac{m_0+e^{2\pi\sqrt{-1}c_1}(m_1+m_2+m_3)}{e^{2\pi\sqrt{-1}c_1}-1}, \ j=2,\dots,n,
\end{aligned}$$

where we put $c_\infty = -\lambda - c_0\tau - c_1t_1 - \cdots - c_nt_n$ and the symbol $\mathrm{reg}(a,b)$ denotes the regularization of the path joining the two points a and b. On the other hand, the group $B_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ of 1-boundaries is generated by the single element

$$\partial A = (e^{2\pi\sqrt{-1}c_0}-1)\gamma_\infty + (1-e^{-2\pi\sqrt{-1}c_\infty})\gamma_0 - \sum_{j=2}^{n} e^{-2\pi\sqrt{-1}(c_1+\cdots+c_j)}(1-e^{2\pi\sqrt{-1}c_j})\gamma_j.$$

Theorem 3.1. *Under the assumptions $c_i \notin \mathbf{Z}$, $i=1,\dots,n$, the twisted homology group $H_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$ is generated by $n+1$ twisted cycles $\gamma_\infty, \gamma_0, \gamma_2, \dots, \gamma_n$ subject to the unique relation*

$$(e^{2\pi\sqrt{-1}c_0}-1)\gamma_\infty + (1-e^{-2\pi\sqrt{-1}c_\infty})\gamma_0 - \sum_{j=2}^{n} e^{-2\pi\sqrt{-1}(c_1+\cdots+c_j)}(1-e^{2\pi\sqrt{-1}c_j})\gamma_j = 0.$$

4. Connection on $H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$

We put $\mathcal{T} = \mathbf{C}^n \times \mathbf{H} \setminus D_{\mathcal{T}}$, where $D_{\mathcal{T}}$ is the divisor on $\mathbf{C}^n \times \mathbf{H}$ defined by

$$D_{\mathcal{T}} = \{(t_1, \dots, t_n, \tau) \in \mathbf{C}^n \times \mathbf{H} \mid \exists\, i \neq j \text{ s.t. } t_i + \Gamma = t_j + \Gamma\}.$$

Let $f : \mathcal{M} \to \mathcal{T}$ be the family of curves over $\mathcal{T}$ whose fiber over $(t_1, \dots, t_n, \tau) \in \mathcal{T}$ is $M(t_1, \dots, t_n, \tau)$. The local system $\mathcal{L} \otimes_{\mathbf{C}} R_\lambda$ on M extends naturally to the one on the total space $\mathcal{M}$ under the constraint $\lambda + c_0\tau + c_1 t_1 + \cdots + c_n t_n + c_\infty = 0$ for a constant $c_\infty \in \mathbf{C}$, which we denote by the same symbol $\mathcal{L} \otimes_{\mathbf{C}} R_\lambda$ by abuse of notation. The coherent $\mathcal{O}_{\mathcal{T}}$-module $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{T}}$ is locally free, and we have the isomorphism

$$R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda) \otimes_{\mathbf{C}} \mathcal{O}_{\mathcal{T}} \cong f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)/\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D), \tag{4.1}$$

where $\nabla_{\mathcal{M}/\mathcal{T}} = d_{\mathcal{M}/\mathcal{T}} + \omega$. We wish to describe a connection on the right-hand side of (4.1) whose horizontal sections form the image of $R^1 f_*(\mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$, namely the Gauss-Manin connection on the de Rham cohomology [13].

For a local section $\varphi(u;\lambda)du$ of the sheaf $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)$, we define differential operators ∇_{t_i} $(i = 1, \dots, n)$ and ∇_τ by

$$\begin{aligned}
\nabla_{t_i}\varphi(u;\lambda)du &= \frac{\partial\varphi}{\partial t_i}(u;\lambda)du + \omega_i(u)\varphi(u;\lambda)du - c_i\frac{\partial\varphi}{\partial\lambda}(u;\lambda)du,\\
\nabla_\tau\varphi(u;\lambda)du &= \frac{\partial\varphi}{\partial\tau}(u;\lambda)du + \omega_0(u)\varphi(u;\lambda)du - c_0\frac{\partial\varphi}{\partial\lambda}(u;\lambda)du\\
&\quad + \frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial\lambda}\nabla_{\mathcal{M}/\mathcal{T}}\varphi(u;\lambda),
\end{aligned}$$

where we put $\omega_i(u) = (\partial/\partial t_i)\log T(u)$ and $\omega_0(u) = (\partial/\partial\tau)\log T(u)$.

Proposition 4.1. *The differential operators ∇_{t_i} and ∇_τ define $\mathbf{C}$-endomorphisms on $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)$ and $\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)$. Consequently, ∇_{t_i} and ∇_τ induce differential operators on $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)/\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)$.*

Proof. Note that $\varphi(u;\lambda)$ has the following quasi-periodicity:

$$\varphi(u+1;\lambda) = \varphi(u;\lambda),\ \varphi(u+\tau;\lambda) = e^{2\pi\sqrt{-1}\lambda}\varphi(u;\lambda),$$

from which we see that $\nabla_{t_i}\varphi(u;\lambda)du$ is also a section of $f_*\Omega^1_{\mathcal{M}/\mathcal{T},\lambda}(*D)$. Besides, we have $\nabla_{t_i}\nabla_{\mathcal{M}/\mathcal{T}} = \nabla_{\mathcal{M}/\mathcal{T}}\nabla_{t_i}$, which implies that $\nabla_{t_i}(\nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)) \subset \nabla_{\mathcal{M}/\mathcal{T}} f_*\mathcal{O}_{\mathcal{M},\lambda}(*D)$. We can also prove the assertions for ∇_τ in a similar way. □

For $\lambda \in P \setminus \{0\}$, we take the basis $\{\mathfrak{s}(u - t_i;\lambda)du\}_{i=1}^n$ of $H^1(M, \mathcal{L} \otimes_{\mathbf{C}} R_\lambda)$ and fix a basis $\{\gamma_{(1)}, \dots, \gamma_{(n)}\}$ of $H_1(M, \check{\mathcal{L}} \otimes_{\mathbf{C}} \check{R}_\lambda)$. We consider the pairing

$$f_i^{(j)}(t_1, \dots, t_n, \tau) = \langle \gamma_{(j)}, \mathfrak{s}(u - t_i;\lambda)du \rangle.$$

By the well-known procedure, this pairing is written by the integral

$$f_i^{(j)}(t_1, \dots, t_n, \tau) = \int_{\gamma_{(j)}} T(u)\mathfrak{s}(u - t_i;\lambda)du.$$

Then, by the action of the Gauss-Manin connection, it follows that

$$\mathbf{f}^{(j)} = \begin{pmatrix} f_1^{(j)} \\ \vdots \\ f_n^{(j)} \end{pmatrix}$$

$(j = 1, \dots, n)$ form a fundamental system of solutions to a system of linear partial differential equations with respect to the independent variables $t_1, \dots, t_n, \tau$. Such a differential system is given by the following (see [15] for details):

(4.2)

$$\begin{cases} \dfrac{\partial f_i}{\partial t_j} = -c_j \mathfrak{s}(t_j - t_i; \lambda) f_j + c_j \rho(t_j - t_i) f_i, \quad j \neq i, \\ \dfrac{\partial f_i}{\partial t_i} = \displaystyle\sum_{k \neq i} c_k \mathfrak{s}(t_k - t_i; \lambda) f_k + (2\pi\sqrt{-1} c_0 - \sum_{k \neq i} c_k \rho(t_k - t_i)) f_i, \\ 2\pi\sqrt{-1} \dfrac{\partial f_i}{\partial \tau} = \displaystyle\sum_{k=1}^{n} \frac{c_k}{2} (\rho(t_i - t_k)^2 - \wp(t_i - t_k)) f_i + \sum_{k=1}^{n} c_k \frac{\partial \mathfrak{s}}{\partial \lambda}(t_k - t_i; \lambda) f_k, \end{cases}$$

where $\rho(u) = \theta'(u)/\theta(u)$, and we assume that λ depends on $t_1, \dots, t_n, \tau$ such that $\lambda + c_0\tau + c_1 t_1 + \cdots + c_n t_n + c_\infty = 0$ for some constant c_∞. This is the system of differential equations satisfied by the Riemann-Wirtinger integral (1.1).

5. The Riemann-Wirtinger integral and integral formulae for solutions of the KZB equations

Felder and Varchenko [9] showed that horizontal sections of the KZB connection admit integral representations which are multiple integrals over a direct product of elliptic curves obtained by integrating power products of theta functions. The Riemann-Wirtinger integral is related to the $\mathfrak{sl}_2$-case of the KZB equation. We put

$$T(u) = e^{2\pi\sqrt{-1} c_0 u} \theta(u - t_1)^{c_1} \cdots \theta(u - t_n)^{c_n};$$

here we do not assume $\sum_{j=1}^n c_j = 0$. Take a function $g(\lambda, \tau)$ which satisfies the following partial differential equation:

$$4\pi\sqrt{-1} \frac{\partial g}{\partial \tau}(\lambda, \tau) = -\sum_{j=1}^{n} c_j \frac{\partial^2 g}{\partial \lambda^2}(\lambda, \tau) + 4\pi\sqrt{-1} c_0 \frac{\partial g}{\partial \lambda}(\lambda, \tau). \tag{5.1}$$

Then we have

Proposition 5.1. *Put*

(5.2)

$$f_i = f_i(t_1, \dots, t_n, \tau; \lambda) = \int_\gamma T(u) g(\lambda - \sum_{j=1}^{n} c_j (u - t_j), \tau) \mathfrak{s}(u - t_i; \lambda) du, \quad i = 1, \dots, n.$$

Then f_i $(i = 1, \dots, n)$ satisfy the following system of partial differential equations:

(5.3)

$$\begin{cases} \dfrac{\partial f_i}{\partial t_j} = c_j \dfrac{\partial f_i}{\partial \lambda} - c_j \mathfrak{s}(t_j - t_i; \lambda) f_j + c_j \rho(t_j - t_i) f_i, \quad j \neq i, \\ \dfrac{\partial f_i}{\partial t_i} = -\displaystyle\sum_{k \neq i} c_k \frac{\partial f_i}{\partial \lambda} + \sum_{k \neq i} c_k \mathfrak{s}(t_k - t_i; \lambda) f_k + (2\pi\sqrt{-1} c_0 - \sum_{k \neq i} c_k \rho(t_k - t_i)) f_i, \\ 2\pi\sqrt{-1} \dfrac{\partial f_i}{\partial \tau} = 2\pi\sqrt{-1} c_0 \dfrac{\partial f_i}{\partial \lambda} - \displaystyle\sum_{k=1}^{n} \frac{c_k}{2} \frac{\partial^2 f_i}{\partial \lambda^2} + \sum_{k=1}^{n} c_k \frac{\partial \mathfrak{s}}{\partial \lambda}(t_k - t_i; \lambda) f_k \\ \qquad\qquad + \displaystyle\sum_{k=1}^{n} \frac{c_k}{2} (\rho(t_i - t_k)^2 - \wp(t_i - t_k) + \frac{1}{3} \frac{\theta'''}{\theta'}) f_i. \end{cases}$$

Proposition 5.1 includes a particular case of the results in [9]. We take the Lie algebra $\mathfrak{g}$ in [9] as $\mathfrak{g} = \mathfrak{sl}_2$. Let α be the positive simple root of $\mathfrak{sl}_2$ with $(\alpha, \alpha) = 2$, and let $\Lambda_1, \dots, \Lambda_n$ be dominant integral weights with $(\Lambda_1 + \cdots + \Lambda_n, \alpha) = 2$. If we put $c_0 = 0$ and $c_j = -(\Lambda_j, \alpha)/\kappa$, then the system of partial differential equations (5.3) essentially coincides with the KZB equation. We can prove Proposition 5.1 by a manner similar to that of Proposition 9 in [9]. Here we remark that we can eliminate the parameter c_0 from (5.3) if $\sum_{j=1}^{n} c_j \neq 0$ because if $g(\lambda, \tau)$ solves the equation (5.1), then $\tilde{g}(\lambda, \tau) = e^{-\pi\sqrt{-1}c_0(2\lambda + c_0\tau)/\sum_{j=1}^{n} c_j} g(\lambda, \tau)$ solves the equation (5.1) for $c_0 = 0$ and correspondingly, if $f_1, \dots, f_n$ satisfy the system (5.3), then $\tilde{f}_1, \dots, \tilde{f}_n$ given by

$$\tilde{f}(t_1, \dots, t_n, \tau; \lambda) = e^{-\pi\sqrt{-1}c_0(2\lambda + c_0\tau + 2\sum_{j=1}^{n} c_j t_j)/\sum_{j=1}^{n} c_j} f_i(t_1, \dots, t_n, \tau; \lambda)$$

satisfy the system (5.3) for $c_0 = 0$.

If we assume $\sum_{j=1}^{n} c_j = 0$, and λ depends on $t_1, \dots, t_n, \tau$ such that $\lambda + c_0\tau + c_1 t_1 + \cdots + c_n t_n + c_\infty = 0$ for some constant c_∞, the system (5.3) turns into the system (4.2). On the other hand, for $\sum_{j=1}^{n} c_j = 0$, the equation (5.1) is solved by $g(\lambda, \tau) = h(\lambda + c_0\tau)$ for an arbitrary function $h(\cdot)$ of one variable. Under the further assumption that $\lambda + c_0\tau + c_1 t_1 + \cdots + c_n t_n + c_\infty = 0$, we have

$$g(\lambda - \sum_{j=1}^{n} c_j(u - t_j), \tau) = h(\lambda + c_0\tau + \sum_{j=1}^{n} c_j t_j) = h(-c_\infty),$$

which is just a constant, and the integral (5.2) coincides with the integral (1.1). That is, Proposition 5.1 includes the Riemann-Wirtinger integral as a special case.

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