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UNIQUENESS OF CRITICAL TRAVELING WAVES FOR NONLOCAL LATTICE EQUATIONS WITH DELAYS

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ABSTRACT. In this paper, we investigate uniqueness (up to translation) of critical traveling waves for delayed lattice equations with monotone or nonmonotone birth functions. Our method requires finding exactly a priori asymptotic behavior of the critical traveling wave. This we accomplish with the help of Ikehara's Theorem.

1. INTRODUCTION

Consider the following system of differential equations on lattice $\mathbb{Z}$:

$$u_n'(t)=D[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)]-du_n(t)+\sum_{j\in\mathbb{Z}}\beta(n-j)b(u_j(t-r)),\ n\in\mathbb{Z}, \tag{1.1}$$

which describes the distribution of matured population of a single species over patch $\mathbb{Z}$; see Weng et al. [14]. Here $\beta(-j)=\beta(j)\geq 0$ with $\sum_{j\in\mathbb{Z}}\beta(j)=1$, $b(0)=dK-b(K)=0$ for some $K>0$. A traveling wave of (1.1) with speed c is a nonnegative bounded solution of the form $u_n(t)=\phi(n+ct)$ satisfying $\phi(-\infty)=0$ and $\liminf_{\xi\to+\infty}\phi(\xi)>0$. Substituting $u_n(t)=\phi(n+ct)$ into (1.1), we have the following wave profile equation:

$$c\phi'(\xi)=D[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)]-d\phi(\xi)+\sum_{j\in\mathbb{Z}}\beta(j)b(\phi(\xi-j-cr)). \tag{1.2}$$

In the case where $b(w)$ is monotone in $w\in[0,K]$, Weng et al. [14] obtained the spreading speed c_* and a monotone traveling wave for the wave speed $c\geq c_*$. When $c>c_*$, Ma and Zou [12] established the uniqueness (up to translation) and stability of traveling waves for the local case (i.e. $\beta(j)=0$ for all $j\neq 0$) of equation (1.1) with monotone birth functions. For the nonmonotone case, Ma et al. [13] established the spreading speed c_* and Ma [10] obtained the existence of traveling waves for the nonlocal lattice equation by the method used in reaction diffusion equation (see Remark 1.4 and Theorem 1.2 in [10]). Recently, for (1.1) with nonmonotone birth functions, Fang et al. [8] further studied the spreading speed c_* by the comparison argument and the fluctuation method, and the existence of traveling waves for $c\geq c_*$ by Schauder's fixed point theorem.

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More recently, the authors of [9] established the uniqueness of traveling waves of (1.1) for $c > c_*$ without assuming that wave profile is monotone, which was based on exactly a priori asymptotic behavior of the wave profile. Although this was accomplished by developing a structure theorem of entire solutions to a class of linear integro-differential equations, the use of the theorem (i.e. Theorem 2.1 in [9]) has a disadvantage; that is, the conclusion holds only when the characteristic equation has different eigenvalues in the belt region. Note that the characteristic equation $\Delta(c,\lambda) = 0$ of the wave profile equation (1.2) has double roots for $c = c_*$, and so Theorem 2.1 in [9] could not be applied. Our object is to give a proof about the uniqueness of traveling waves of (1.1) for $c = c_*$ with the help of Ikehara's Theorem. This technique was also used in [2] to prove the uniqueness of traveling waves for some monostable integrodifferential equations. To my knowledge, no result on uniqueness of critical traveling waves for delayed lattice systems has been reported. Other methods to prove uniqueness of *noncritical* traveling waves for other types of evolution systems can be found in [1, 3, 4, 5, 6, 7, 11, 12]. In [8], it is also shown that for (1.1), the minimal wave speed c_* coincides with the spreading speed and the linear determinacy holds for (1.1), meaning that c_* is fully determined by the characteristic equation of the linearization of (1.1) at the trivial equilibrium.

2. Uniqueness of critical traveling waves

In this section, we show the uniqueness of traveling waves of (1.1) for $c = c_*$. In order to accomplish this, we find exactly a priori asymptotic behavior of the critical traveling wave with the help of Ikehara's Theorem. Throughout this section, we assume that $\phi(n+c_*t)$ is a nonnegative bounded critical traveling wave (wave shape is monotone or nonmonotone) of (1.1) with $\phi(-\infty) = 0$ and $\liminf_{\xi\to+\infty}\phi(\xi) > 0$.

Assume that the function $b(u)$ is differentiable at $u = 0$. Define the characteristic equation

$$\Delta(c,\lambda) := c\lambda - D(e^{\lambda} + e^{-\lambda} - 2) + d - b'(0)\sum_{j\in\mathbb{Z}}\beta(j)e^{-\lambda(j+cr)},$$

where c is regarded as a parameter. We make the following assumptions on functions β and b:

- (H1) $\beta(j) = \beta(-j) \geq 0$ for all $j \in \mathbb{Z}$; $\sum_{j\in\mathbb{Z}}\beta(j) = 1$; there exists $\lambda^\sharp > 0$ such that $\chi(\lambda) =: \sum_{j\in\mathbb{Z}}\beta(j)e^{-\lambda j}$ is convergent when $\lambda \in [0,\lambda^\sharp)$ and $\lim_{\lambda\to\lambda^\sharp}\chi(\lambda) = +\infty$, where $\lambda^\sharp$ may be $+\infty$.
- (H2) b is continuous from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $b'(0) > d$; there exist $a > 0$, $\delta > 0$ and $\sigma > 1$ such that $b(u) \geq b'(0)u - au^\sigma$ for all $u \in [0,\delta]$.
- (H3) For all $u_1, u_2 \geq 0$, $|b(u_1) - b(u_2)| \leq b'(0)|u_1 - u_2|$.

From Lemma 3.1 in [9] and Lemma 2.2 in [13], we have

Lemma 2.1. *Assume that* (H1) *holds and* $b'(0) > d$. *Then, there exists a unique pair of* $c_* > 0$ *and* $\lambda_* > 0$ *such that the following assertions hold:*

- (i) $\Delta(c_*,\lambda_*) = 0$. $\frac{\partial\Delta(c,\lambda)}{\partial\lambda}|_{c=c_*,\lambda=\lambda_*} = 0$.
- (ii) *For any* $c > c_*$, $\Delta(c,\lambda) = 0$ *has exactly two positive roots* $\lambda_1 < \lambda_2 < \lambda^\sharp$ *and* $\Delta(c,\lambda) > 0$ *for any* $\lambda \in (\lambda_1,\lambda_2)$.
- (iii) $\Delta(c_*,\lambda)$ *does not have any zeros with* $\Re\lambda = \lambda_*$ *other than* $\lambda = \lambda_*$.

We recall a version of Ikehara's Theorem.

Lemma 2.2 ([2], Proposition 2.3)**.** *Let* $F(\lambda) = \int_0^{+\infty} u(x)e^{-\lambda x}dx$, *with* u *being a positive decreasing function. Assume that* $F(\lambda)$ *has the representation*

$$F(\lambda) = \frac{h(\lambda)}{(\lambda+\alpha)^{k+1}},$$

where $k > -1$ *and* h *is analytic in the strip* $-\alpha \le \mathrm{Re}\lambda < 0$. *Then*

$$\lim_{x\to+\infty} \frac{u(x)}{x^k e^{-\alpha x}} = \frac{h(-\alpha)}{\Gamma(\alpha+1)} > 0.$$

Theorem 2.1 (Asymptotic behavior)**.** *Assume that* (H1)-(H2) *hold. Let* $\phi(n+c_*t)$ *be a traveling wave of* (1.1) *with* $\phi(-\infty) = 0$. *Then*

$$\lim_{\xi\to-\infty} \frac{\phi(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists.} \tag{2.1}$$

Proof. We claim that ϕ is positive. Suppose to the contrary that there exists $\xi_1 \in \mathbb{R}$ such that $\phi(\xi_1) = 0$. Since ϕ is a nonnegative bounded traveling wave with $\phi(-\infty) = 0$ and $\liminf_{\xi\to+\infty} \phi(\xi) > 0$, $\xi_0 := \sup\{\xi \in \mathbb{R} \,|\, \phi(\xi) = 0\}$ is well defined and $\phi(\xi_0) = \phi'(\xi_0) = 0$. Thus,

$$\begin{aligned}0 = c\phi'(\xi_0) &= D[\phi(\xi_0+1)+\phi(\xi_0-1)-2\phi(\xi_0)] - d\phi(\xi_0) + \sum_{j\in\mathbb{Z}} \beta(j)b(\phi(\xi_0-j-cr))\\ &\ge D\phi(\xi_0+1) + D\phi(\xi_0-1) \ge 0,\end{aligned}$$

which implies that $\phi(\xi_0+1) = 0$. This contradicts the definition of ξ_0.

Similar to the proof of Lemma 3.4 in [9], the two-sided Laplace transform of ϕ by

$$L(\lambda) \equiv \int_{\mathbb{R}} \phi(\theta)e^{-\lambda\theta}d\theta$$

is analytic for any $\Re\lambda \in (0, \lambda_*)$ and $L(\lambda)$ has a singularity at $\lambda = \lambda_*$. Since

$$\int_{\mathbb{R}} e^{-\lambda\theta}[\phi(\theta+1)+\phi(\theta-1)-2\phi(\theta)]d\theta = L(\lambda)(e^{\lambda}+e^{-\lambda}-2)$$

and

$$\begin{aligned}&c_*\phi'(\xi) - D[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)] + d\phi(\xi) - b'(0)\sum_{j\in\mathbb{Z}} \beta(j)\phi(\xi-j-c_*r)\\ &= -b'(0)\sum_{j\in\mathbb{Z}} \beta(j)\phi(\xi-j-c_*r) + \sum_{j\in\mathbb{Z}} \beta(j)b(\phi(\xi-j-c_*r)) := R(\phi)(\xi),\end{aligned}$$

it follows that

$$\Delta(c_*,\lambda)L(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\theta}R(\phi)(\theta)d\theta. \tag{2.2}$$

Similarly to the argument in the proof of Lemma 3.4 in [9], there exists $\eta > 0$ such that the right hand side of (2.2) is analytic for $\lambda \in (0, \lambda_* + \eta)$. We rewrite (2.2) as

$$\int_{-\infty}^{0} \phi(\theta)e^{-\lambda\theta}d\theta = \frac{\int_{\mathbb{R}} e^{-\lambda\theta}R(\phi)(\theta)d\theta}{\Delta(c_*,\lambda)} - \int_0^{\infty} \phi(\theta)e^{-\lambda\theta}d\theta.$$

Note that $\int_0^{\infty} \phi(\theta)e^{-\lambda\theta}d\theta$ is analytic for $\Re\lambda > 0$. Also, by Lemma 2.1 (iii), $\Delta(c_*,\lambda) = 0$ does not have any zero with $\Re\lambda = \lambda_*$ other than $\lambda = \lambda_*$.

Assume that $\phi(\xi)$ is increasing for large $-\xi > 0$. Then we can choose a translation of ϕ such that it is increasing for $\xi < 0$. Letting $u(\xi) = \phi(-\xi)$ and $T(u)(\xi) := -b'(0)\sum_{j\in\mathbb{Z}}\beta(j)u(\xi+j+c_*r) + \sum_{j\in\mathbb{Z}}\beta(j)b(u(\xi+j+c_*r))$, it is clear that $u(\xi)$ is decreasing $\xi > 0$ and

$$\int_0^\infty u(\theta)e^{\lambda\theta}d\theta = \frac{\int_{\mathbb{R}} e^{\lambda\theta}T(u)(\theta)d\theta}{\Delta(c_*,\lambda)} - \int_{-\infty}^0 u(\theta)e^{\lambda\theta}d\theta := \frac{h(\lambda)}{(\lambda-\lambda_*)^{k+1}},$$

where $k = 1$ and

$$h(\lambda) = \frac{(\lambda-\lambda_*)^2\int_{\mathbb{R}} e^{\lambda\theta}T(u)(\theta)d\theta}{\Delta(c_*,\lambda)} - (\lambda-\lambda_*)^2\int_{-\infty}^0 u(\theta)e^{\lambda\theta}d\theta.$$

By Lemma 2.1 (i), λ_* is a double root of $\Delta(c_*,\lambda) = 0$, and hence $\lim_{\lambda\to\lambda_*} h(\lambda)$ exists. Therefore, $h(\lambda)$ is analytic for all $0 < \Re\lambda \le \lambda_*$. Then Lemma 2.2 implies that

$$\lim_{\xi\to+\infty}\frac{u(\xi)}{\xi e^{-\lambda_*\xi}} \text{ exists;}$$

that is,

$$\lim_{\xi\to-\infty}\frac{\phi(\xi)}{|\xi|e^{\lambda_*\xi}} \text{ exists.}$$

Assume that $\phi(\xi)$ is not monotone for large $-\xi > 0$. Letting $p = \frac{2D+d}{c_*}$ and $\hat{\phi}(\xi) = \phi(\xi)e^{p\xi} > 0$, it follows that

$$c_*\hat{\phi}'(\xi) = D[\hat{\phi}(\xi+1)e^{-p} + \hat{\phi}(\xi-1)e^{p}] + (c_*p - 2D - d)\phi(\xi) + b(\phi(\xi - c_*r))e^{p\xi} > 0,$$

which implies that $\hat{\phi}'(\xi) > 0$ for any $\xi \in \mathbb{R}$. Letting $\hat{u}(\xi) = \hat{\phi}(-\xi)$, it is obvious that $\hat{u}(\xi)$ is decreasing $\xi > 0$. Let $\widehat{L}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\xi}\hat{\phi}(\xi)d\xi$. Noting that $\widehat{L}(\lambda) = L(\lambda - p)$ and repeating the above argument, we have that

$$\lim_{\xi\to+\infty}\frac{\hat{u}(\xi)}{\xi e^{-(p+\lambda^*)\xi}} = \lim_{\xi\to-\infty}\frac{\phi(\xi)}{|\xi|e^{\lambda^*\xi}} \text{ exists.}$$

This completes the proof. □

Remark 2.1. Similarly to the proof of Theorem 2.1, without assuming the wave profile is monotone, we can also obtain that

$$\lim_{\xi\to-\infty}\frac{\phi(\xi)}{e^{\lambda_1\xi}} \text{ exists}$$

for $c > c_*$, where λ_1 is given in Lemma 2.1.

Theorem 2.2 (Uniqueness)**.** *Assume that* (H1)-(H2) *hold and that function β has compact support. Let φ, ψ be two traveling waves of* (1.1) *with the speed c_*. Then ϕ is a translation of ψ; more precisely, there exists $\bar{\xi} \in \mathbb{R}$ such that $\phi(\xi) = \psi(\xi + \bar{\xi})$.*

Proof. Let φ, ψ be two traveling waves for $c = c_*$. According to Theorem 2.1, there exist some positive numbers θ_1 and θ_2 such that

$$\lim_{\xi\to-\infty}\frac{\phi(\xi)}{|\xi|e^{\lambda_*\xi}} = \theta_1 \text{ and } \lim_{\xi\to-\infty}\frac{\psi(\xi)}{|\xi|e^{\lambda_*\xi}} = \theta_2.$$

Define

$$w_\epsilon(\xi) := \frac{\phi(\xi) - \psi(\xi+\bar{\xi})}{(\varepsilon|\xi|+1)e^{\lambda_*\xi}}, \tag{2.3}$$

where $\bar{\xi} = \frac{1}{\lambda_*} \ln \frac{\theta_1}{\theta_2}$. Then $w_\varepsilon(\pm\infty) = 0$, and hence $\max_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\}$ and $\min_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\}$ exist. Without loss of generality, we assume $\max_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} \geq |\min_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\}|$ (otherwise, we may consider $w_\epsilon(\xi) := \frac{\psi(\xi+\bar{\xi})-\phi(\xi)}{(\varepsilon|\xi|+1)e^{\lambda_*\xi}}$). So, if $w_\epsilon(\xi) \not\equiv 0$, there exists ξ_0^ϵ such that

$$w_\epsilon(\xi_0^\epsilon) = \max_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} > 0 \text{ and } w_\epsilon'(\xi_0) = 0.$$

We first suppose that $\xi_0^\epsilon \to \infty$ as $\epsilon \to 0$. Choose $\epsilon > 0$ sufficiently small such that $\xi_0^\epsilon > \sup\{j : j \in \operatorname{supp}\beta(j)\} + \max\{1, c_*r\}$. Note that

$$\phi'(\xi_0^\epsilon) - \psi'(\xi_0^\epsilon + \bar{\xi}) = w_\epsilon'(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi} + w_\epsilon(\xi_0^\epsilon)\epsilon e^{\lambda_*\xi} + w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\lambda_* e^{\lambda_*\xi},$$

and for all $u, v \geq 0$,

$$|b(u) - b(v)| \leq b'(0)|u - v|. \tag{2.4}$$

Thus, we have

$$\begin{aligned}
&c_* w_\epsilon(\xi_0^\epsilon)\epsilon e^{\lambda_*\xi} + c_* w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\lambda_* e^{\lambda_*\xi} \\
&= D\{[\phi(\xi_0^\epsilon + 1) - \psi(\xi_0^\epsilon + \bar{\xi} + 1)] + [\phi(\xi_0^\epsilon - 1) - \psi(\xi_0^\epsilon + \bar{\xi} - 1)] \\
&\quad - 2[\phi(\xi_0^\epsilon) - \psi(\xi_0^\epsilon + \bar{\xi})]\} - d[\phi(\xi_0^\epsilon) - \psi(\xi_0^\epsilon + \bar{\xi})] \\
&\quad + \sum_{j\in\mathbb{Z}} \beta(j)[b(\phi(\xi_0^\epsilon - j - c_*r)) - b(\psi(\xi_0^\epsilon + \bar{\xi} - j - c_*r))] \\
&\leq D\{w(\xi_0^\epsilon + 1)[\epsilon(\xi_0^\epsilon + 1) + 1]e^{\lambda_*(\xi_0^\epsilon+1)} + w(\xi_0^\epsilon - 1)[\epsilon(\xi_0^\epsilon - 1) + 1]e^{\lambda_*(\xi_0^\epsilon-1)} \\
&\quad - 2w(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi_0^\epsilon}\} - dw(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi_0^\epsilon} \\
&\quad + b'(0)\sum_{j\in\mathbb{Z}} \beta(j)|w(\xi_0^\epsilon - j - c_*r)|[\epsilon(\xi_0^\epsilon - j - c_*r) + 1]e^{\lambda_*(\xi_0^\epsilon - j - c_*r)}.
\end{aligned} \tag{2.5}$$

It follows from (2.5), $\Delta(c_*, \lambda_*) = 0$ and

$$c_* = D(e^{\lambda_*} - e^{-\lambda_*}) - b'(0)\sum_{j\in\mathbb{Z}} \beta(j)(j + c_*r)e^{-\lambda_*(j+c_*r)} \tag{2.6}$$

that

$$w_\epsilon(\xi_0^\epsilon) = w_\epsilon(\xi_0^\epsilon \pm 1).$$

Indeed, assume that $w_\epsilon(\xi_0^\epsilon) > w_\epsilon(\xi_0^\epsilon + 1)$ or $w_\epsilon(\xi_0^\epsilon) > w_\epsilon(\xi_0^\epsilon - 1)$. Then

$$\begin{aligned}
&c_* w_\epsilon(\xi_0^\epsilon)\epsilon e^{\lambda_*\xi} + c_* w_\epsilon(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)\lambda_* e^{\lambda_*\xi} \\
&< D\{w_\epsilon(\xi_0^\epsilon)[\epsilon(\xi_0^\epsilon + 1) + 1]e^{\lambda_*(\xi_0^\epsilon+1)} + w_\epsilon(\xi_0^\epsilon)[\epsilon(\xi_0^\epsilon - 1) + 1]e^{\lambda_*(\xi_0^\epsilon-1)} \\
&\quad - 2w(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi_0^\epsilon}\} - dw(\xi_0^\epsilon)(\epsilon\xi_0^\epsilon + 1)e^{\lambda_*\xi_0^\epsilon} \\
&\quad + b'(0)\sum_{j\in\mathbb{Z}} \beta(j)w_\epsilon(\xi_0^\epsilon)[\epsilon(\xi_0^\epsilon - j - c_*r) + 1]e^{\lambda_*(\xi_0^\epsilon - j - c_*r)},
\end{aligned} \tag{2.7}$$

which implies that

$$c_* < D(e^{\lambda_*} - e^{-\lambda_*}) - b'(0)\sum_{j\in\mathbb{Z}} \beta(j)(j + c_*r)e^{-\lambda_*(j+c_*r)}.$$

This is a contradiction to (2.6). By the bootstrapping arguments, we have $w_\epsilon(\xi_0^\epsilon) = w_\epsilon(\xi_0^\epsilon - j)$ for $j \in \mathbb{Z}$, which implies that w_ϵ is a constant. Since $w_\epsilon(+\infty) = 0$, we get $\phi \equiv \psi$.

Next we assume that $\xi_0^\epsilon \to -\infty$ as $\epsilon \to 0$. Then $w_\epsilon(\xi_0^\epsilon) \to 0$, as $\epsilon \to 0$. Since

$$\lim_{\epsilon\to 0} w_\epsilon(\xi) = w(\xi) := \frac{\phi(\xi) - \psi(\xi+\bar{\xi})}{e^{\lambda_* \xi}} \quad \text{for all } \xi \in \mathbb{R} \tag{2.8}$$

and $w_\epsilon(x) \le w_\epsilon(\xi_0^\epsilon)$, we have $w(\xi) \le 0$ for all $\xi \in \mathbb{R}$. Note that $w_\epsilon(\xi_0^\epsilon) > 0$ implies $\phi(\xi_0^\epsilon) - \psi(\xi_0^\epsilon + \bar{\xi}) > 0$ and hence $w(\xi_0^\epsilon) > 0$, which gives a contradiction.

Lastly, we assume $\{\xi_0^\epsilon\}$ is bounded. Then we can take a subsequence $\xi_0^\epsilon \to \xi_1$ as $\epsilon \to 0$, for some finite ξ_1. From uniform convergence of w_ϵ to w on compact sets, $w_\epsilon(\xi_0^\epsilon) \to w(\xi_1)$ as $\epsilon \to 0$, where $w(\xi)$ is defined by (2.8). Thus, $w(\xi) = \lim_{\epsilon\to 0} w_\epsilon(\xi) \le \lim_{\epsilon\to 0} w_\epsilon(\xi_0^\epsilon) = w(\xi_1)$ for all $\xi \in \mathbb{R}$. Now we begin with repeating the above argument for $w(\xi)$. It is obvious that $w(\xi_1) = \max_{\xi\in\mathbb{R}}\{w(\xi)\} \ge 0$ and $w'(\xi_1) = 0$. Since $\max\limits_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\} \ge |\min\limits_{\xi\in\mathbb{R}}\{w_\epsilon(\xi)\}|$, we have $\max\limits_{\xi\in\mathbb{R}}\{w(\xi)\} \ge |\min\limits_{\xi\in\mathbb{R}}\{w(\xi)\}|$. We claim that $w(\xi_1) = w(\xi_1 \pm 1)$. Assume, for the sake of contradiction, that either $w(\xi_1+1) < w(\xi_1)$ or $w(\xi_1-1) < w(\xi_1)$. According to (2.4) and $\phi'(\xi_1) - \psi'(\xi_1+\bar{\xi}) = w'(\xi_1)e^{\lambda_*\xi_1} + \lambda_* w(\xi_1)e^{\lambda_*\xi_1}$, we obtain

$$\begin{aligned}
c_*\lambda_* w(\xi_1)e^{\lambda_*\xi_1} =& D[w(\xi_1+1)e^{\lambda_*(\xi_1+1)} + w(\xi_1-1)e^{\lambda_*(\xi_1-1)} - 2w(\xi_1)e^{\lambda_*\xi_1}]\\
&- dw(\xi_1)e^{\lambda_*\xi_1} + \sum_{j\in\mathbb{Z}} \beta(j)[b(\phi(\xi_0^\epsilon - j - c_* r)) - b(\psi(\xi_0^\epsilon + \bar{\xi} - j - c_* r))]\\
\le& [w(\xi_1+1)e^{\lambda_*(\xi_1+1)} + w(\xi_1-1)e^{\lambda_*(\xi_1-1)} - 2w(\xi_1)e^{\lambda_*\xi_1}]\\
&- dw(\xi_1)e^{\lambda_*\xi_1} + b'(0)\sum_{j\in\mathbb{Z}} \beta(j)|w(\xi_1 - j - c_* r)|e^{\lambda_*(\xi_1 - j - c_* r)}\\
<& [w(\xi_1)e^{\lambda_*(\xi_1+1)} + w(\xi_1)e^{\lambda_*(\xi_1-1)} - 2w(\xi_1)e^{\lambda_*\xi_1}]\\
&- dw(\xi_1)e^{\lambda_*\xi_1} + b'(0)\sum_{j\in\mathbb{Z}} \beta(j)w(\xi_1)e^{\lambda_*(\xi_1 - j - c_* r)},
\end{aligned}$$

which is a contradiction to $\Delta(c_*, \lambda_*) = 0$. Repeating the above argument and by $w(+\infty) = 0$, we have $\phi \equiv \psi$. This completes the proof. ☐

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