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ON THE RECURSION FORMULA FOR DOUBLE HURWITZ NUMBERS

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Abstract. In this paper, we will give a recursion formula for double Hurwitz numbers by the cut-join analysis. This recursion formula can be considered as a generalized version of the recursion formula for simple Hurwitz numbers derived by Mulase and Zhang. As a direct application, we get a polynomial identity for Goulden-Jackson-Vakil's conjectural intersection numbers and an explicit recursion formula for the computation of these intersection numbers with only ψ-classes.

1. Introduction

In this paper, we will follow the notation used in [7]. Let $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_n)$ be two partitions of a positive integer d. The double Hurwitz number $H^g_{\mu,\nu}$ is the number of genus g branched covers of $\mathbb{CP}^1$ with branching corresponding to μ and ν over 0 and ∞ respectively. By the Riemann-Hurwitz formula, such covers must have $r^g_{\mu,\nu} = (2g - 2 + m + n)$ fixed simple branched points. As in [7] , we take the extra condition that the points mapping to 0 and ∞ are labelled. Thus $H^g_{\mu,\nu}$ would be $|\operatorname{Aut}(\mu)| \cdot |\operatorname{Aut}(\nu)|$ bigger than the conventional definition [2, 8].

To understand the structure of double Hurwitz numbers is an interesting topic involving geometry, algebra and combinatorics.

We define $H^g_\mu := H^g_{\mu,(1,..,1)}$. Then H^g_μ is the simple Hurwitz number which has a rich structure.

The generating function of simple Hurwitz numbers H^g_μ is defined by

$$H(\lambda, p) = \sum_{g\geq 0} \sum_{|\mu|>0} \frac{H^g_\mu}{|\operatorname{Aut}(\mu)|} p_\mu \frac{\lambda^{r^g_\mu}}{r^g_\mu!},$$

which satisfies the following cut-join equation [6, 12, 19]:

$$\frac{\partial H}{\partial \lambda} = \frac{1}{2} \left(\sum_{i,j\geq 1} (i+j) p_i p_j \frac{\partial H}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2 H}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial H}{\partial p_i} \cdot \frac{\partial H}{\partial p_j} \right). \tag{1}$$

The celebrated Ekedahl-Lando-Shapiro-Vainshtein [2] formula relates Hurwitz numbers to the Hodge integrals on $\overline{\mathcal{M}}_{g,m}$ which is the Deligne-Mumford compactification of the moduli space of genus g curves with m marked points such that

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$2g-2+m>0$,

$$H_\mu^g = r_\mu^g! \prod_{i=1}^m \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,m}} \frac{1-\lambda_1+\lambda_2+\cdots+\pm\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_m\psi_m)}, \tag{2}$$

where λ_k and ψ_k are the λ-class and ψ-class defined as follows: Denote by $\pi_{g,m+1} : \overline{\mathcal{M}}_{g,m+1} \to \overline{\mathcal{M}}_{g,m}$ the natural projection. Let $\omega_{\pi_{g,m+1}}$ be the relative dualizing sheaf of $\pi_{g,m+1}$. The Hodge bundle on $\overline{\mathcal{M}}_{g,m}$ is defined by $\mathbb{E} = (\pi_{g,m+1})_*\omega_{\pi_{g,m+1}}$. Then

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,m}, \mathbb{Q}).$$

Let $\sigma : \overline{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_{g,m+1}$ be the i-th section of $\pi_{g,m+1}$, and $\mathcal{L}_i = \sigma_i^*(\pi_{g,m+1})$. Then

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,m}, \mathbb{Q}).$$

The integrand on the right-hand side of formula (2) should be expanded formally, and then taking the terms of degree $3g-3+n$ capped with the fundamental class $\overline{\mathcal{M}}_{g,m}$. We introduce the Witten notation of *(linear) Hodge integrals*:

$$\langle \tau_{b_1}\tau_{b_2}\cdots\tau_{b_m}\lambda_k\rangle_g = \int_{\overline{\mathcal{M}}_{g,m}} \psi_1^{b_1}\cdots\psi_m^{b_m}\lambda_k, \tag{3}$$

which are zeros unless $\sum_{i=1}^m b_i + k = 3g-3+m$. We note that the original ELSV formula [2] includes a factor $|\operatorname{Aut}(\mu)|$, but here we consider the points over ∞ to be labelled.

The moduli spaces of curves have played an important role in diverse fields such as algebraic geometry, combinatorics, representation theory and mathematics physics. Many topological properties of the moduli spaces of curves are related to the computation of certain Hodge integrals [17, 3, 9, 4, 5, 13, 14, 18].

As stated above, the Hurwitz numbers relate to the linear Hodge integrals via the ELSV formula (2). Moreover, the generating function of Hurwitz numbers should satisfy the cut-join equation (1). Therefore, it is a natural idea to use the cut-and-join restriction of Hurwitz numbers to get recursion formulas for linear Hodge integrals (3). Much progress has been achieved in this direction [16, 11, 8, 1, 10, 20, 21, 15]. By cut-join analysis, due to Mulase and Zhang [15], formula (1) is equivalent to the following recursion formula for simple Hurwitz numbers:

$$\begin{aligned}(4)\quad &(2g-2+l(\mu)+|\mu|)H_g(\mu)\\ &= \sum_{i<j}(\mu_i+\mu_j)H_g((\mu(\hat{i},\hat{j}),\mu_i+\mu_j)) + \frac{1}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta\Bigg(H_{g-1}((\mu(\hat{i}),\alpha,\beta))\\ &\qquad + \sum_{\xi_1\coprod\xi_2=\mu(\hat{i})}\sum_{g_1+g_2=g}H_{g_1}(\xi_1,\alpha)H_{g_2}(\xi_2,\beta)\Bigg),\end{aligned}$$

where $H_g(\mu) = \frac{H_\mu^g}{r_\mu^g!}$. Via some transformation of variables, they obtained a polynomial identity for linear Hodge integrals (Theorem 5.1 in [15]). As an application, they showed that this recursion implies the Witten-Kontsevich theorem when restricted to the top degree terms and the λ_g formula when reduced to the lowest degree terms.

Now, we come back to the double Hurwitz number $H^g_{\mu,\nu}$. We define the following generating function of $H^g_{\mu,\nu}$:

$$H = \sum_{g\geq 0}\sum_{d\geq 1}\sum_{\mu,\nu\vdash d} y^d z^d p_\alpha q_\beta u^{l(\beta)} \frac{H^g_{\mu,\nu}}{r^g_{\mu,\nu}!|Aut(\mu)|\cdot|Aut(\nu)|}.$$

The following lemma is mentioned in [8].

Lemma 1.1 (Cut-join equation)**.**

$$\left(\sum_{j\geq 1} p_i\frac{\partial}{\partial p_i} + u\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial y} - 2\right)H \tag{5}$$
$$= \frac{1}{2}\left(\sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial H}{\partial p_{i+j}} + ijp_{i+j}y\frac{\partial^2 H}{\partial p_i\partial p_j} + ijp_{i+j}\frac{\partial H}{\partial p_i}\cdot\frac{\partial H}{\partial p_j}\right)$$

with initial conditions $[z^ip_iq_iu]H = \frac{1}{i}$ *for* $i\geq 1$.

With the same analysis used in [15], we obtain a recursion formula for $H^g_{\mu,\nu}$ which is equivalent to formula (5).

Theorem 1.2. *The functions* $H_g(\mu,\nu)$ *defined by* $H_g(\mu,\nu) = \frac{H^g_{\mu,\nu}}{r^g_{\mu,\nu}!}$ *satisfy the following recursion formula:*

(6)

$$\begin{aligned}&(2g-2+l(\mu)+l(\nu))H_g(\mu,\nu)\\ &= \sum_{i<j}(\mu_i+\mu_j)H_g((\mu(\hat{i},\hat{j}),\mu_i+\mu_j),\nu) + \frac{1}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta\Bigg(H_{g-1}((\mu(\hat{i}),\alpha,\beta),\nu)\\ &\qquad + \sum_{\xi_1\coprod\xi_2=\mu(\hat{i})}\sum_{g_1+g_2=g}\sum_{\nu_1\coprod\nu_2=\nu}H_{g_1}((\xi_1,\alpha),\nu_1)H_{g_2}((\xi_2,\beta),\nu_2)\Bigg),\end{aligned}$$

where the notation $(\mu(\hat{i},\hat{j}),\mu_i+\mu_j)$ *and* $(\mu(\hat{i}),\alpha,\beta)$ *will be introduced at the beginning of Section 2.*

Remark 1.3. When ν is the partition of d with profile $\nu=(1,\ldots,1)$, then

$$H_g(\mu,(1,\ldots,1)) = d!H_g(\mu), \tag{7}$$

where $H_g(\mu)$ is a quantity related to simple Hurwitz numbers defined in [15] (formula (3.5) in [15]).

Substituting (7) into (6), the recursion formula (6) is reduced to the recursion formula for simple Hurwitz numbers (4).

Example 1.4. When $g=0$ and $\nu=(d)$,

$$(l(\mu)-1)H_0(\mu,(d)) = \sum_{i<j}(\mu_i+\mu_j)H_0((\mu(\hat{i},\hat{j}),\mu_i+\mu_j),(d)) \tag{8}$$

with the initial value $H_0((d),(d)) = \frac{1}{d}$. Solving the above recursion (8), we get $H_0(\mu,(d)) = d^{l(\mu)-2}$. Hence, $H^0_{(d),\mu} = (l(\mu)-1)d^{l(\mu)-2}$, which was calculated in [7].

When $\nu = (d)$, Goulden, Jackson and Vakil [7] conjectured that there existed a moduli space $\overline{Pic}_{g,n}$ and an ELSV-type formula for Hurwitz numbers $H^g_{\mu,(d)}$.

Conjecture 1.5. *For each $g \geq 0, n \geq 1$, $(g,n) \neq (0,1),(0,2)$, $\mu \vdash d$ and the length of μ, $l(\mu)$ is l,*

$$H^g_{\mu,(d)} = r^g_{\mu,(d)}!d\int_{\overline{Pic}_{g,l}} \frac{\Lambda_0 - \Lambda_2 + \cdots + \pm\Lambda_{2g}}{(1-\mu_1\psi_1)\cdots(1-\mu_l\psi_l)},$$

where $\overline{Pic}_{g,l}$ is the conjectured moduli space with a fundamental class $[\overline{Pic}_{g,l}]$ of dimension $4g-3+l$. ψ_i and Λ_i are two natural classes defined on $\overline{Pic}_{g,l}$.

We introduce the notation of *linear GJV intersection numbers*,

$$\langle\langle \tau_{b_1}\tau_{b_2}\cdots\tau_{b_l}\Lambda_k\rangle\rangle_g = \int_{\overline{Pic}_{g,l}} \psi_1^{b_1}\cdots\psi_l^{b_l}\Lambda_k,$$

which are zeros unless $\sum_{i=1}^l b_i + k = 4g-3+l$. We refer the reader to the original paper [7] for a more precise description of the conjectured moduli space $\overline{Pic}_{g,l}$ and the GJV intersection numbers.

With the similar variable transformation technique used in [15], we obtain the following polynomial identity:

Theorem 1.6. *The linear GJV intersection numbers satisfy the following polynomial identity:*

(9)

$$\begin{aligned}
&(2g-1+l)\sum_{b_L\geq 0}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{b_L}\Lambda_{2k}\rangle\rangle_g\psi_{b_L}(y_L)\\
&= \sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\sum_{i<j}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda_{2k}\rangle\rangle_g\psi_{b_{L\setminus\{i,j\}}}(y_{L\setminus\{i,j\}})\\
&\times\left[\psi_{a+1}(y_i)\frac{y_i(y_j-1)}{y_i-y_j} - \psi_{a+1}(y_j)\frac{y_j(y_i-1)}{y_i-y_j}\right]\\
&+\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda_{2k}\rangle\rangle_{g-1}\psi_{b_{L\setminus\{i\}}}(y_{L\setminus\{i\}})\psi_{a_1+1}(y_i)\psi_{a_2+1}(y_i),
\end{aligned}$$

where we have used the multiple index $\tau_{b_L} = \prod_{i=1}^l \tau_{b_i}$ and $\psi_{b_L}(y_L) = \prod_{i=1}^l \psi_{b_i}(y_i)$. $\psi_{b_i}(y_i)$ is a polynomial of y_i defined by

$$\psi_{b_i}(y_i) = \left(\left(y_i^2 - y_i\right)\frac{d}{dy_i}\right)^{b_i}(y_i - 1)$$

for $b_i \geq 0$.

Restricting the above polynomial identity to the top degree terms, we obtain a recursion formula for the GJV's pure ψ-class intersection numbers.

Theorem 1.7. *We have the following recursion formula for intersection numbers* $\langle\langle \tau_{b_L} \rangle\rangle_g$*:*

$$\langle\langle \tau_{b_L} \rangle\rangle_g = \frac{1}{2g-1+l} \Bigg[\sum_{i<j} \langle\langle \tau_{b_i+b_j-1} \tau_{b_{L\setminus\{i,j\}}} \rangle\rangle_g \frac{(b_i+b_j)!}{b_i! b_j!} + \frac{1}{2} \sum_{i=1}^{l} \sum_{a_1+a_2=b_i-3} \langle\langle \tau_{a_1} \tau_{a_2} \tau_{b_{L\setminus\{i\}}} \rangle\rangle_{g-1} \frac{(a_1+1)!(a_2+1)!}{b_i!} \Bigg]. \tag{10}$$

Example 1.8. When $g = 0$, the above recursion is

$$\langle\langle \tau_{b_L} \rangle\rangle_0 = \frac{1}{l-1} \sum_{i<j} \langle\langle \tau_{b_i+b_j-1} \tau_{b_{L\setminus\{i,j\}}} \rangle\rangle_0 \frac{(b_i+b_j)!}{b_i! b_j!}$$

with the solution

$$\langle\langle \tau_{b_L} \rangle\rangle_0 = \binom{l-3}{b_1, \ldots, b_l}.$$

This paper will be arranged as follows. After introducing the identities that will be used in the cut-join analysis, we obtain the recursion formula of Theorem 1.2 for double Hurwitz numbers in Section 2. Then, in Section 3, we show how to derive the polynomial identity for linear GJV intersection numbers (9) and the explicit formula for $\langle\langle \tau_{b_L} \rangle\rangle_g$ (10).

2. Recursion formula for double Hurwitz numbers

In this section, we present some combinatorial identities which will be used in the proof of our main result.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l) = (1^{m_1(\mu)} 2^{m_2(\mu)} \cdots)$ be a partition of d with length l. Then $|\operatorname{Aut}(\mu)| = \prod_{j\geq 1} m_j(\mu)$. We have the following formulas related to partition μ.

i) We denote by $(\mu(\hat{i}, \hat{j}), \mu_i + \mu_j)$ the partition obtained from μ by erasing μ_i, μ_j and adding $\mu_i + \mu_j$. Then if $\mu_i < \mu_j$,

$$(\mu(\hat{i}, \hat{j}), \mu_i + \mu_j) = 1^{m_1(\mu)} 2^{m_2(\mu)} \cdots \mu_i^{m_{\mu_i}(\mu)-1} \cdots \mu_j^{m_{\mu_j}(\mu)-1} \cdots (\mu_i+\mu_j)^{m_{\mu_i+\mu_j}+1} \cdots ;$$

hence by definition,

$$\begin{aligned} |\operatorname{Aut}(\mu(\hat{i}, \hat{j}), \mu_i + \mu_j)| &= (m_{\mu_i}(\mu) - 1)(m_{\mu_j}(\mu) - 1)(m_{\mu_i+\mu_j}(\mu) + 1) \prod_{k \neq i,j} m_{\mu_k}(\mu) \\ &= |\operatorname{Aut}(\mu)| \cdot \frac{m_{\mu_i+\mu_j}(\mu) + 1}{m_{\mu_i}(\mu) m_{\mu_j}(\mu)}. \end{aligned}$$

The analogous identity also holds for the case $\mu_i > \mu_j$.

When $\mu_i = \mu_j$,

$$(\mu(\hat{i}, \hat{j}), 2\mu_i) = 1^{m_1(\mu)} 2^{m_2(\mu)} \cdots \mu_i^{m_{\mu_i}(\mu)-2} \cdots (2\mu_i)^{m_{2\mu_i}(\mu)+1} \cdots ,$$

$$|\operatorname{Aut}(\mu(\hat{i}, \hat{j}), 2\mu_i)| = |\operatorname{Aut}(\mu)| \cdot \frac{m_{2\mu_i}(\mu) + 1}{(m_{\mu_i}(\mu) - 1)(m_{\mu_i}(\mu))}.$$

With a similar analysis, we have

ii) Let $(\mu(\hat{i}),a,b)$ be a partition obtained by replacing μ_i with two elements a,b such that $a+b=\mu_i$. Then we have

$$|\operatorname{Aut}(\mu(\hat{i}),a,b)| = \begin{cases} |\operatorname{Aut}(\mu)|\cdot\frac{(m_a(\mu)+1)(m_b(\mu)+1)}{m_{a+b}(\mu)}, & a\neq b,\\ |\operatorname{Aut}(\mu)|\cdot\frac{(m_a(\mu)+1)(m_a(\mu)+2)}{m_{2a}(\mu)}, & a=b.\end{cases}$$

iii) If ν_1 and ν_2 are two partitions such that their join is μ, i.e. $\nu_1\amalg\nu_2=\mu$, then

$$|\operatorname{Aut}(\nu_1)|\cdot|\operatorname{Aut}(\nu_2)| = |\operatorname{Aut}(\mu)|\cdot\frac{1}{\prod_{j\geq 1}\binom{m_j(\mu)}{m_j(\nu_1)}}.$$

iv) If $\nu_1\amalg\nu_2=\mu(\hat{i})$, and $a+b=\mu_i$, then

$$|\operatorname{Aut}(\nu_1,a)|\cdot|\operatorname{Aut}(\nu_2,b)| = \begin{cases} |\operatorname{Aut}(\mu)|\cdot\frac{(m_a(\mu)+1)(m_b(\mu)+1)}{m_{a+b}(\mu)}\cdot\frac{1}{\prod_{j\geq1}\binom{m_j(\mu(\hat{i}),a,b)}{m_j(\nu_1,a)}}, & a\neq b,\\ |\operatorname{Aut}(\mu)|\cdot\frac{(m_a(\mu)+1)(m_a(\mu)+2)}{m_{2a}(\mu)}\cdot\frac{1}{\prod_{j\geq1}\binom{m_j(\mu(\hat{i}),a,b)}{m_j(\nu_1,a)}}, & a=b.\end{cases}$$

Substituting H to the cut-join equation (5) and collecting the coefficients of $y^gz^dp_\alpha q_\beta u^{l(\beta)}$, we have

$$\left[y^gz^dp_\alpha q_\beta u^{l(\beta)}\right]LHS = (2g-2+l(\mu)+l(\nu))\frac{H_g(\mu,\nu)}{|\operatorname{Aut}(\mu)|\cdot|\operatorname{Aut}(\nu)|},$$

$$\left[y^gz^dp_\alpha q_\beta u^{l(\beta)}\right]\left(\frac{1}{2}\sum_{i,j\geq1}(i+j)p_ip_j\frac{\partial H}{\partial p_{i+j}}\right)$$
$$=\frac{1}{2}\sum_{i<j}(\mu_i+\mu_j)\frac{H_g((\mu(\hat{i},\hat{j}),\mu_i+\mu_j),\nu)}{|\operatorname{Aut}(\mu(\hat{i},\hat{j}),\mu_i+\mu_j)|\cdot|\operatorname{Aut}(\nu)|}\times\begin{cases}2\cdot\frac{m_{\mu_i+\mu_j}(\mu)+1}{m_{\mu_i}(\mu)\cdot m_{\mu_j}(\mu)}, & \mu_i\neq\mu_j,\\ \frac{m_{2\mu_i}(\mu)+1}{\binom{m_{\mu_i}(\mu)}{2}}, & \mu_i=\mu_j.\end{cases}$$

Also,

$$\left[y^gz^dp_\alpha q_\beta u^{l(\beta)}\right]\left(\frac{1}{2}\sum_{i,j\geq1}ijp_{i+j}y\frac{\partial^2H}{\partial p_ip_j}\right)$$
$$=\frac{1}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta$$
$$\times\frac{H_{g-1}((\mu(\hat{i}),\alpha,\beta),\nu)}{|\operatorname{Aut}(\mu(\hat{i}),\alpha,\beta)|\cdot|\operatorname{Aut}(\nu)|}\times\begin{cases}\frac{(m_\alpha(\mu)+1)(m_\beta(\mu)+1)}{m_{\mu_i}(\mu)}, & \alpha\neq\beta,\\ \frac{(m_\alpha(\mu)+1)(m_\alpha(\mu)+2)}{m_{\mu_i}(\mu)}, & \alpha=\beta.\end{cases}$$

Moreover,

$$\begin{aligned}
&\left[y^g z^d p_\alpha q_\beta u^{l(\beta)}\right]\left(\frac{1}{2}\sum_{i,j\geq 1} ijp_{i+j}y\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j}\right)\\
&=\frac{1}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta\\
&\quad\times\sum_{\xi_1\amalg\xi_2=\mu(\hat{i})}\sum_{g_1+g_2=g}\sum_{\nu_1\amalg\nu_2=\nu}\frac{H_{g_1}((\xi_1,\alpha),\nu_1)H_{g_2}((\xi_2,\beta),\nu_2)}{|\operatorname{Aut}(\xi_1,\alpha)|\cdot|\operatorname{Aut}(\xi_2,\beta)|\cdot\cdot|\operatorname{Aut}(\nu_1)|\cdot|\operatorname{Aut}(\nu_2)|}\\
&\cdot\frac{1}{\prod_{j\geq 1}\binom{m_j(\nu)}{m_j(\nu_1)}}\times\begin{cases}\frac{m_\alpha(\xi_1,\alpha)m_\beta(\xi_2,\beta)}{m_{\alpha+\beta}(\mu)}\cdot\frac{1}{\prod_{j\geq 1}\binom{m_j(\mu(\hat{i}))}{m_j(\xi_1)}}, & \alpha\neq\beta,\\ \frac{m_\alpha(\xi_1,\alpha)m_\alpha(\xi_2,\beta)}{m_{2\alpha}(\mu)}\cdot\frac{1}{\prod_{j\geq 1}\binom{m_j(\mu(\hat{i}))}{m_j(\xi_1)}}, & \alpha=\beta.\end{cases}
\end{aligned}$$

Combining the above analysis, we obtain Theorem 2.1.

3. Polynomial recursion formula for Goulden-Jackson-Vakil's intersection numbers

Now, we focus on the special case for $\nu=(d)$. Then the formula (6) can be written as

$$(2g-1+l)H_g(\mu,(d))=\sum_{i<j}(\mu_i+\mu_j)H_g((\mu(\hat{i},\hat{j}),\mu_i+\mu_j),(d))\\ +\frac{1}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta H_{g-1}((\mu(\hat{i}),\alpha,\beta),(d)). \tag{11}$$

According to Conjecture 1.5, we have

$$\begin{aligned}
H_g(\mu,(d))&=\frac{H^g_{\mu,(d)}}{r^g_{\mu,(d)}!}\\
&=d\cdot\int_{\overline{Pic}_{g,l}}\frac{\Lambda_0-\Lambda_2+\cdots\pm\Lambda_{2g}}{(1-\mu_1\psi_1)\cdots(1-\mu_2\psi_2)}\\
&=d\cdot\sum_{b_i\geq 0}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{b_1}\tau_{b_2}\cdots\tau_{b_n}\Lambda_{2k}\rangle\rangle_g\mu_1^{b_1}\mu_2^{b_2}\cdots\mu_n^{b_n}.
\end{aligned}$$

As $H_{\mu,(d)}$ is a symmetric function of $\mu_1,\ldots,\mu_l$, it is natural to extend its definition to $\mu\in\mathbb{N}^l$.

Let us introduce the generating function

$$H_g(x_1,\ldots,x_l)=\sum_{\mu\in\mathbb{N}^l}H_g(\mu,(d))\cdot\prod_{i=1}^{l}x_i^{\mu_i}.$$

Then

$$\begin{aligned} H_g(x_1,x_2,\ldots,x_l) &= \sum_{\mu\in\mathbb{N}^l} d \sum_{b_L\geq 0}\sum_{k=0}^{g} \langle\langle \tau_{b_L}\Lambda_{2k}\rangle\rangle_g \mu_L^{b_L} x_L^{\mu_L} \\ &= d\cdot \sum_{b_L\geq 0}\sum_{k=0}^{g}(-1)^k \langle\langle \tau_{b_L}\Lambda_{2k}\rangle\rangle_g \phi_{b_L}(x_L), \end{aligned}$$

where $\phi_{b_L}(x_L)=\prod_{i=1}^{l}\phi_{b_i}(x_i)$, and

$$\phi_{b_i}(x_i)=\sum_{\mu_i\geq 1}\mu_i^{b_i}x_i^{\mu_i}=\left(x_i\frac{d}{dx_i}\right)^{b_i}\left(\sum_{\mu_i\geq 1}x_i^{\mu_i}\right)=\left(x_i\frac{d}{dx_i}\right)^{b_i}\left(\frac{1}{1-x_i}-1\right),$$

when $b_i\geq 0$.

We now introduce the new variable $y_i=\frac{1}{1-x_i}$. It is easy to show that

$$(y_i^2-y_i)\frac{d}{dy_i}=x_i\frac{d}{dx_i}.$$

With the new variable y_i,

$$\phi_{b_i}(x_i)=\left(\left(y_i^2-y_i\right)\frac{d}{dy_i}\right)^{b_i}(y_i-1).$$

We let

$$\psi_{b_i}(y_i)=\left[\left(y_i^2-y_i\right)\frac{d}{dy_i}\right]^{b_i}(y_i-1),$$

which is a polynomial of y_i.

Thereby,

$$H_g(x_1,...,x_l)=d\sum_{b_L\geq 0}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{b_L}\Lambda_{2k}\rangle\rangle_g\psi_{b_L}(y_L).$$

Now, let us apply the above variable transformation procedure to formula (6). The left-hand side is

$$\begin{aligned} &\sum_{\mu\in\mathbb{N}^l}(2g-1+l)H_g(\mu,(d))x_1^{\mu_1}\cdots x_l^{\mu_l} \\ &=(2g-1+l)\cdot d\cdot\sum_{b_L\geq 0}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{b_L}\Lambda_{2k}\rangle\rangle_g\psi_{b_L}(y_L). \end{aligned}\tag{12}$$

In order to calculate the first term on the right-hand side, we first need the following lemma:

Lemma 3.1.

$$\sum_{\mu_i,\mu_j\geq 1}(\mu_i+\mu_j)^{a+1}x_i^{\mu_i}x_j^{\mu_j}=\psi_{a+1}(y_i)\frac{y_i(y_j-1)}{y_i-y_j}-\psi_{a+1}(y_j)\frac{y_j(y_i-1)}{y_i-y_j}.$$

Proof. Prove by direct calculation,

$$\begin{aligned}
&\sum_{\mu_i,\mu_j\geq 1}(\mu_i+\mu_j)^{a+1}x_i^{\mu_i}x_j^{\mu_j}\\
&=\sum_{\mu_i,\mu_j\geq 1}(\mu_i+\mu_j)^{a+1}x_i^{\mu_i+\mu_j}\left(\frac{x_j}{x_i}\right)^{\mu_j}\\
&=\sum_{\mu_i,\mu_j\geq 0}(\mu_i+\mu_j)^{a+1}x_i^{\mu_i+\mu_j}\left(\frac{x_j}{x_i}\right)^{\mu_j}-\sum_{\mu_i\geq 1}\mu_i^{a+1}x_i^{\mu_i}-\sum_{\mu_j\geq 1}\mu_j^{a+1}x_j^{\mu_j}\\
&=\sum_{\nu\geq 0}\sum_{\mu_j=0}^{\nu}\nu^{a+1}x_i^{\nu}\left(\frac{x_j}{x_i}\right)^{\mu_j}-\phi_{a+1}(x_i)-\phi_{a+1}(x_j)\\
&=\phi_{a+1}(x_i)\frac{x_i}{x_i-x_j}-\phi_{a+1}(x_j)\frac{x_j}{x_i-x_j}-\phi_{a+1}(x_i)-\phi_{a+1}(x_j)\\
&=\phi_{a+1}(x_i)\frac{x_j}{x_i-x_j}-\phi_{a+1}(x_j)\frac{x_i}{x_i-x_j}\\
&=\psi_{a+1}(y_i)\frac{y_i(y_j-1)}{y_i-y_j}-\psi_{a+1}(y_j)\frac{y_j(y_i-1)}{y_i-y_j}.
\end{aligned}$$

□

Hence, by Lemma 3.1, we finish the calculation of the first term on the right-hand side,

(13)

$$\begin{aligned}
&\sum_{\mu\in\mathbb{N}^l}\sum_{i<j}(\mu_i+\mu_j)H_g((\mu(\hat{i},\hat{j}),\mu_i+\mu_j),(d))x_1^{\mu_1}x_2^{\mu_2}\cdots x_l^{\mu_l}\\
&=d\cdot\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\sum_{i<j}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda_{2k}\rangle\rangle_g\sum_{\mu\in\mathbb{N}^l}\mu_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}}x_{L\setminus\{i,j\}}^{\mu_{L\setminus\{i,j\}}}(\mu_i+\mu_j)^{a+1}x_i^{\mu_i}x_j^{\mu_j}\\
&=d\cdot\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\sum_{i<j}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda_{2k}\rangle\rangle_g\psi_{b_{L\setminus\{i,j\}}}(y_{L\setminus\{i,j\}})\\
&\quad\times\left[\psi_{a+1}(y_i)\frac{y_i(y_j-1)}{y_i-y_j}-\psi_{a+1}(y_j)\frac{y_j(y_i-1)}{y_i-y_j}\right].
\end{aligned}$$

Lastly, we need to calculate the second term on the right-hand side,

(14)

$$
\begin{aligned}
&\sum_{\mu\in\mathbb{N}^l}\frac{1}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta H_{g-1}((\mu(\hat{i}),\alpha,\beta),(d))x_1^{\mu_1}x_2^{\mu_2}\cdots x_l^{\mu_l}\\
&=\frac{1}{2}d\cdot\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\mu\in\mathbb{N}^l}\sum_{\alpha+\beta=\mu_i}\alpha\cdot\beta\\
&\quad\times\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda_{2k}\rangle\rangle_{g-1}\alpha^{a_1}\beta^{a_2}\mu_{L\setminus\{i\}}^{b_{L\setminus\{i\}}}x_{L\setminus\{i\}}^{\mu_{L\setminus\{i\}}}x_i^{\alpha}x_j^{\beta}\\
&=\frac{1}{2}d\cdot\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda_{2k}\rangle\rangle_{g-1}\psi_{b_{L\setminus\{i\}}}(y_{L\setminus\{i\}})\psi_{a_1+1}(y_i)\psi_{a_2+1}(y_i).
\end{aligned}
$$

Combining the formulas (12), (13) and (14) together, we obtain Theorem 1.6

As an application, a recursion formula for pure ψ-class intersection numbers $\langle\langle\tau_{b_L}\rangle\rangle_g$ will be derived from Theorem 1.6.

We need to take a close look at the polynomial $\psi_b(y)$ first. By definition,

$$\psi_0(y)=y-1,\psi_1(y)=y^2-y,\psi_2(y)=2y^3-3y^2+y.$$

More generally, $\psi_b(y)$ takes the following form:

$$\psi_b(y)=\sum_{i=1}^{b+1}f(b,i)y^i.$$

By induction, we have the recursion relation for the coefficients $f(b,i)$,

$$f(b,b+1)=f(b-1,b)b,f(b,i)=f(b-1,i-1)(i-1)-f(b,i)i$$

for $b\geq 1$, $1\geq i\leq b$. Thus, for a fixed b, all the coefficients $f(b,i)(1\leq i\leq b)$ can be calculated through the above recursion. Particularly, we have $f(b,b+1)=b!$.

From the above analysis, we know that $\psi_b(y)$ is a polynomial with degree $b+1$. So as a polynomial identity of variables $\{y_i\}$, formula (9) has highest degree $4g-3+2l$ when $\sum_{i=1}^{l}b_i=4g-3+l$.

Let $F_k(g(y_1,\ldots,y_l))$ be an operator by taking all the terms of the polynomial $g(y_1,\ldots,y_n)\in Q[y_1,\ldots,y_n]$ with degree k.

With this notation,

$$F_{4g-3+2l}(LHS\ of\ (9)) = (2g-1+l)\sum_{b_L\geq 0}\langle\langle\tau_{b_L}\rangle\rangle_g b_L! y_L^{b_L+1}, \tag{15}$$

$$F_{4g-3+2l}(First\ term\ of\ RHS\ of\ (9)) \tag{16}$$

$$= \sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\sum_{i<j}\langle\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\rangle\rangle_g b_{L\setminus\{i,j\}}!(a+1)! y_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}\sum_{k=0}^{a+1} y_i^{a+2-k}y_j^{k+1},$$

$$F_{4g-3+2l}(Second\ term\ of\ RHS\ of\ (9)) \tag{17}$$

$$= \frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{k=0}^{g}(-1)^k\langle\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\rangle\rangle_{g-1} b_{L\setminus\{i\}}!(a_1+1)!(a_2+1)! y_{L\setminus\{i\}}^{b_{L\setminus\{i\}}+1}y_i^{a_1+a_2+4}.$$

We denote by

$$[y_1^{k_1}\cdots y_n^{k_n}]g(y_1,\dots,y_n)$$

the coefficient of $y_1^{k_1}\cdots y_n^{k_n}$ in the polynomial $g(y_1,\dots,y_n)\in Q[y_1,\dots,y_n]$.

Then,

$$[y_1^{b_1+1},\dots,y_l^{b_l+1}](15) = (2g-1+l)\langle\langle\tau_{b_L}\rangle\rangle_g b_L!, \tag{18}$$

$$[y_1^{b_1+1},\dots,y_l^{b_l+1}](16) = \sum_{i<j}\langle\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\rangle\rangle_g b_{L\setminus\{i,j\}}!(b_i+b_j)!, \tag{19}$$

(20)

$$[y_1^{b_1+1},\dots,y_l^{b_l+1}](17) = \frac{1}{2}\sum_{i=1}^{l}\sum_{a_1+a_2=b_i-3}\langle\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\rangle\rangle_{g-1} b_{L\setminus\{i\}}!(a_1+1)!(a_2+1)!.$$

Combining the formulas (18), (19) and (20) together, we obtain Theorem 1.7.

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