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EXTENSION OF THE BORSUK THEOREM ON NON-EMBEDDABILITY OF SPHERES

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ABSTRACT. It is proved by elementary techniques that the suspension $\sum M$ of a closed n-dimensional manifold M, $n \geq 1$, does not embed in a product of $n+1$ curves. Thus we get in particular an elementary proof of a far-reaching generalization of the Borsuk theorem on non-embeddability of the sphere $\mathbb{S}^{n+1}$ in a product of $n+1$ curves. The ultimate results are even more general; they complement and extend some principal results of Koyama, Krasinkiewicz, and Spież.

1. Introduction

All *spaces* discussed in this paper are metrizable and all *mappings* (also called *maps*) are continuous. By a *compactum* we mean a compact metric space, by a *continuum* a connected compactum, and by a *curve* a 1-dimensional continuum.

In 1975 K. Borsuk [B] discovered a remarkable property of spheres (this property provided an answer to a problem of J. Nagata [N, p. 163]):

Theorem 1.1 (Borsuk). *The n-sphere $\mathbb{S}^n$, $n \geq 2$, does not embed in a product of n curves.*

In this article we present a simple elementary proof of the following extension of this theorem.

Theorem 1.2. *The suspension $\sum M$ of any closed n-manifold M, where $n \geq 1$, does not embed in a product of $n+1$ curves.*

The proof is given in Section 4. Moreover, we shall prove the following more general result (see Section 6 for a proof).

Theorem 1.3. *The suspension $\sum X$ of any locally connected quasi-n-manifold X, where $n \geq 1$, does not embed in a product of $n+1$ curves.*

The proof is based on the Second Factorization Theorem 3.2 and the Second Structure Theorem 6.1, which are stronger forms of the corresponding results from [K-K-S]. In addition we show that the Second Factorization Theorem holds for ramified 3-manifolds; see Theorem 5.3.

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2. On quasi-manifolds, weak manifolds and ramified manifolds

An n-dimensional compactum is said to be an *n-manifold at a point x* if there is an open neighborhood W of x in X that is an open n-disc. Obviously, X is an n-manifold at every point $x \in X$ if and only if X is a closed n-manifold. The property discussed in the next paragraph follows from the Borsuk Separation Theorem, which relates compact sets separating $\mathbb{S}^n$, $n \geq 1$, to essential mappings from those sets into $\mathbb{S}^{n-1}$ (cf. [E-S, p. 302]).

For an n-dimensional compactum X which is an n-manifold at a point $x_0 \in X$ there is an open neighborhood W of x_0 in X such that for every open set $U \ni x$ with $\overline{U} \subset W$ there is an essential map $\partial U \to \mathbb{S}^{n-1}$. (In fact, this holds for every neighborhood W that is an open n-disc.) One can use this property to define a larger class of n-dimensional compacta. Namely, an n-dimensional compactum X is said to be a *quasi-n-manifold at a point $x \in X$* if there is an open neighborhood W of x in X such that for every open set $U \ni x$ with $\overline{U} \subset W$ and $\dim \partial U \leq n-1$ there is an essential map $\partial U \to \mathbb{S}^{n-1}$; equivalently, $\widetilde{H}^{n-1}(\partial U) \neq 0$. (By H^* we denote the Čech cohomology with integer coefficients $\mathbb{Z}$.) If X is a quasi-n-manifold at x, then every neighborhood of x is n-dimensional. A compactum X is said to be a *quasi-n-manifold* if it is a quasi-n-manifold at every point. For polyhedra we have the following characterization.

Theorem 2.1 ([K-K-S, Theorem 2.2]). *For a compact n-dimensional polyhedron $X = |K|$ and a point $x \in X$ the following are equivalent:*

(i) *X is a quasi-n-manifold at x;*

(ii) *$\widetilde{H}^{n-1}(|\operatorname{lk}(x, L)|) \neq 0$ for every triangulation L of X with x as a vertex;*

(iii) *either $\dim \sigma(x) = n$ or $\dim \sigma(x) < n$ and $\widetilde{H}^{n-\dim \sigma(x)-1}(|\operatorname{lk}(\sigma(x), K)|) \neq 0$, where $\sigma(x)$ is the carrier of x in K;*

(iv) *$H^n(X, X \setminus \{x\}) \neq 0$.*

Let X be a compact polyhedron. A triangulation K of X is said to be *fine* if for each point $x \in X$ there is a vertex v of K such that (X, x) is homeomorphic to (X, v), written $(X, x) \approx (X, v)$. Every polyhedron has a fine triangulation, e.g., the barycentric subdivision of any triangulation. We have also the following description of quasi-n-manifolds among n-dimensional polyhedra. The proof is similar to the proof of the above theorem.

Theorem 2.2. *For a compact n-dimensional polyhedron X the following are equivalent:*

(i) *X is a quasi-n-manifold;*

(ii) *there is a fine triangulation of X such that the links of vertices have nontrivial $(n-1)$-cohomology.*

A compact pair (X, A) is said to *extend* to a pair (X', A') if there is an embedding $h : X \to X'$ such that $h(A) \subset A'$.

A compact n-dimensional space X is said to be a *weak n-manifold at a point $x \in X$* if there is an open neighborhood W of x in X such that no pair $(\overline{U}, \partial U)$, where $U \ni x$ is open and $\overline{U} \subset W$, extends to a compact pair (P, Q) of absolute retracts with $\dim(P, Q) = (n, n-1)$ (that is, $\dim P = n$ and $\dim Q = n-1$). An n-dimensional compactum is said to be a *weak n-manifold* if it is a weak n-manifold at every point. If X is a quasi-n-manifold at a point x, then X is a weak n-manifold at x as well.

Let S be a subset of a compactum X. Then X is said to be an *n-manifold* (*quasi-n-manifold, weak n-manifold,* resp.) *off* S if X is n-dimensional at every point and X is an n-manifold (quasi-n-manifold, weak n-manifold, resp.) at each point $x \in X \setminus S$.[1]

A cell complex[2] K is called a *ramified n-complex* if K consists of a finite collection of n-cells and their faces with each $(n-1)$-cell being a face of at least two n-cells. (A ramified 0-complex is a 0-complex with at least two 0-cells.) The underlying polyhedron $X = |K|$ of a ramified n-complex K is called a *ramified n-manifold.* Any polyhedron that is a quasi-n-manifold is a ramified n-manifold. For $n \leq 2$ the converse is also true.[3] Every ramified n-manifold $X = |K|$ is a quasi-n-manifold off $|K^{(n-2)}|$.

Lemma 2.3. *Let X be a weak n-manifold off a set with dimension $\leq n-2$. If U is a nonvoid open subset of X and $h : U \to \mathbb{R}^n$ is an embedding such that $h(U)$ is closed in $\mathbb{R}^n$, then $h(U) = \mathbb{R}^n$.*

Proof. Suppose $h(U)$ is a proper subset of $\mathbb{R}^n$. Assume X is a weak n-manifold off a set S with $\dim S \leq n-2$. There is a point $a \in h(U) \setminus h(S)$ because X is n-dimensional at each point. Pick any point $b \in \mathbb{R}^n \setminus h(U)$. Since $\dim h(S) \leq n-2$, the Mazurkiewicz theorem [E, Theorem 1.8.18, p. 62] implies that there is a continuum $C \subset \mathbb{R}^n \setminus h(S)$ connecting a with b. Let C_0 be the component of $C \setminus h(U)$ containing b. It follows that $\overline{C_0} \cap \partial h(U) \neq \emptyset$ [Kur, p. 172]. Pick any point $c \in \overline{C_0} \cap \partial h(U)$. Then $h^{-1}(c) \in U \setminus S$. Let B_r denote the open n-ball in $\mathbb{R}^n$ with radius $r > 0$ and center at c, and let S_r denote its boundary. Since X is a weak n-manifold at $h^{-1}(c)$ there is an open neighborhood W of $h^{-1}(c)$ in X satisfying the definition of weak n-manifold at $h^{-1}(c)$. Then there is a positive number $r < d(c, b)$ such that $\overline{V} \subset h(U \cap W)$ for $V = B_r \cap h(U)$. Then $\partial_{h(U)} V \subset S_r \setminus C_0$. Since $C_0 \cap S_r \neq \emptyset$ the pair $(\overline{V}, \partial_{h(U)} V)$ extends to a pair of absolute retracts $(\overline{B_r}, D)$ with dimension $(n, n-1)$, where D is a closed $(n-1)$-disc in S_r. Since $h^{-1}(V)$ is an open neighborhood of $h^{-1}(c)$ whose closure lies in W and h transforms $(\overline{h^{-1}(V)}, \partial h^{-1}(V))$ homeomorphically onto $(\overline{V}, \partial_{h(U)} V)$, we get a contradiction. □

Theorem 2.4. *Let X be a weak n-manifold off a set with dimension $\leq n-2$. If $f : X \to |K|$ is an embedding, where K is an n-dimensional cell complex, then $f(X) = |L|$, where L is a ramified n-subcomplex of K. In particular, the conclusion holds for ramified n-manifolds.*

The proof is based on the preceding lemma and runs along the lines of the proof of Theorem 2.7 in [K-K-S].

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[1]It is not excluded that X has the discussed properties at certain points of S.

[2]The notion of *cell complex* and other related notions are understood in the sense of [R-S]. By $K^{(m)}$ we denote the mth skeleton of K; if $m < 0$, then $K^{(m)} = \emptyset$ by definition.

[3]For $n = 3$ it fails (see [K-K-S]).

3. Second factorization theorem

In [K-K-S] the following theorem has been established.

Factorization Theorem 3.1. *Let X be a locally connected weak n-manifold with $H^1(X)$ of finite rank. If $f = (f_1, \cdots, f_n) : X \to Y_1 \times \cdots \times Y_n$ is an embedding in a product of n curves, then there exist mappings $g = (g_1, \cdots, g_n) : X \to P_1 \times \cdots \times P_n$ and $h = h_1 \times \cdots \times h_n : P_1 \times \cdots \times P_n \to Y_1 \times \cdots \times Y_n$ such that $f_i = h_i \circ g_i$ for each $i = 1, \cdots, n$ (hence $f = h \circ g$), where $g_i : X \to P_i$ is a monotone surjection, P_i is a graph with no endpoint, and $h_i : P_i \to Y_i$ is 0-dimensional. In particular, g is an embedding.*

Let us recall the main steps in the proof of this theorem. The existence of the mappings follows from the Whyburn factorization theorem [W, p. 141] (which is applicable to any compactum). The local connectedness of X plus rank $H^1(X) < \infty$ ensure that P_i is a compact 1-dimensional ANR with

(*) *$H^1(P_i)$ isomorphic to a subgroup of $H^1(X)$.*

Then the following basic observation was made:

(**) *if $g_i(x)$ is an endpoint of P_i, then X is not a weak n-manifold at x.*

Let NWM(X) denote the subset of X composed of all points at which X is not a weak n-manifold, and let $E(P_i)$ denote the set of endpoints of P_i. From (**) it follows that

(***) $\#E(P_i) \leq \#\,\mathrm{NWM}(X)$.

One easily sees that if P is a compact 1-dimensional ANR and the set $E(P)$ of its endpoints is finite, then P is a graph. Therefore, by a slight modification of the discussed proof, one obtains the following refined version of the Factorization Theorem:

Second Factorization Theorem 3.2. *Let X be a locally connected weak n-manifold off a finite set with $H^1(X)$ of finite rank. If $f = (f_1, \cdots, f_n) : X \to Y_1 \times \cdots \times Y_n$ is an embedding in a product of n curves, then there exist mappings $g = (g_1, \cdots, g_n) : X \to P_1 \times \cdots \times P_n$ and $h = h_1 \times \cdots \times h_n : P_1 \times \cdots \times P_n \to Y_1 \times \cdots \times Y_n$ such that $f_i = h_i \circ g_i$ for each $i = 1, \cdots, n$, where $g_i : X \to P_i$ is a monotone surjection onto a graph satisfying (**), and $h_i : P_i \to Y_i$ is 0-dimensional. Consequently, $\#E(P_i) \leq \#\,\mathrm{NWM}(X)$ for each i, and g is an embedding.*

4. Proof of Theorem 1.2

Suppose $\sum M$ embeds in a product of $n+1$ curves. We may assume that M is connected. Then, by the Second Factorization Theorem 3.2, there is a mapping $g = (g_1, \cdots, g_{n+1}) : \sum M \to P_1 \times \cdots \times P_{n+1}$ such that each $g_i : \sum M \to P_i$ is a monotone surjection with the following properties: P_i is a compact connected 1-dimensional ANR, $H^1(P_i) = 0$, and P_i has at most two endpoints (because $\sum M$ is an $(n+1)$-manifold off the vertices). It follows that P_i is an arc (because it is a dendrite with at most two endpoints). Let K denote the natural cell structure on $P_1 \times \cdots \times P_{n+1}$ with just one $(n+1)$-cell. Since $g : \sum M \to |K|$ is an embedding and $\sum M$ is an $(n+1)$-manifold off the vertices, by Theorem 2.4, $g(\sum M) = |K|$ is a ramified $(n+1)$-manifold which contradicts $|K|$ being an $(n+1)$-cell.

Remark. Notice that Theorem 1.2 also follows from [D-K], but the above proof is more elementary.

5. Factorization Theorem for ramified 3-manifolds

Lemma 5.1. *If $X = |K|$ is a ramified n-manifold with a triangulation K, then for every simplex $\sigma \in K$ with $\dim \sigma < n$, the link $\mathrm{lk}(\sigma, K)$ is a ramified $(n - \dim \sigma - 1)$-complex. Also, X is a quasi-n-manifold off $|K^{(n-3)}|$ (hence a fortiori a weak n-manifold off this set).*

Proof. The proof of the first assertion is straightforward. To prove the second one, first notice that X is a quasi-n-manifold off $|K^{(n-2)}|$ (because $X \setminus |K^{(n-2)}|$ is a union of open n-discs). Next consider the case where $x \in |K^{(n-2)}| \setminus |K^{(n-3)}|$. Then $\dim \sigma(x) = n-2$, where $\sigma(x)$ is the carrier of x in K. Hence $|\,\mathrm{lk}(\sigma(x), K)|$ is a ramified 1-manifold. Consequently, $H^1(|\,\mathrm{lk}(\sigma(x), K)|) \neq 0$. The conclusion follows now from Theorem 2.1(iii). □

Corollary 5.2. *Let $X = |K|$ be a ramified 3-manifold with a triangulation K. Then X is a quasi-3-manifold off the vertices of K.*

Combining this corollary with the Second Factorization Theorem 3.2 we obtain the following.

Theorem 5.3. *If a ramified 3-manifold embeds in a product of three curves, then it embeds in a product of three graphs as well. Moreover, if $f = (f_1, f_2, f_3) : X \to Y_1 \times Y_2 \times Y_3$ is an embedding in a product of curves, then there exist an embedding $g = (g_1, g_2, g_3) : X \to P_1 \times P_2 \times P_3$ and a mapping $h = h_1 \times h_2 \times h_3 : P_1 \times P_2 \times P_3 \to Y_1 \times Y_2 \times Y_3$ such that $f_i = h_i \circ g_i$ for each $i = 1, 2, 3$, where $g_i : X \to P_i$ is a monotone surjection onto a graph, and $h_i : P_i \to Y_i$ is 0-dimensional.*

6. Product structure of locally connected weak manifolds lying in products of curves

Using the Second Factorization Theorem in the proof of the Structure Theorem 5.1 in [K-K-S] we obtain the following improvement of the Structure Theorem. Denote

$$J_X = \{j \in \{1, \cdots, n\} : pr_j(X) \text{ is a circle}\}.$$

Second Structure Theorem 6.1. *Let X be a locally connected weak n-manifold off a finite set, with $H^1(X)$ of finite rank, lying in a product $Y_1 \times \cdots \times Y_n$ of n curves, $n \geq 1$. Then X is a polyhedron and the following conditions are fulfilled:*

(1) $\mathrm{rank}\, H^1(X) \geq n$.

(2) *If* $\mathrm{rank}\, H^1(X) = n+k$, *where $k < n$, then J_X contains at least $n-k$ elements. In particular, if* $\mathrm{rank}\, H^1(X) = n$, *then $J_X = \{1, \cdots, n\}$.*

(3) *If $J_X = \{1, \cdots, n\}$, then $X = pr_1(X) \times \cdots \times pr_n(X)$ is an n-torus.*

(4) *If J_X is a proper nonvoid subset of $\{1, \cdots, n\}$, then $X = (\prod_{j \in J_X} pr_j(X)) \times pr_{J_X^c}(X)$, where the first factor is an n_{J_X}-torus, and $pr_{J_X^c}(X)$ is a polyhedron that is a weak $n_{J_X^c}$-manifold in $\prod_{j \in J_X^c} Y_j$ having no projection onto a circle.*

If, in addition, X is a quasi-n-manifold, then $pr_{J_X^c}(X)$ is a quasi-$n_{J_M^c}$-manifold.

7. Proof of Theorem 1.3

Let X be a locally connected quasi-n-manifold, $n \geq 1$, and suppose the suspension $\sum X$ embeds in a product of $n+1$ curves. Without loss of generality we may assume that X is connected. By Corollary 2.3 in [K-K-S] the suspension is a quasi-$(n+1)$-manifold off the vertices. Hence $\sum X$ is a locally connected, weak $(n+1)$-manifold off the vertices, and $H^1(\Sigma X) = 0$. Then, by the Second Structure Theorem 6.1(1), we reach a contradiction.

We end this paper with the following.

Problem. Does the Factorization Theorem hold for ramified n-manifolds in place of connected weak n-manifolds?

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