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LINK BETWEEN NOETHERIANITY AND THE WEIERSTRASS DIVISION THEOREM ON SOME QUASIANALYTIC LOCAL RINGS

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ABSTRACT. In the setting of well-behaved quasianalytic differentiable systems, we prove that the Weierstrass Division Theorem holds in such system if, and only if, the system is Noetherian.

1. INTRODUCTION

Let $\mathcal{C}_k$, $k = 1, 2, \ldots$, be local quasianalytic rings of germs, at the origin in $\mathbb{R}^k$, of smooth functions. We suppose that the system $\mathcal{C} = \{\mathcal{C}_k,\ k \in \mathbb{N}\}$ satisfies some natural properties; see Section 2. We know by [6] that the Weierstrass Division Theorem never holds in such a system if $\mathcal{C}_k$, $k = 1, 2, \ldots$, is not contained in the ring of germs of real analytic functions. Because of the lack of a Weierstrass Division Theorem, many problems remain open for such rings. For example, we do not know if the $\mathcal{C}_k$ are Noetherian rings. The present study may be regarded as an inquiry as to what differences exist between a system $\mathcal{C} = \{\mathcal{C}_k,\ k \in \mathbb{N}\}$ in which we suppose that a Weierstrass Division Theorem holds and a system in which the $\mathcal{C}_k$, $k = 1, 2, \ldots$, are Noetherian rings. This question is supported by the following conjecture, asked in the setting of quasianalytic Denjoy-Carlemen classes (see [4, Ch. 7, An algebraic question]).

Conjecture. It may be that a Weierstrass Division Theorem holds in the system $\mathcal{C} = \{\mathcal{C}_k,\ k \in \mathbb{N}\}$ if and only if the $\mathcal{C}_k$ are Noetherian rings.

We remark that in light of the result of [6], where this conjecture is established, we would know that some quasianalytic rings are not Noetherian.

It is clear that if a Weierstrass Division Theorem holds in the system $\mathcal{C} = \{\mathcal{C}_k,\ k \in \mathbb{N}\}$, then by repeating standard arguments, we show that the $\mathcal{C}_k$ are Noetherian rings; see [12]. The aim of this paper is to prove the converse for some Noetherian local rings.

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2. Differentiable system

Definition 2.1. A differentiable system is a sequence $\mathcal{C} = \{\mathcal{C}_n\,,\, n \in \mathbb{N}\}$ such that, for each $n \in \mathbb{N}$, $\mathcal{C}_n \subset \mathcal{E}_n$ is a local subring of the ring of germs, at the origin of $\mathbb{R}^n$, of C^∞ functions. We suppose that, for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is closed under taking derivatives, and the following hold:

- (C_1) $\mathbb{R}[x_1, \ldots, x_n] \subset \mathcal{C}_n \subset \mathcal{E}_n$, for each $n \in \mathbb{N}$, where $\mathbb{R}[x_1, \ldots, x_n]$ is the ring of polynomials with coefficients in $\mathbb{R}$.
- (C_2) The system $\mathcal{C}$ is closed under composition. This means that if $g \in \mathcal{C}_k$ and $f = (f_1, \ldots, f_k) \in (\mathcal{C}_n)^k$ with $f(0) = 0$, then $g \circ f \in \mathcal{C}_n$.
- (C_3) For each $n \in \mathbb{N}$, $\mathcal{C}_n$ is closed under division by coordinates. This means that if $f \in \mathcal{C}_n$ and $f = (x_i - \alpha)g$, where $g \in \mathcal{E}_n$ and $\alpha \in \mathbb{R}$, then $g \in \mathcal{C}_n$.
- (C_4) The Implicit Function Theorem for $\mathcal{C}_n$ holds in the following sense: Suppose that $f = (f_1, \ldots, f_m) \in (\mathcal{C}_{n+m})^m$ with $f(0,0) = 0$. Put $y = (y_1, \ldots, y_m)$ and suppose that
$$det(\frac{\partial f_i}{\partial y_j}(0,0))_{i,j=1,\ldots,m} \neq 0.$$
Then there is a (unique) $g = (g_1, \ldots, g_m) \in (\mathcal{C}_n)^m$ with $g(0) = 0$ such that $f(x, g(x)) = 0$.

Call
$$\hat{.} : \mathcal{C}_n \to \mathbb{R}[[x_1, \ldots, x_n]]$$
the map which associates to each $f \in \mathcal{C}_n$ its Taylor expansion at the origin. We consider the following conditions:

- (C_5) $\hat{.}$ is an injective homomorphism.
- (C_6) $\mathcal{C}_n$ is a Noetherian ring for each $n \in \mathbb{N}$.

Definition 2.2. A differentiable system $\mathcal{C} = \{\mathcal{C}_n\,,\, n \in \mathbb{N}\}$ is called *quasianalytic* [resp. *Noetherian*] if condition (C_5) holds [resp. if condition (C_6) holds].

Remark 2.3. It is clear that every Noetherian differentiable system is a quasianalytic system.

Example 2.4.

i) If for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is the ring of germs, at the origin in $\mathbb{R}^n$, of Nash functions, i.e. algebraic on the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$, the system $\mathcal{C} = \{\mathcal{C}_n,\, n \in \mathbb{N}\}$ is a Noetherian differentiable system.
ii) If for each $n \in \mathbb{N}$, $\mathcal{C}_n$ is the ring of germs, at the origin in $\mathbb{R}^n$, of real analytic functions, the system $\mathcal{C} = \{\mathcal{C}_n,\, n \in \mathbb{N}\}$ is a Noetherian differentiable system.
iii) Let $\mathcal{R}$ be a polynomially bounded o-minimal structure which is an expansion of the ordered field of reals. For more details about an o-minimal structure over the field of reals, we refer the reader to [13]. We denote by $\mathcal{D}_n$ the ring of germs, at the origin in $\mathbb{R}^n$, of C^∞ definable functions in a neighborhood of the origin in $\mathbb{R}^n$. By [8], the system $\mathcal{D} = \{\mathcal{D}_n,\, n \in \mathbb{N}\}$ is a quasianalytic differentiable system.
iv) Let $M = \{M_p\}_{p=0}^\infty$ be an increasing sequence of positive real numbers. We denote by $C_n(M) \subset \mathcal{E}_n$ the subring of germs of C^∞ functions in a neighborhood of the origin which are in the class M; see [4] and [10]. If we suppose that the class is quasianalytic, then the system $C(M) = \{C_n(M),\, n \in \mathbb{N}\}$ is a quasianalytic differentiable system.

In the following, for a differentiable quasianalytic system, we will not distinguish notationally by $\hat{}$ between the germ and its image, i.e. its Taylor expansion at the origin.

In particular these conditions on a quasianalytic system imply that the maximal ideal of $\mathcal{C}_n$ is $\underline{m}_n = \{f \in \mathcal{C}_n \,/\, f(0) = 0\} = (x_1, \dots, x_n)\mathcal{C}_n$ and its completion with the $\underline{m}_n$-adic topology is the ring of formal series $\mathbb{R}[[x_1, \dots, x_n]]$.

3. Artin approximation property for a Noetherian system

Let $\mathcal{C} = \{\mathcal{C}_n\,,\, n \in \mathbb{N}\}$ be a differentiable Noetherian system. For each $n \in \mathbb{N}$, the completion of the ring $\mathcal{C}_n$ with respect to the $\underline{m}_n$-topology is the ring of formal power series $\mathbb{R}[[x_1, \dots, x_n]]$, which is a regular ring. By [11, Proposition 24], $\mathcal{C}_n$ is also a regular ring; hence $\mathcal{C}_n$ is an excellent ring [2, Theorem 24]. We see then that the morphism

$$\mathcal{C}_n \to \mathbb{R}[[x_1, \dots, x_n]]$$

is a regular homomorphism [12, Section 1]. The condition (C_4) means that $\mathcal{C}_n$ is a Henselian ring. By [12, Theorem 2.4], the *Artin approximation property* holds for the pair $(\mathcal{C}_n, \mathbb{R}[[x_1, \dots, x_n]])$. This means that for every system of polynomial equations $f = 0$, where $f = (f_1, \dots, f_q)$ with $f_i \in \mathcal{C}_n[Y]$ and $Y = (Y_1, \dots, Y_N)$ a set of variables, for each $\nu \in \mathbb{N}$ and each formal solution $\hat{g} = (\hat{g}_1, \dots, \hat{g}_N) \in (\mathbb{R}[[x_1, \dots, x_n]])^N$, so that $f(\hat{g}) = 0$, we can find a solution $g = (g_1, \dots, g_N) \in (\mathcal{C}_n)^N$ such that

$$f(g) = 0 \ \text{ and } \ g - \hat{g} \in \underline{m}_n^{\nu}\mathbb{R}[[x_1, \dots, x_n]]^N.$$

3.1. Monomialization lemma for Noetherian system.

We recall a result proved by Eakin-Harris [5, Lemma 5.1] for convergent power series. Here we give its analogue for formal series; the proof is the same.

Lemma 3.1. *Let $f \in \mathbb{R}[[x_1, \dots, x_n]]$. Then there exists $H = (x_1, M_2x_2, M_3x_3, \dots, M_nx_n)$ where for each $j = 2, \dots, n$, M_j is a monomial in only the variables $x_1, \dots, x_j$ such that*

$$f(x_1, M_2x_2, M_3x_3, \dots, M_nx_n) = x_{i_1}^{\alpha_{\mu_1}} \dots x_{i_r}^{\alpha_{\mu_r}} Q$$

for some unit $Q \in \mathbb{R}[[x_1, \dots, x_n]]$, $i_1, \dots i_r \in \{1, \dots, n\}$ and $\alpha_{\mu_1}, \dots, \alpha_{\mu_r} \in \mathbb{N}$.

Proposition 3.2. *Let $\mathcal{C} = \{\mathcal{C}_n\,,\, n \in \mathbb{N}\}$ be a differentiable Noetherian system, and let $\varphi \in \mathcal{C}_n$. Then there exits $H = (x_1, M_2x_2, M_3x_3, \dots, M_nx_n)$ where for each $j = 2, \dots, n$, M_j is a monomial in only the variables $x_1, \dots, x_j$ such that*

$$\varphi(x_1, M_2x_2, M_3x_3, \dots, M_nx_n) = x_{i_1}^{\alpha_{\mu_1}} \dots x_{i_r}^{\alpha_{\mu_r}} Q$$

for some unit $Q \in \mathcal{C}_n$, $i_1, \dots i_r \in \{1, \dots, n\}$ and $\alpha_{\mu_1}, \dots, \alpha_{\mu_r} \in \mathbb{N}$.

Proof. If $\varphi \in \mathcal{C}_n$, by Lemma 3.1, there exists $H = (x_1, M_2x_2, M_3x_3, \dots, M_nx_n)$ where for each $j = 2, \dots, n$, M_j is a monomial in only the variables $x_1, \dots, x_j$ such that

$$\hat{\varphi}(x_1, M_2x_2, M_3x_3, \dots, M_nx_n) = x_{i_1}^{\alpha_{\mu_1}} \dots x_{i_r}^{\alpha_{\mu_r}} \hat{Q}$$

for some unit $\hat{Q} \in \mathbb{R}[[x_1, \dots, x_n]]$, $i_1, \dots i_r \in \{1, \dots, n\}$ and $\alpha_{\mu_1}, \dots, \alpha_{\mu_r} \in \mathbb{N}$. We consider the equation $E(Z)$:

$$x_{i_1}^{\alpha_{\mu_1}} \dots x_{i_r}^{\alpha_{\mu_r}} Z - \varphi(x_1, M_2x_2, M_3x_3, \dots, M_nx_n) = 0.$$

We see that $E(Z) \in \mathcal{C}_n[Z]$ and $\hat{Q}$ is a formal solution of this equation. By the Artin Approximation Theorem, if $\nu \in \mathbb{N}^*$, there exists $Q \in \mathcal{C}_n$ that is a solution of the equation $E(Z)$ such that $Q - \hat{Q} \in \underline{m}_n^\nu \mathbb{R}[[x_1, \dots, x_n]]$, hence the result since $\nu \geq 1$. □

4. Eakin-Harris property

In this section, we fix a quasianalytic differentiable system $\mathcal{C} = \{\mathcal{C}_n \, , \, n \in \mathbb{N}\}$. We put $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_k)$. We are concerned here with local homomorphisms $\Phi : \mathcal{C}_n \to \mathcal{C}_k$, i.e. homomorphisms such that $\Phi(\underline{m}_n) \subset \underline{m}_k$. The *generic rank* of Φ, denoted $rk(\Phi)$, is the rank of the Jacobian matrix $[\frac{\partial \Phi(x_i)}{\partial y_j}]$, considered as a matrix over the quotient field of $\mathcal{C}_k$. Recall that $\mathcal{C}_k$ is a domain by condition (C_5). Since Φ is a local homomorphism, we consider its natural extension to the completion $\hat{\Phi} : \mathbb{R}[[x_1, \dots, x_n]] \to \mathbb{R}[[y_1, \dots, y_k]]$. It is well known that if $rk(\Phi) = n$, then Φ is injective. In [9], we can find an example (Osgood's example) of an injective homomorphism Φ for which $rk(\Phi) < n$. Thus the condition is not necessary.

We let
$$\Phi_* : \frac{\mathbb{R}[[x_1, \dots, x_n]]}{\mathcal{C}_n} \to \frac{\mathbb{R}[[y_1, \dots, y_k]]}{\mathcal{C}_k}$$
be the homomorphism of groups induced by Φ and $\hat{\Phi}$ in the obvious manner.

Definition 4.1. We say that the homomorphism Φ is *strongly injective* if the homomorphism Φ_* is injective.

In the analytic setting, i.e. when each $\mathcal{C}_n$ is the ring of germs of real analytic functions, Eakin and Harris [5] showed that Φ is strongly injective if and only if $rk(\Phi) = n$. Their result extends a result of Abhyankar and van der Put [1].

This result justifies the following definition:

Definition 4.2. We say that a local morphism $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ has the *Eakin-Harris property* if $rk(\Phi) = n$ implies Φ is strongly injective.

It is shown in [7] that if every morphism $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ has the Eakin-Harris property, then the system $\mathcal{C} = \{\mathcal{C}_n\}$ is contained in the analytic system.

Lemma 4.3. *Let $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ and $\Psi : \mathcal{C}_k \to \mathcal{C}_l$ be local homomorphisms. If $\Psi \circ \Phi$ is strongly injective, then Φ is strongly injective.*

Proof. Follows from the definition. □

Remark 4.4. Isomorphisms are strongly injective.

Recall that a local homomorphism $u : A \to B$ between local rings is called finite if B is a finite module over the ring $u(A)$.

Lemma 4.5 ([1])**.** *Let $\mathcal{C} = \{\mathcal{C}_n \, , \, n \in \mathbb{N}\}$ be a differentiable Noetherian system. If $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ is injective and finite, then $\hat{\Phi}$ is injective and finite and Φ is strongly injective.*

Proof. Since each $\mathcal{C}_k$ is a Zariski ring, the lemma follows from [13, Chapter VIII, Theorem 9].

In the following we identify local homomorphisms $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ defined by $\Phi(f) = f(\varphi_1, \dots, \varphi_n)$, $\varphi_i \in \mathcal{C}_k$, $i = 1, \dots, n$, by $(\varphi_1, \dots, \varphi_n)$ and we write $\Phi = (\varphi_1, \dots, \varphi_n)$.

We consider the local homomorphism

$$e = (y_1 y_2, y_2, \ldots, y_k) : \mathcal{C}_k \to \mathcal{C}_k.$$

For $d \in \mathbb{N}^*$ we also consider the local homomorphism

$$r_d = (y_1^d, y_2, \ldots, y_k) : \mathcal{C}_k \to \mathcal{C}_k.$$

It is clear that homomorphisms e, r_d ($d \in \mathbb{N}$) are injective and $rk(r_d) = rk(e) = k$. □

Remark 4.6. Let $P = \sum_\omega a_\omega y_1^{\omega_1} y_2^{\omega_2} \ldots y_k^{\omega_k}$ be a polynomial. There exists $N \in \mathbb{N}$ such that, for each $\omega = (\omega_1, \ldots, \omega_k) \in \mathbb{N}^k$ with $a_\omega \neq 0$, $N + \omega_2 \geq \omega_1$. We have then $y_2^N P = e(Q)$, where $Q = \sum_\omega a_\omega y_1^{\omega_1} y_2^{N+\omega_2-\omega_1} y_3^{\omega_3} \ldots y_k^{\omega_k}$.

Definition 4.7. We say that the quasianalytic differentiable system $\mathcal{C} = \{\mathcal{C}_n\ ,\ n \in \mathbb{N}\}$ is *well behaved* if the homomorphisms e and r_d are strongly injective.

Proposition 4.8. *The systems considered in* i) *and* ii) *of Example* 2.4 *are well behaved.*

Proof.

(1) Analytic system.

Let $\hat{f} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, be a formal power series such that $e(\hat{f})$ is a convergent series. Hence there exist $r, M \in \mathbb{R}_+^*$, $r < 1$, such that for every $\alpha \in \mathbb{N}^n$, $|a_\alpha| r^{|\alpha|+\alpha_1} \leq M$, where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We have then, for every $\alpha \in \mathbb{N}^n$, $|a_\alpha| r^{2|\alpha|} \leq M$, which proves that $\hat{f}$ is a convergent power series; hence the homomorphism e is strongly injective. It is also an elementary calculation to show that the homomorphism r_d, $d \in \mathbb{N}$, is strongly injective. Hence the analytic system is well behaved.

(2) Nash system.

Let $\hat{f} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be a formal power series such that $e(\hat{f})$ is algebraic on the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$. It is known already that $e(\hat{f})$ is a convergent series (the ring of convergent power series is algebraically closed in the ring of formal power series); hence $\hat{f}$ is also a convergent power series. We have:

$$(*) \qquad P_q(e(\hat{f}))^q + \ldots + P_1 e(\hat{f}) + P_0 = 0,$$

with all the polynomials $P_j \neq 0$. By Remark 4.6, there exist $m \in \mathbb{N}$ and polynomials Q_j, $j = 1, \ldots, q$, such that $x_2^m P_j = e(Q_j)$. By multiplying the equation $(*)$ by x_2^m and since e is injective, we have $Q_q \hat{f}^q + \ldots + Q_0 = 0$; hence $\hat{f}$ is algebraic on the ring of polynomials, which proves that the morphism e is strongly injective.

Now, let $\hat{f} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be a formal power series such that $r_d(\hat{f})$ is algebraic on the ring of polynomials. Since the Weierstrass Division Theorem is true in the ring of Nash functions (see [3, 8.2.8]), we divide $r_d(\hat{f})$ by the polynomial $x_1^d - T$, where T is an auxiliary variable,

$$r_d(\hat{f}) = (x_1^d - T) Q(x, T) + \sum_{j=1}^{d} b_j(x_2, \ldots, x_n, T) x_1^{d-j},$$

where $Q(x, T)$, $b_j(x_2, \ldots, x_n, T)$ are algebraic on the ring of polynomials, $j = 1, \ldots, d$.

We also divide the formal series $\hat{f}$ by the polynomial $x_1 - T$ in the ring of formal series $\mathbb{R}[[x_1, \dots, x_n, T]]$,

$$(**) \qquad \hat{f} = (x_1 - T)Q_1(x, T) + W(x_2, x_3, \dots, x_n, T),$$

where $Q_1(x, T) \in \mathbb{R}[[x_1, \dots, x_n, T]]$, $W(x_2, x_3, \dots, T) \in \mathbb{R}[[x_2, \dots, x_n, T]]$.

We have then

$$\hat{f}(x_1^d, x_2, \dots, x_n) = (x_1^d - T)Q_1(x_1^d, x_2, \dots, x_n, T) + W(x_2, x_3, \dots, x_n, T).$$

Since the division is unique in $\mathbb{R}[[x_1, \dots, x_n, T]]$, we see that

$$Q(x, T) = Q_1(x_1^d, x_2, \dots, x_n, T)$$

and

$$\sum_{j=1}^{d} b_j(x_2, \dots, x_n, T)x_1^{d-j} = W(x_2, x_3, \dots, x_n, T).$$

Hence $b_1 = \dots = b_{d-1} = 0$, and $W(x_2, x_3, \dots, x_n, T) = b_d(x_2, \dots, x_n, T)$, but, from $(**)$, we have $W(x_2, x_3, \dots, x_n, x_1) = \hat{f}(x_1, \dots, x_n)$, hence the result. □

Remark 4.9. For each $l \in \{1, 2, \dots, k-1\}$, the local homomorphism $H = (y_1, y_2, \dots, y_l, y_l y_{l+1}, \dots, y_l y_k) : \mathcal{C}_k \to \mathcal{C}_k$ is a finite compositions of e and permutations.

5. Eakin-Harris property for differentiable Noetherian system

We prove in this section a version of a theorem proved by Eakin and Harris [5]. This result is about local homomorphisms of rings in a given well-behaved differentiable Noetherian system. The proof is inspired by the proof of a similar result in the setting of an analytic system, proved by Eakin and Harris [5].

We put $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_k)$. If $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is a quasianalytic differentiable system, let $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ be a local homomorphism.

Definition 5.1. *An admissible modification* of Φ is a homomorphism $\tilde{\Phi}$ related to Φ in one of the following ways:

(i) There is an isomorphism $\Gamma : \mathcal{C}_n \to \mathcal{C}_n$ such that $\Phi \circ \Gamma = \tilde{\Phi}$.
(ii) There is a homomorphism $\Psi : \mathcal{C}_k \to \mathcal{C}_k$ with $rk(\Psi) = k$ such that $\Psi \circ \Phi = \tilde{\Phi}$.
(iii) There is a strongly injective homomorphism $\omega : \mathcal{C}_n \to \mathcal{C}_n$ with $rk(\omega) = n$ such that $\Phi = \tilde{\Phi} \circ \omega$.

Remark 5.2. Let $\tilde{\Phi}$ be an admissible modification of Φ. Then

(i) $rk(\tilde{\Phi}) = rk(\Phi)$.
(ii) $\tilde{\Phi}$ strongly injective $\Rightarrow$ Φ strongly injective.

From now on, we suppose that the system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is a well-behaved Noetherian differentiable system. Let $\Phi : \mathcal{C}_n \to \mathcal{C}_k$, $\Phi = (\varphi_1, \dots, \varphi_n)$, be a local homomorphism. We suppose $\varphi_1 \neq 0$.

Lemma 5.3. *There exist $d \in \mathbb{N}^*$ and a homomorphism $H : \mathcal{C}_k \to \mathcal{C}_k$ such that*

$$H(\varphi_1) = c_1 y_1^d + c_2 y_2^d + \dots + c_k y_k^d + \psi$$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$, $c_1 \neq 0$ and $\psi \in \mathcal{C}_k$ such that $\hat{\psi} = \{$mixed terms of degree $d\}$ $+ \sum_{|\omega|>d} a_\omega y_1^{\omega_1} \dots y_k^{\omega_k}$.

Proof. $\hat{\varphi}_1 = \sum_{\nu=0}^{\infty} P_\nu(y_1, \dots, y_k)$, where $P_\nu(y_1, \dots, y_k)$ is a homogeneous polynomial of degree ν, $\nu = 0, 1, \dots$. Let $d \in \mathbb{N}$ be the least integer such that $P_d \neq 0$. We have then

$$\hat{\varphi}_1 = P_d(y_1, \dots, y_k) + \sum_{\nu > d}^{\infty} P_\nu(y_1, \dots, y_k). \tag{1}$$

Let $(\lambda_{ij=1})_{i,j=1}^k$ be a nonsingular matrix of scalars such that $P_d(\lambda_{11}, \dots, \lambda_{k1}) \neq 0$. We define an isomorphism $H : \mathcal{C}_k \to \mathcal{C}_k$ by

$$H = (\sum_{l=1}^k \lambda_{1l} y_l, \sum_{l=2}^k \lambda_{2l} y_l, \dots, \sum_{l=1}^k \lambda_{kl} y_l).$$

We put $\tilde{\varphi_1} := H(\varphi_1) \in \mathcal{C}_k$. We see then that

$$\hat{\tilde{\varphi}}_1 = c_1 y_1^d + c_2 y_2^d + \dots + c_k y_k^d + \{\text{mixed terms of degree } d\} + \sum_{|\omega| > d} b_\omega y_1^{\omega_1} \dots y_k^{\omega_k},$$

where $c_1 = P_d(\lambda_{11}, \dots, \lambda_{k1})$, $c_2 = P_d(\lambda_{12}, \dots, \lambda_{k2}), \dots, c_k = P_d(\lambda_{1k}, \dots, \lambda_{kk})$.

If we put $\hat{\gamma} := \sum_{|\omega|>d} b_\omega y_1^{\omega_1} \dots y_k^{\omega_k}$, we see that $\hat{\gamma}$ is a formal solution of the equation

$$Z + c_1 y_1^d + c_2 y_2^d + \dots + c_k y_k^d + \{\text{mixed terms of degree } d\} - \tilde{\varphi_1} = 0. \tag{2}$$

By the Artin Approximation Theorem, if $\nu \in \mathbb{N}$, $\nu > d$, there exists $\psi \in \mathcal{C}_k$ that is a solution of the equation (2) such that $\hat{\psi} - \hat{\gamma} \in \underline{m}_n^\nu \mathbb{R}[[x_1, \dots, x_n]]$, which proves the lemma. □

We consider $H_1 = (y_1, y_1 y_2, \dots, y_1 y_k) : \mathcal{C}_k \to \mathcal{C}_k$. We have

$$H_1 H(\varphi_1)(y_1, \dots, y_k) = y_1^d \psi_1,$$

for some unit $\psi_1 \in \mathcal{C}_k$.

Since ψ_1 is a unit, there is $\psi_2 \in \mathcal{C}_k$ such that $\psi_2^d \psi_1 = 1$. Now we define $H_2 := (y_1 \psi_2, y_2, \dots, y_k) : \mathcal{C}_k \to \mathcal{C}_k$. We have $H_2 H_1 H(\varphi_1)(y_1, \dots, y_k) = y_1^d$.

Remark 5.4. We remark that the homomorphism

$$(y_1^d, H_2 H_1 H(\varphi_1), \dots, H_2 H_1 H(\varphi_n))$$

is an admissible modification of Φ. Since our system is well behaved, we notice, also, that the homomorphism $(y_1, H_2 H_1 H(\varphi_1), \dots, H_2 H_1 H(\varphi_n))$ is an admissible modification of $(y_1^d, H_2 H_1 H(\varphi_1), \dots, H_2 H_1 H(\varphi_n))$.

Theorem 5.5. *If $rk(\Phi) = s$, then there is a finite sequence $\Phi_1, \dots, \Phi_l$ of homomorphisms $\Phi_i : \mathcal{C}_n \to \mathcal{C}_k$ such that $\Phi_1 = \Phi$, Φ_{i+1} is related to Φ_i by an admissible modification, $i = 1, \dots, l-1$, and $\Phi_l = (y_1, \dots, y_s, 0, 0, \dots 0)$.*

Proof. By the above remark, we can suppose that $\Phi = (y_1, \varphi_2, \dots \varphi_n)$. Assume we have found a sequence $(\Phi_i)_{i=1}^N$ of admissible modifications such that $\Phi_N = (y_1, y_2, \dots, y_j, \varphi_{j+1}, \dots, \varphi_n)$ for some integer j. Notice $j \leq s$, since $rk(\Phi_N) = s$. We consider two cases:

Case $j < s$. We may assume that one of the germs, $\varphi_{j+1}, \dots, \varphi_k$, depends on at least one of the remaining variables $y_{j+1}, \dots, y_k$. If not, we consider the homomorphism $\Gamma = (x_1, \dots, x_j, x_{j+1} - \varphi_{j+1}, x_{j+2}, \dots, x_n)$. We see then that $\Phi_N \circ \Gamma = (y_1, \dots, y_j, 0, \varphi_{j+2}, \dots, \varphi_n)$ is an admissible modification of Φ_N. We continue with φ_{j+2}. At the end we find an admissible modification of Φ_N of the form $(y_1, \ldots, y_j, 0 \ldots, 0)$, which is a contradiction since $rk(\Phi_N) = s$.

We can then suppose that φ_{j+1} depends on at least one of the variables $y_{j+1}, \ldots, y_k$. We have

$$\hat{\varphi}_{j+1} = P_d(y_{j+1}, \ldots, y_k) + \hat{\psi},$$

where $P_d \in \mathcal{C}_j[y_{j+1}, \ldots, y_k]$ is a homogenous polynomial of degree d, $\hat{\psi} \in (y_{j+1}, \ldots, y_k)^{d+1}\mathbb{R}[[y_1, \ldots, y_k]]$. We have then $\hat{\psi} = \sum_{l=1}^{q} \lambda_l \hat{\beta}_l$, where $\lambda_l \in (y_{j+1}, \ldots, y_k)^{d+1} \in \mathbb{R}[y_1, \ldots, y_k]$ and $\hat{\beta}_l \in \mathbb{R}[[y_1, \ldots, y_k]]$. We put

$$E(Z, Z_1, \ldots, Z_q) = (\varphi_{j+1} - P_d(y_{j+1}, \ldots, y_k) - Z, Z - \sum_{l=1}^{q} \lambda_l Z_l),$$

$E(Z, Z_1, \ldots, Z_q) \in (\mathcal{C}_k[Z, Z_1, \ldots, Z_q])^2$.

We see that $(\hat{\psi}, \hat{\beta}_1, \ldots, \hat{\beta}_q)$ is a formal solution of the equation

$$E(Z, Z_1, \ldots, Z_q) = 0.$$

By the Artin Approximation Theorem, there exists $(\psi, \beta_1, \ldots, \beta_q) \in (\mathcal{C}_k)^{q+1}$ that is a solution of the equation $E(Z, Z_1, \ldots, Z_q) = 0$. We have then that

$$\varphi_{j+1} = P_d(y_{j+1}, \ldots, y_k) + \psi,$$

with $\psi \in (y_{j+1}, \ldots, y_k)^{d+1}\mathcal{C}_k$.

Let $(\lambda_{i,\nu})_{\nu=j+1}^{k}$ be a nonsingular matrix of scalars and define the homomorphism $H : \mathcal{C}_k \to \mathcal{C}_k$ by

$$H = (y_1, \ldots, y_j, \sum_{\nu=j+1}^{k} \lambda_{i,\nu} y_\nu, \ldots, \sum_{\nu=j+1}^{k} \lambda_{k,\nu} y_\nu).$$

We see then that

$$H(\varphi_{j+1}) = c_{j+1} y_{j+1}^d + \ldots + c_k y_k^d + \{\text{mixed terms of degree } d\} + H(\psi), \tag{3}$$

where

$$c_l = P_d(\lambda_{l,j+1}, \lambda_{l,j+2}, \ldots, \lambda_{l,k}),\ l = j+1, \ldots, k.$$

We put

$$Q = c_{j+2} y_{j+2}^d + \ldots + c_k y_k^d + \{\text{mixed terms of degree } d\}.$$

Let us note that each term of Q is divisible by at least one of $y_{j+2}, \ldots, y_k$.

We may choose the matrix $(\lambda_{i,\nu})_{\nu=j+1}^{k}$ so that $P_d(\lambda_{j+1,j+1}, \lambda_{j+1,j+2}, \ldots, \lambda_{j+1,k}) \neq 0$.

Define $G : \mathcal{C}_k \to \mathcal{C}_k$ by

$$G = (y_1, \ldots, y_{j+1}, y_{j+1} y_{j+2}, \ldots, y_{j+1} y_k).$$

We see that y_{j+1}^d divides $G \circ H(\varphi_{j+1})$. Since $G \circ H(\Phi_N)$ is an admissible modification of Φ_N, we can suppose that y_{j+1}^d divides φ_{j+1} and φ_{j+1} is still of the form (3).

On the other hand, by Proposition 3.2, there exists $H_1 : \mathcal{C}_k \to \mathcal{C}_k$,

$$H_1 = (y_1, y_1^{e_{1,2}} y_2, y_1^{e_{1,3}} y_2^{e_{2,3}} y_3, \ldots, y_1^{e_{1,k}} y_2^{e_{2,k}} y_3^{e_{3,k}} \ldots y_{k-1}^{e_{k-1,k}} y_k),$$

such that $H_1(\varphi_{j+1}) = (\text{monomial}) \cdot Q_1$ for some unit $Q_1 \in \mathcal{C}_k$.

From (3), we have

$$H_1(\varphi_{j+1}) = H_1(c_{j+1})(y_1^{e_{1,j+1}} y_2^{e_{2,j+1}} \ldots y_j^{e_{j,j+1}} y_{j+1})^d + H_1(Q_1) + H_1(\psi). \tag{4}$$

$H_1(c_{j+1}) \in \mathcal{C}_j$, $H_1(c_{j+1}) \neq 0$, and each term of $H_1(Q_1)$ is divisible by at least one of $y_{j+2}, \ldots, y_k$ and $H_1(\psi) \in (y_{j+1}, \ldots, y_k)^{d+1}\mathcal{C}_k$.

By (4), we see then that none of $y_{j+2}, \ldots, y_k$ divides $H_1(\varphi_{j+1})$. On the other hand, we know that y_{j+1}^d divides φ_{j+1}, and hence y_{j+1}^d divides $H_1(\varphi_{j+1})$. But we have $H_1(\varphi_{j+1}) = (\text{monomial}) \cdot Q_1$; hence

$$H_1(\varphi_{j+1}) = (y_1^{\epsilon_{1,j+1}} y_2^{\epsilon_{2,j+1}} \cdots y_j^{\epsilon_{j,j+1}} y_{j+1}^{\epsilon_{j+1,j+1}})Q_1.$$

Thus we have modified Φ_N to an $H_1(\Phi_N)$ of the form

$$H_1(\Phi_N)=(y_1, y_1^{e_{1,2}}y_2, \ldots, y_1^{e_{1,j}}y_2^{e_{2,j}} \cdots y_{j-1}^{e_{j-1,j}}y_j, y_1^{\epsilon_{1,j+1}} \cdots y_{j+1}^{\epsilon_{j+1,j+1}}Q_1, \varphi_{j+2}, \ldots, \varphi_n).$$

Now we may absorb the unit Q_1 into y_{j+1} as above. By successive applications of morphisms e and r_d we have modified Φ_N to the form

$$\Phi_{N'} = (y_1, \ldots, y_j, y_{j+1}, \varphi_{j+2}, \varphi_n).$$

Case $j = s$. If $\Phi_N = (y_1, \ldots, y_s, \varphi_{s+1}, \ldots, \varphi_n)$, then for each $l = s+1, \ldots, n$, φ_l is independent of all $y_{s+1}, \ldots, y_k$. Otherwise we could modify Φ_N by the above procedure to $\Phi_N = (y_1, \ldots, y_s, y_{s+1}, \varphi_{s+2}, \ldots, \varphi_n)$, but $rk(\Phi_N) = s+1 > s$.

Define $\Gamma : \mathcal{C}_n \to \mathcal{C}_n$ by

$$\Gamma = (x_1, \ldots, x_s, x_{s+1} - \varphi_{s+1}(x_1, \ldots, x_s), \ldots, x_n - \varphi_n(x_1, \ldots, x_s)).$$

We see then that $\Phi_n \circ \Gamma$ is defined by $(y_1, \ldots, y_s, 0, \ldots, 0)$, and hence the theorem is proved. □

Corollary 5.6. *Let $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ be a local homomorphism. Suppose that $rk(\Phi) = n$. Then Φ is strongly injective.*

Proof. By the above theorem we can modify Φ by successive admissible modifications to the form $(y_1, y_2, \ldots, y_n)$, which is trivially strongly injective. □

Corollary 5.7. *The Weierstrass Division Theorem holds in the well-behaved Noetherian differentiable system $\mathcal{C} = \{\mathcal{C}_n,\, n \in \mathbb{N}\}$.*

Proof. First, every Noetherian differentiable system is a quasianalytic differentiable system. By Corollary 5.6, every local homomorphism $\Phi : \mathcal{C}_n \to \mathcal{C}_k$ with $rk(\Phi) = n$ is strongly injective, hence the result by Theorem 4.1 of [7]. □

Corollary 5.8. *Every well-behaved Noetherian differentiable system is continued in the analytic system.*

Proof. By Corollary 5.7, the Weierstrass Division Theorem holds in this system. We deduce the result by [6]. □

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