

ENTROPY, WEIL-PETERSSON TRANSLATION DISTANCE AND GROMOV NORM FOR SURFACE AUTOMORPHISMS

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ABSTRACT. Thanks to a theorem of Brock on the comparison of Weil-Petersson translation distances and hyperbolic volumes of mapping tori for pseudo-Anosovs, we prove that the entropy of a surface automorphism in general has linear bounds in terms of a Gromov norm of its mapping torus from below and an inbounded geometry case from above. We also prove that the Weil-Petersson translation distance does the same from both sides in general. The proofs are in fact immediately derived from the theorem of Brock, together with some other strong theorems and small observations.

1. INTRODUCTION

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus g with n punctures, $\mathcal{T} = \mathcal{T}_{g,n}$ be the Teichmüller space of Σ and $\mathcal{M} = \mathcal{M}_{g,n}$ be the (orientation preserving) mapping class group of Σ . An element of \mathcal{T} is defined by a pair (R, f) of a Riemann surface R homeomorphic to Σ with a marking homeomorphism $f : \Sigma \rightarrow R$. A mapping class $\varphi \in \mathcal{M}$ naturally acts on $(R, f) \in \mathcal{T}$ by $(R, f \circ \varphi^{-1})$. Throughout this paper, we assume that $3g - 3 + n \geq 1$ or $(g, n) = (0, 3)$, and fix g and n . Also, we omit a marking homeomorphism f to indicate a marked Riemann surface in \mathcal{T} .

In this paper, we discuss three invariants measuring the complexity of a conjugacy class of a mapping class $\varphi \in \mathcal{M}$. One is the infimum of topological entropies of surface automorphisms in φ , which we denote by $\text{ent } \varphi$. As we will point out in Corollary 10, it is equal to the Teichmüller translation distance of $\varphi \in \mathcal{M}$,

$$\|\varphi\|_{\mathcal{T}} = \inf_{R \in \mathcal{T}} d_{\mathcal{T}}(R, \varphi(R)),$$

where $d_{\mathcal{T}}$ denotes the Teichmüller distance on \mathcal{T} . The second one is the Weil-Petersson translation distance defined by

$$\|\varphi\|_{\text{WP}} = \inf_{R \in \mathcal{T}} d_{\text{WP}}(R, \varphi(R)),$$

where d_{WP} denotes the Weil-Petersson distance on \mathcal{T} . The last one through three dimensional manifolds is the Gromov norm $\|\cdot\|_{\text{Gr}}$ of the mapping torus of φ ,

$$N_{\varphi} = \Sigma \times [0, 1] / (x, 1) \sim (\varphi(x), 0).$$

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Here the notation N_φ indicates only a mapping class because the topological type of N_φ does not depend on the choice of representatives of φ . In fact, it depends only on the conjugacy class of φ . Hence any topological invariant of N_φ such as the Gromov norm is an invariant of a mapping class φ .

The starting point of this paper is a comparison of the Weil-Petersson translation distances of pseudo-Anosovs and volumes of their mapping tori by Brock in [4]. We discussed similar inequalities in Theorem 3.2 of [9], where we replaced the Weil-Petersson translation distance in Brock's work by the entropy. Here we generalize it further to one for general surface automorphisms as follows.

Theorem 1. *There is a constant $A = A(g, n) > 0$ which depends only on the genus g and the number n of punctures such that the inequality*

$$A \|N_\varphi\|_{\text{Gr}} \leq \text{ent } \varphi$$

holds for any $\varphi \in \mathcal{M}_{g,n}$. Furthermore for any $\varepsilon > 0$, there is a constant $B = B(g, n, \varepsilon) > 0$ which depends only on g , n and ε such that

$$\text{ent } \varphi \leq B \|N_\varphi\|_{\text{Gr}}$$

holds for any $\varphi \in \mathcal{M}_{g,n}$ of which the hyperbolic pieces of N_φ contain no closed geodesics of length $< \varepsilon$.

We also generalize the theorem of Brock to one for general surface automorphisms as follows.

Theorem 2. *There exists a constant $C = C(g, n)$ which depends only on g and n such that*

$$C^{-1} \|N_\varphi\|_{\text{Gr}} \leq \|\varphi\|_{\text{WP}} \leq C \|N_\varphi\|_{\text{Gr}}$$

holds for any $\varphi \in \mathcal{M}_{g,n}$.

Both theorems are immediately derived from Brock's theorem together with some existing strong theorems and small observations. We collect necessary materials first with a reasonable account of explanations as packages in the next section, and then present a reducing argument in the last section. For the reader's convenience, we duplicate some of the arguments in [9].

2. PACKAGES

2.1. Entropy. Let us review the topological entropy defined by Adler, Konheim and McAndrew in [1] for a self-map $f : X \rightarrow X$ of a compact topological space X . Choose an open covering \mathcal{U} of X , and let $N(m, \mathcal{U})$ be the minimal number of members in a refined covering $\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \dots \vee f^{-(m-1)}(\mathcal{U})$ by intersections that cover Σ . Then

$$h(f) = \sup_{\mathcal{U}} \left(\limsup_{m \rightarrow \infty} \frac{1}{m} \log N(m, \mathcal{U}) \right)$$

is called the topological entropy of f . It is shown in [1] to always exist and to satisfy the identity

$$h(f^n) = n h(f)$$

for any positive integer n .

Our topological space Σ may not be compact; however, if we suppose that the covering \mathcal{U} contains open sets including each puncture in their interior, then it can

be handled as a compact space and every argument in [1] works equally well. For a mapping class $\varphi \in \mathcal{M}$, we define the entropy of φ by

$$\text{ent } \varphi = \inf_{f \in \varphi} h(f).$$

In particular, we have the identity

$$(2.1) \quad \text{ent } \varphi^n = n \text{ent } \varphi.$$

2.2. Thurston's classification of surface automorphisms. According to a seminal work of Thurston in [20] (see also [7]), the elements of \mathcal{M} are classified into three types: periodic, pseudo-Anosov and reducible. This classification has become standard, but it is better to note that the classification is not in trichotomy. In fact, there is a class which is periodic and also reducible.

A pseudo-Anosov mapping class φ is known to have a good deal of structures. It has a nice representative $f : \Sigma \rightarrow \Sigma$ which leaves a pair of stable and unstable foliations invariant. The expanding factor λ of the unstable foliation is called the dilatation of f . Thurston [20] (see also [7]) showed that if $\varphi \in \mathcal{M}$ is pseudo-Anosov, then

$$\text{ent } \varphi = \log \lambda.$$

2.3. Bers' classification of surface automorphisms. Bers gave an alternative proof of Thurston's classification in [2] by adopting an extremal approach in quasiconformal mappings. Since Bers' classification is more suitable for us, we briefly review it.

Bers calls φ elliptic if $\|\varphi\|_{\mathcal{T}} = 0$, and there is a Riemann surface $R \in \mathcal{T}$ such that $\varphi(R) = R$. This is identical with Thurston's periodic case. Call φ hyperbolic if $\|\varphi\|_{\mathcal{T}} > 0$, and there is a Riemann surface $R \in \mathcal{T}$ which attains the infimum. This is also identical with Thurston's pseudo-Anosov case. The remaining cases in Bers' classification are called parabolic if $\|\varphi\|_{\mathcal{T}} = 0$ or pseudo-hyperbolic if $\|\varphi\|_{\mathcal{T}} > 0$, where there are no Riemann surfaces in \mathcal{T} which attain the infimum for either cases. These are reducible but not periodic in Thurston's classification.

Then Bers showed that if φ is hyperbolic (= pseudo-Anosov), then there is a unique invariant Teichmüller geodesic $\ell_{\mathcal{T}}(\varphi)$ for the action of φ on \mathcal{T} , and the identity

$$\log \lambda = d_{\mathcal{T}}(R, \varphi(R)) = \|\varphi\|_{\mathcal{T}}$$

holds for any $R \in \ell_{\mathcal{T}}(\varphi)$.

2.4. Gromov norm. According to Section 6.5 of [19], we recall the definition of the Gromov norm for a 3-manifold with toral boundary. Let M be a compact orientable 3-manifold whose boundary consists of tori. Then, the Gromov norm is defined to be the limit infimum of an ℓ_1 -norm of cycles representing the relative fundamental class $[M, \partial M] \in H_3(M, \partial M; \mathbb{R})$ with vanishing boundary condition, namely,

$$\|[M]\|_{\text{Gr}} = \liminf_{\varepsilon \rightarrow 0} \{ \|c\|_1 ; [c] = [M, \partial M], \text{ and } \|\partial c\| \leq \varepsilon \}.$$

In [18], Soma proved the splitting theorem for the Gromov norm for compact orientable 3-manifolds whose boundary consists of tori. His theorem, together with the solution of the geometrization conjecture by Perelman [16, 17], establishes a simpler statement than the original in [18] as follows. Let $v_3 = 1.01494\dots$ be the volume of the regular ideal hyperbolic 3-simplex.

Theorem 3 (Soma [18]). *The Gromov norm of a compact orientable 3-manifold M whose boundary consists of tori is equal to v_3 times the sum of volumes of hyperbolic pieces in the geometric decomposition of M .*

Thus the term $\|N_\varphi\|_{\text{Gr}}$ in the inequalities of the theorems can be replaced by the sum of volumes of hyperbolic parts of N_φ in the geometric decomposition.

One of the useful properties of the Gromov norm for us is that if $\widetilde{M} \rightarrow M$ is an n -fold covering, then

$$(2.2) \quad \|\widetilde{M}\|_{\text{Gr}} = n\|M\|_{\text{Gr}}.$$

This fact will be used to reduce the problem in question in the next subsection.

2.5. Reduction. To each mapping class $\varphi \in \mathcal{M}$ is associated a reduced maximal family \mathcal{C} of disjoint simple closed curves on Σ which is invariant by the action of φ . This means more precisely that $\Sigma - \mathcal{C}$ is divided into two groups, U and V , which consist of components of $\Sigma - \mathcal{C}$ so that φ restricts to U is periodic and to V is pseudo-Anosov, where U and V may be disconnected.

We know from (2.1) and (2.2) that ent and $\|\cdot\|_{\text{Gr}}$ behave well under taking powers. The same identities hold for the Weil-Petersson translation distance and Teichmüller translation distance by definition; namely,

$$\|\varphi^n\|_{\text{WP}} = n\|\varphi\|_{\text{WP}} \quad \text{and} \quad \|\varphi^n\|_{\text{T}} = n\|\varphi\|_{\text{T}}$$

hold for any positive integer n . Thus, it is sufficient to show the inequalities in the theorems in the introduction for some power of φ , and we may assume by taking some power of φ that φ (with the same notation) preserves each component of $\Sigma - \mathcal{C}$. Then Σ will be a union of subsurfaces $S_1 \cup S_2 \cup \cdots \cup S_k$ split by \mathcal{C} , and the restriction of φ to S_i defines a mapping class φ_i on each component S_i . If some φ_i is periodic, again taking further power, we may assume that it is the identity. Thus, relabeling suffices, and letting j be the number of pseudo-Anosov components, we assume in the sequel that

Reduction Hypothesis 4. *The restriction φ_i of φ to S_i in $\Sigma - \mathcal{C} = S_1 \cup S_2 \cup \cdots \cup S_k$ preserves S_i and is pseudo-Anosov if $i \leq j$ ($\leq k$) and the identity if $i > j$.*

Note that j could range from zero to k . Also, k could be 1.

2.6. Geometric decomposition of N_φ . Thurston proved in [21] (cf. [15]) that the mapping torus of a pseudo-Anosov admits a hyperbolic structure. The hyperbolic structure in dimension ≥ 3 is known to be unique by Mostow rigidity. Under the Reduction Hypothesis 4, the geometric decomposition of N_φ is quite simple and the hyperbolic pieces consist of N_{φ_i} 's for $i \leq j$. Thus by Theorem 3 of Soma ([18]), we have

$$(2.3) \quad \|N_\varphi\|_{\text{Gr}} = \sum_{i=1}^j \|N_{\varphi_i}\|_{\text{Gr}} = \frac{1}{v_3} \sum_{i=1}^j \text{vol } N_{\varphi_i}.$$

2.7. Teichmüller metric. The Teichmüller distance between two Riemann surfaces $R, R' \in \mathcal{T}$ is originally defined by

$$d_{\text{T}}(R, R') = \inf_h \sup_{x \in R} \log K_h(x),$$

where $h : R \rightarrow R'$ is a quasiconformal map and $K_h(x)$ is a dilatation of h at x . To see the infinitesimal form of d_T , we recall a little analysis by quoting a few well known results from the text book by Gardiner and Lakic [8].

Let R be a Riemann surface. Also, let $T^{1,0}R$ and $T^{0,1}R$ be the holomorphic part and the anti-holomorphic part of the complex cotangent bundle over R respectively. A Beltrami differential μ on R , which is a section of a line bundle $(T^{0,1}R) \otimes (T^{1,0}R)^*$, represents an infinitesimal deformation of the complex structure of R including holomorphically trivial ones caused by diffeomorphisms. We let $L_\infty(R)$ be the space of uniformly bounded Beltrami differentials, namely $\{\mu : \|\mu\|_\infty < \infty\}$. This is an infinite dimensional space.

Another object of interest is a quadratic differential q which is a section of a line bundle $(T^{1,0})^{\otimes 2}$. Let $Q(R)$ be the space of holomorphic quadratic differentials with bounded L_1 norm $\|q\| = \int_R |q| < \infty$. By Riemann-Roch, the dimension of $Q(R)$ is equal to $3g - 3 + n$. Then, there is a natural pairing defined by

$$(2.4) \quad (\mu, q) = \int_R \mu q.$$

If we let

$$N = \{\mu : (\mu, q) = 0 \text{ for all } q \in Q(R)\},$$

then $L_\infty(R)/N$ can be identified with the tangent space $T_R\mathcal{T}$ of \mathcal{T} at R . A standard argument in functional analysis implies that

Lemma 5 (cf. 3.1, Theorem 2 in [8]). *The natural pairing (2.4) induces an isomorphism of $Q(R)^*$ to $L_\infty(R)/N \cong T_R\mathcal{T}$.*

Lemma 6 (cf. 4.12 Theorem 13 in [8]). *The norm $\|\mu\| = \sup\{(\mu, q) : \|q\| \leq 1\}$ dual to the L_1 norm on $Q(R) \cong T_R^*\mathcal{T}$ is the infinitesimal form of the Teichmüller distance d_T .*

2.8. Weil-Petersson metric. The Weil-Petersson metric comes from the L^2 inner product on $Q(R) \cong T_R^*\mathcal{T}$ defined by

$$\langle q, q' \rangle_{\text{WP}} = \int_X \frac{\bar{q}q'}{\rho^2},$$

where $q, q' \in Q(R)$ and $\rho(z)|dz|$ is the hyperbolic metric on Σ conformally equivalent to the complex structure on R . The Weil-Petersson metric is defined to be a Riemannian part of the dual Hermitian metric to the above co-metric on the cotangent space.

The Weil-Petersson metric is known to have negatively curved sectional curvature by the work of several authors; see for example Wolpert [23]. Also, it is known to be incomplete by Wolpert in [22] but geodesically convex also by Wolpert in [24]: for each pair of points, there exists a unique distance realizing joining curve.

Daskalopoulos and Wentworth [6] showed that if φ itself is pseudo-Anosov, then it admits the unique invariant Weil-Petersson geodesic $\ell_{\text{WP}}(\varphi)$ of the action by φ on \mathcal{T} , and the identity

$$d_{\text{WP}}(R, \varphi(R)) = \|\varphi\|_{\text{WP}}$$

holds for any $R \in \ell_{\text{WP}}(\varphi)$.

2.9. Teichmüller versus Weil-Petersson. Lemma 6 says that the L_1 norm on $Q(R)$ is the dual norm to the infinitesimal form of the Teichmüller distance through the natural pairing established in Lemma 5. Then, by the Cauchy-Schwarz inequality, we have

$$\|q\|^2 = \left(\int_R |q|\right)^2 = \left(\int_R \rho \cdot \frac{|q|}{\rho}\right)^2 \leq \int_R \rho^2 \cdot \int_R \frac{\bar{q}q}{\rho^2} = \text{Area } \Sigma \cdot \langle q, q \rangle_{\text{WP}}.$$

Thus, this infinitesimal inequality on the dual space implies the inequality of the other direction in two distances,

$$(\sqrt{\text{Area } \Sigma})^{-1} d_{\text{WP}} \leq d_{\text{T}},$$

which was originally proved by Linch in [10]. This implies

Lemma 7. *There is a constant $D = D(g, n)$ which depends only on g, n such that the inequality*

$$(2.5) \quad \|\varphi\|_{\text{WP}} \leq D \|\varphi\|_{\text{T}}$$

holds for any $\varphi \in \mathcal{M}$.

Proof. Choose a Riemann surface R on the Teichmüller geodesic $\ell_{\text{T}}(\varphi)$ of φ in \mathcal{T} . Then, we have

$$\|\varphi\|_{\text{T}} = d_{\text{T}}(R, \varphi(R)) \geq (\sqrt{\text{Area } \Sigma})^{-1} d_{\text{WP}}(R, \varphi(R)) \geq (\sqrt{\text{Area } \Sigma})^{-1} \|\varphi\|_{\text{WP}},$$

where the last inequality holds because R may not be on $\ell_{\text{WP}}(\varphi)$. □

On the other hand, since the Weil-Petersson metric is incomplete and there is a point at infinity which is of a finite distance from an interior point, the inequality of the other direction to (2.5) cannot be established in general. However, there is such a case under some bounded geometry condition. For any $\varepsilon > 0$, there is a constant $\delta > 0$ such that if $\varphi \in \mathcal{M}$ is pseudo-Anosov and if a hyperbolic structure on N_φ contains no closed geodesics of length $< \varepsilon$, then both the Teichmüller geodesic $\ell_{\text{T}}(\varphi)$ and the Weil-Petersson geodesic $\ell_{\text{WP}}(\varphi)$ are located in the subset \mathcal{T}^δ of \mathcal{T} consisting of hyperbolic surfaces with no closed geodesics of length $< \delta$. This fact is proved for the Teichmüller metric by Minsky in [13] and for the Weil-Petersson metric by Brock-Masur-Minsky in [5]. This implies

Lemma 8. *For any $\varepsilon > 0$, there is a constant $E = E(g, n, \varepsilon) > 0$ such that*

$$(2.6) \quad E \|\varphi\|_{\text{T}} \leq \|\varphi\|_{\text{WP}}$$

for any pseudo-Anosov $\varphi \in \mathcal{M}$ so that N_φ contains no closed geodesics of length $< \varepsilon$.

Proof. The subset \mathcal{T}^δ is invariant by the action of \mathcal{M} , and the quotient $\mathcal{T}^\delta/\mathcal{M}$ is compact by a theorem of Mumford in [14]. Hence both Teichmüller and Weil-Petersson metrics on \mathcal{T}^δ are pullbacks of metrics on the compact space $\mathcal{T}^\delta/\mathcal{M}$, and there is a constant $E = E(g, n, \varepsilon)$ which depend only on g, n and ε such that

$$E d_{\text{T}} \leq d_{\text{WP}}$$

on \mathcal{T}^δ . Choose a Riemann surface R on the Weil-Petersson geodesic $\ell_{\text{WP}}(\varphi)$ of φ on \mathcal{T} . Then

$$\|\varphi\|_{\text{WP}} = d_{\text{WP}}(R, \varphi(R)) \geq E d_{\text{T}}(R, \varphi(R)) \geq E \|\varphi\|_{\text{T}},$$

where the last inequality holds because R may not be on $\ell_{\text{T}}(\varphi)$. □

2.10. Weil-Petersson translation distance. The augmented structure of the completion $\overline{\mathcal{T}}$ with respect to the Weil-Petersson metric was extensively studied by Masur in [11]. Using this, together with Wolpert's convexity, Masur and Wolf provided a very clear geodesically embedded picture of the frontier of $\overline{\mathcal{T}}$ in Subsection 1.2 of [12].

One important observation there is that the intrinsic Weil-Petersson metric defined on the frontier of \mathcal{T} is identical with the induced metric on the metric completion of \mathcal{T} with respect to the Weil-Petersson metric. In particular, if we let $\mathcal{O}(\mathcal{C})$ be the product $\mathcal{T}(S_1) \times \mathcal{T}(S_2) \times \cdots \times \mathcal{T}(S_k)$ of Teichmüller spaces of S_i 's, together with the product metric of the Weil-Petersson metrics on each Teichmüller space $\mathcal{T}(S_i)$, then $\mathcal{O}(\mathcal{C})$ is isometrically embedded in $\overline{\mathcal{T}}$ as the corresponding frontier of \mathcal{T} .

Recall that Daskalopoulos and Wentworth [6] established the existence of a Weil-Petersson geodesic for the action of a pseudo-Anosov φ on \mathcal{T} . The following lemma concerns the other case, which was discussed earlier in [25]. We here provide its quick proof for the reader's convenience.

Lemma 9. *The following hold:*

- (1) *If $j = 0$, then φ fixes $\mathcal{O}(\mathcal{C})$ and $\|\varphi\|_{\text{WP}} = 0$.*
- (2) *If $k \geq 2$ and $1 \leq j \leq k$, then there is no invariant geodesic of the action by φ in \mathcal{T} , but there is at least one in $\mathcal{O}(\mathcal{C}) \subset \overline{\mathcal{T}}$, and*

$$(2.7) \quad \|\varphi\|_{\text{WP}} = \sqrt{\sum_{i=1}^j \|\varphi_i\|_{\text{WP}}^2}.$$

The invariant geodesic is not unique if and only if $j < k$ and there exist $i > j$ so that S_i is not a thrice punctured sphere.

Proof. (1) is clear.

To see (2), choose a point $R_i \in \mathcal{T}(S_i)$ on the invariant geodesic of the action by φ_i for $i \leq j$ and any point $R_i \in \mathcal{T}(S_i)$ if $i > j$. Then the point $R = (R_1, R_2, \dots, R_k)$ on $\mathcal{O}(\mathcal{C})$ will be on the geodesic ℓ in $\mathcal{O}(\mathcal{C}) \subset \overline{\mathcal{T}}$ invariant by the action of φ . The identity (2.7) will be obvious by the structure of the metric on $\mathcal{O}(\mathcal{C})$.

To see the non-existence of an invariant geodesic of the action by φ in \mathcal{T} , suppose to the contrary that we have one ℓ' lying in \mathcal{T} . Since \mathcal{T} with the Weil-Petersson metric has negative sectional curvature and is geodesically convex, it is a CAT(0) space. Then by Chapter II, Corollary 3.11 in [3], the metric completion $\overline{\mathcal{T}}$ is also a CAT(0) space. Since ℓ and ℓ' are both invariant by the action of φ , it is easy to see that they provide asymptotic geodesics, namely, there are a constant K and arclength parameterizations for ℓ and ℓ' so that $d_{\text{WP}}(\ell(r), \ell'(t)) < K$ for all $t \in \mathbb{R}$. This contradicts the flat strip theorem, Chapter II, Theorem 2.13 in [3], since ℓ' is located in the part with negative sectional curvature. \square

2.11. Teichmüller translation distance versus entropy. By applying Theorem 4 in [1] to our setting, we obtain the identity,

$$(2.8) \quad \text{ent } \varphi = \max_{1 \leq i \leq j} \text{ent } \varphi_i.$$

This is the only result in this subsection that we will use in the next section. However, we would like to add a few more lines that are worth noting.

Theorem 9 in [2] can be simply stated in our setting by

$$\|\varphi\|_T = \max_{1 \leq i \leq j} \|\varphi_i\|_T.$$

Since φ_i is pseudo-Anosov for $i \leq j$, $\text{ent } \varphi_i = \|\varphi_i\|_T$ holds for $i \leq j$, and we have the following corollary,

Corollary 10. *The identity*

$$\text{ent } \varphi = \|\varphi\|_T$$

holds for any $\varphi \in \mathcal{M}$.

This should be well known to the experts; however, it seems to have not been in the literature so far.

3. PROOFS

3.1. Setting up. If $j = 0$, then the entropy of φ , the Weil-Petersson translation distance of φ and the Gromov norm of N_φ all are obviously zero, and we are done for both theorems. Thus assume that $j \geq 1$. Then by the theorem of Brock in [4], we have a set of inequalities,

$$(3.1) \quad C_i^{-1} \|\varphi_i\|_{\text{WP}} \leq \text{vol } N_{\varphi_i} \leq C_i \|\varphi_i\|_{\text{WP}}, \quad i = 1, 2, \dots, j.$$

Each C_i depends only on the topology of the surface S_i . Now, there are only finitely many topologies which can appear for surfaces obtained by splitting Σ along a system of essential curves. Thus, there is a constant $F_1 = F_1(g, n)$ which bounds the Brock constant for a surface appearing in any essential splitting of Σ . In particular, F_1 depends only on g, n and does not depend on any particular splitting of Σ . Replacing C_i by F_1 in (3.1), summing all inequalities in (3.1), and replacing the middle term by the identity (2.3), we obtain

$$(3.2) \quad F_1^{-1} \sum_{i=1}^j \|\varphi_i\|_{\text{WP}} \leq v_3 \|N_\varphi\|_{\text{Gr}} \leq F_1 \sum_{i=1}^j \|\varphi_i\|_{\text{WP}}.$$

3.2. Proof of Theorem 1. By Lemma 7 and Lemma 8, we have a set of inequalities

$$E_i \text{ent } \varphi_i = E_i \|\varphi_i\|_T \leq \|\varphi_i\|_{\text{WP}} \leq D_i \|\varphi_i\|_T = D_i \text{ent } \varphi_i \quad i = 1, 2, \dots, j.$$

By the same reasoning as in subsection 3.1, we can find constants $F_2 = F_2(g, n, \varepsilon)$ and $F_3 = F_3(g, n)$ which do not depend on any splitting of Σ such that the equalities

$$(3.3) \quad F_2 \sum_{i=1}^j \text{ent } \varphi_i \leq \sum_{i=1}^j \|\varphi_i\|_{\text{WP}} \leq F_3 \sum_{i=1}^j \text{ent } \varphi_i$$

hold. The left inequality is established only under the bounded geometry condition controlled by ε .

On the other hand, since j , which is the number of pseudo-Anosov components, is at most $n + 2g - 2$, there is an obvious constant $F_4 = F_4(g, n) > 0$ which depends only on g, n such that the inequalities

$$F_4^{-1} \max_{1 \leq i \leq j} \text{ent } \varphi_i \leq \sum_{i=1}^j \text{ent } \varphi_i \leq F_4 \max_{1 \leq i \leq j} \text{ent } \varphi_i$$

hold. Thus, by letting $F_5 = F_2F_4^{-1}$ and $F_6 = F_3F_4$, we have

$$(3.4) \quad F_5 \text{ ent } \varphi \leq \sum_{i=1}^j \|\varphi_i\|_{\text{WP}} \leq F_6 \text{ ent } \varphi.$$

Theorem 1 is immediate from the comparisons of (3.2) and (3.4).

3.3. Proof of Theorem 2. Again since $j \leq n+2g-2$, there is an obvious constant $F_7 = F_7(g, n)$ which does not depend on any splitting of Σ such that

$$(3.5) \quad F_7^{-1} \sqrt{\sum_{i=1}^j \|\varphi_i\|_{\text{WP}}^2} \leq \sum_{i=1}^j \|\varphi_i\|_{\text{WP}} \leq F_7 \sqrt{\sum_{i=1}^j \|\varphi_i\|_{\text{WP}}^2}.$$

This, together with the identity (2.7) and the comparison (3.2), immediately implies Theorem 2. \square

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