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ON EXISTENCE OF GLOBAL SOLUTIONS OF SCHRÖDINGER EQUATIONS WITH SUBCRITICAL NONLINEARITY FOR $\widehat{L}^p$-INITIAL DATA

RYOSUKE HYAKUNA AND MASAYOSHI TSUTSUMI

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Abstract. We construct a local theory of the Cauchy problem for the nonlinear Schrödinger equations

$$\begin{aligned} &iu_t + u_{xx} \pm |u|^{\alpha-1}u = 0, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ &u(0,x) = u_0(x) \end{aligned}$$

with $\alpha \in (1,5)$ and $u_0 \in \widehat{L}^p(\mathbb{R})$ when p lies in an open neighborhood of 2. Moreover we prove the global existence for the initial value problem when p is sufficiently close to 2.

1. Introduction

Consider the following Cauchy problem:

$$(1) \qquad (\mathbf{NLS}^\alpha) \begin{cases} iu_t + u_{xx} \pm |u|^{\alpha-1}u = 0, & t \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$ is an unknown function and $1 < \alpha < 5$. We are interested in the problem of whether or not a global solution of $(\mathbf{NLS}^\alpha)$ exists when $u_0 \in L^p(\mathbb{R})$ or $u_0 \in \widehat{L}^p(\mathbb{R})$, where $\widehat{L}^p$ is defined by

$$\widehat{L}^p := \{\varphi : \widehat{\varphi} \in L^{p'}\},$$

and we note that by the Hausdorff-Young inequality, $L^p \subseteq \widehat{L}^p$ if $p \leq 2$ and $\widehat{L}^p \subseteq L^p$ if $p \geq 2$. As is well-known, the Cauchy problem $(\mathbf{NLS}^\alpha)$ is globally well-posed if $p = 2$. When $p \neq 2$, although not many authors study this subject, there are some recent works in terms of the L^p- and $\widehat{L}^p$-local well-posedness of $(\mathbf{NLS}^\alpha)$ in the cubic case ($\alpha = 3$): Zhou [13] obtained local well-posedness in L^p for $1 < p < 2$. Grünrock [5] proved the local well-posedness in $\widehat{L}^p$ for $1 < p < \infty$ and global existence for $\frac{5}{3} < p < 2$. For general $\alpha \in (1,5)$, the authors studied the case of $u_0 \in L^p$ with $p < 2$ and constructed a global solution of $(\mathbf{NLS}^\alpha)$ in [6] when $1 < \alpha \leq 3$ and p is close to 2.

In this paper we investigate $(\mathbf{NLS}^\alpha)$ when $u_0 \in \widehat{L}^p$ and p is near 2. Two types of estimates play an important role in our study. One is the inequality of type

$$(2) \qquad \|e^{it\partial_x^2} f\|_{L^qL^r} \lesssim \|f\|_{\widehat{L}^p}, \quad \frac{2}{q} + \frac{1}{r} = \frac{1}{p}.$$

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When $q = r$ and $p > \frac{4}{3}$, the inequality (2) is valid and is known as the Fefferman-Stein estimate, which goes back to [3]. It is not difficult to see that the inequality leads to the local well-posedness of the cubic NLS when $\frac{4}{3} < p < 2$ (see [4]). In the present paper we prove (2) for more general pairs, especially for the case of $q \neq r$ in order to adapt to the setting of our paper. Moreover, we also use the Strichartz estimates for inhomogeneous equations:

$$\left\| \int_0^t e^{i(t-\tau)\partial_x^2} F(\cdot,\tau) d\tau \right\|_{L^q_{[0,T]} L^r_x} \le C \|F\|_{L^\gamma_{[0,T]} L^\rho_x}. \tag{3}$$

Although this estimate is well-known when both (q, r) and (γ', ρ') are admissible, we need a more general version. Indeed, when $p > 2$, we seek a solution of $(\mathbf{NLS}^\alpha)$ in $L^q_{[0,T]}(L^r)$ spaces with $q > \frac{2r}{r-2}$ and in such cases the admissible version of (3) is no longer applicable. Here in this paper, we will use the result by Kato [7]. For updated results of the Strichartz estimate of type (3) including multidimensional and endpoint cases, see Vilela [11].

Once we construct a local theory of $(\mathbf{NLS}^\alpha)$, we then extend these solutions globally, especially when p is close enough to 2. Our strategy is motivated by the work of Vargas and Vega [12], and here we split the initial data $u_0 \in \widehat{L}^p$ using an interpolation lemma for $\widehat{L}^p$-spaces:

$$u_0 = \varphi_N + \psi_N \quad \text{and} \quad \|\varphi_N\|_{L^2} \sim N^{\frac{1-\theta}{\theta}}, \quad \|\psi_N\|_{\widehat{L}^{p_0}} \lesssim N^{-1}, \quad N > 1 \tag{4}$$

with

$$\frac{1}{p'} = \frac{1}{p'_\theta} = \frac{1-\theta}{p'_0} + \frac{\theta}{2}, \quad 0 < \theta < 1, \tag{5}$$

where p_0 will be determined later. Then we apply Bourgain's method [1] to find θ for which a global solution of $(\mathbf{NLS}^\alpha)$ with $u_0 \in \widehat{L}^{p_\theta}$ exists.

To state our results we define for $\alpha \in (1, 5)$,

$$r_\alpha := \begin{cases} \alpha + 1 & \text{if} \quad \alpha \in (1,3), \\ 4 & \text{if} \quad \alpha \in [3, \frac{11}{3}], \\ \frac{12}{11}\alpha & \text{if} \quad \alpha \in (\frac{11}{3}, 5); \end{cases}$$

and for $p > 1$, $q_{\alpha,p}$ is defined by

$$\frac{2}{q_{\alpha,p}} + \frac{1}{r_\alpha} = \frac{1}{p}.$$

Our main result is as follows.

Theorem 1. (i) *Let $u_0 \in \widehat{L}^p(\mathbb{R})$ with*

$$2 > p > \max\left(\frac{4}{3}, \frac{\alpha-1}{2}, \frac{\alpha+1}{\alpha}\right).$$

Then there exist $T > 0$ and a solution u of the Cauchy problem $(\mathbf{NLS}^\alpha)$ in $[-T, T]$ with

$$u \in C([-T,T] : \widehat{L}^p(\mathbb{R})) \cap L^{q_{\alpha,p}}([-T,T] : L^{r_\alpha}(\mathbb{R})).$$

Moreover, the mapping $u_0 \mapsto u(t)$ is locally Lipschitz from $\widehat{L}^p$ to $C([-T,T] : \widehat{L}^p) \cap L^{q_{\alpha,p}}([-T,T] : L^{r_\alpha})$.

(ii) *Let* $u_0 \in \widehat{L}^p$ *with*

$$2 < p < \min\left(\frac{2(\alpha+1)}{\alpha-1}, \alpha+1\right).$$

Then there exist $T > 0$ *and a solution* u *of the Cauchy problem* ($\mathbf{NLS}^\alpha$) *in* $[-T, T]$ *with*

$$u \in C([-T,T] : \widehat{L}^p(\mathbb{R})) \cap L^q([-T,T] : L^{\alpha+1}(\mathbb{R})),$$

where

$$\frac{2}{q} + \frac{1}{\alpha+1} = \frac{1}{p}.$$

Moreover, the mapping $u \mapsto u(t)$ *is locally Lipschitz from* $\widehat{L}^p$ *to* $C([-T,T] : \widehat{L}^p) \cap L^q([-T,T] : L^{\alpha+1})$.

Furthermore, the solution in Theorem 1 can be extended globally when p is near 2. For $\alpha \in (1, 5)$ we define

$$q_\alpha^{\min} := \begin{cases} \frac{2(\alpha+1)}{\alpha-1} & \text{if} \quad \alpha \in (1,3), \\ 4 & \text{if} \quad \alpha \in [3, \frac{10}{3}], \\ \frac{8\alpha}{10-\alpha} & \text{if} \quad \alpha \in (\frac{10}{3}, \frac{11}{3}], \\ \frac{24}{19}\alpha & \text{if} \quad \alpha \in (\frac{11}{3}, 5). \end{cases}$$

Our global result is the following.

Theorem 2. (i) *Let* $u_0 \in \widehat{L}^p(\mathbb{R})$ *with*

$$2 > p > \begin{cases} \frac{\alpha+1}{\alpha} & \textit{if} \quad 1 < \alpha < \frac{3+\sqrt{57}}{6}, \\ \frac{5\alpha+3}{2(\alpha+2)} & \textit{if} \quad \frac{3+\sqrt{57}}{6} \le \alpha < 3, \\ \frac{9(\alpha-1)}{2(2\alpha-1)} & \textit{if} \quad 3 \le \alpha \le \frac{10}{3}, \\ \frac{(\alpha-1)(3\alpha+5)}{2(\alpha^2-2\alpha+5)} & \textit{if} \quad \frac{10}{3} \le \alpha < 5. \end{cases}$$

Then there exists $\tilde{q} \in (q_\alpha^{\min}, q_{\alpha,p})$ *such that the Cauchy problem* ($\mathbf{NLS}^\alpha$) *has a unique global solution* u *of the form*

$$u = v + w \in L_{loc}^{q_{\alpha,2}}(\mathbb{R} : L^{r_\alpha}(\mathbb{R})) \cap L_{loc}^\infty(\mathbb{R} : L^2(\mathbb{R})) + L_{loc}^{\tilde{q}}(\mathbb{R} : L^{r_\alpha}(\mathbb{R})).$$

(ii) *Let* $u_0 \in \widehat{L}^p(\mathbb{R})$ *with*

$$2 < p < \begin{cases} \alpha+1 & \textit{if} \quad 1 < \alpha \le \frac{1+\sqrt{41}}{4}, \\ \frac{3\alpha+5}{2\alpha} & \textit{if} \quad \frac{1+\sqrt{41}}{4} < \alpha < 5. \end{cases} \tag{6}$$

Then there is a sufficiently large number Q *such that the Cauchy problem* ($\mathbf{NLS}^\alpha$) *has a unique global solution* u *of the form*

$$u = v + w \in L_{loc}^{\frac{4(\alpha+1)}{\alpha-1}}(\mathbb{R} : L^{\alpha+1}(\mathbb{R})) \cap L_{loc}^\infty(\mathbb{R} : L^2(\mathbb{R})) + L_{loc}^Q(\mathbb{R} : L^{\alpha+1}(\mathbb{R})).$$

We give some notation which is used throughout this paper.

Notation. (i) For an arbitrary $a \in [1, \infty]$, a' is the conjugate of a, namely

$$\frac{1}{a} + \frac{1}{a'} = 1.$$

(ii) Let $I \subseteq \mathbb{R}$ be an interval. For the sake of simplicity, we abbreviate $L^q(I : L^r(\mathbb{R}))$ to $L_I^q L_x^r$. In particular, we write $L_{\mathbb{R}}^q L^r = L^q L^r$, when $I = \mathbb{R}$.

(iii) Let f, F be functions on $\mathbb{R}_x$ and $\mathbb{R}_t \times \mathbb{R}_x$, respectively. Denote $\hat{f}, \hat{F}$ by the Fourier transform of f, F with respect to the space variable.

(iv) Define the two complex-valued functions G and $\tilde{G}$ by

$$G(v,w) = |v+w|^{\alpha-1}(v+w) - |v|^{\alpha-1}v, \quad v, w \in \mathbb{C},$$
$$\tilde{G}(v,w_1,w_2) = G(v,w_1) - G(v,w_2), \quad v, w_1, w_2 \in \mathbb{C}.$$

Note that $G(v,w) = \tilde{G}(v,w,0)$.

(v) For any $T > 0$, put $I_T = [0,T]$.

2. Preliminaries

In this section we mainly treat Strichartz type estimates for homogeneous and inhomogeneous equations and their corollaries. We introduce the geometric notation of Kato [7], [8]. Consider the closed unit square R in $\mathbb{R}^2$:

$$R := \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x, y \le 1\}.$$

Then, we introduce some special points in R:

$$O = (0,0),\ B = \big(\frac{1}{2}, 0\big),\ C = \big(0, \frac{1}{4}\big),\ E = \big(0, \frac{1}{2}\big),\ F = \big(\frac{1}{4}, \frac{1}{4}\big),$$
$$O' = \big(1,1\big), B' = \big(\frac{1}{2}, 1\big),\ C' = (1, \frac{3}{4}),\ E' = \big(1, \frac{1}{2}\big),\ F' = \big(\frac{3}{4}, \frac{3}{4}\big),$$

and define the triangles $\widehat{T}_1 := \triangle OBE$ and $\widehat{T}_2 := \triangle O'B'E'$, which are open except that B and B' are included. Also we define $\widehat{T}_3$ as the triangle BFC which is open except that the segment $\{(x, \frac{1}{4}) \,:\, 0 < x < \frac{1}{4}\}$ is included. Obviously, we have

$$BC := \{(x,y) \in R \,:\, (\frac{1}{y}, \frac{1}{x}) \text{ is admissible}\},$$
$$B'C' := \{(x,y) \in R \,:\, (\frac{1}{y'}, \frac{1}{x'}) \text{ is admissible}\}.$$

Interpolating the operator $e^{it\partial_x^2} \circ \mathcal{F}^{-1}$ between O and any point on the admissible line BC, we get

Lemma 3. *Let $p \ge 2$. Let $(\frac{1}{r}, \frac{1}{q})$ lie in the closed triangle $\triangle OBC$ and satisfy*

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{p}.$$

Then, there is a positive constant $C_1 > 0$ such that we have

$$\|e^{it\partial_x^2} f\|_{L^q_{I_T} L^r} \le C_1 \|f\|_{\widehat{L}^p} \tag{7}$$

for any $T > 0$ and $f \in \widehat{L}^p$.

As we noted in the introduction, the inequality (7) still holds true when $q = r$ and $\frac{4}{3} < p < 2$. We then generalize this inequality for pairs with $(\frac{1}{r}, \frac{1}{q}) \in \widehat{T}_3$. This off-diagonal generalization is one of our key inequalities to both our local and global results. We first need the following lemma.

Lemma 4. *Suppose $4 < r < \infty$ and*

$$\frac{1}{2} + \frac{1}{r} = \frac{1}{p}.$$

Then, there is a positive constant $C_1 > 0$ such that we have

$$\|e^{it\partial_x^2} f\|_{L^4_{I_T} L^r} \le C_1 \|f\|_{\widehat{L}^p} \tag{8}$$

for any $T > 0$ and $f \in \widehat{L}^p$.

Proof. We follow the proofs of Lemmas 3.1 and 3.3 in [4], where the Airy version of (8) is shown. By the Sobolev embedding theorem and the Plancherel identity, we have

$$\begin{aligned}\|e^{it\partial_x^2} f\|^4_{L^4_t L^r_x} = \left\| |e^{it\partial_x^2} f|^2 \right\|^2_{L^2_t L^{\frac{r}{2}}_x} &\le C \left\| I^{\varepsilon_r} |e^{it\partial_x^2} f| \right\|^2_{L^2_t L^2_x} \\ &= \left\| |\xi|^{\varepsilon_r} \cdot (\widehat{e^{it\partial_x^2} f}) * (\widehat{\overline{e^{it\partial_x^2} f}}) \right\|^2_{L^2_t L^2_\xi},\end{aligned}$$

where $I^{\varepsilon_r}\varphi$ is defined by $\widehat{(I^{\varepsilon_r}\varphi)}(\xi) := |\xi|^{\varepsilon_r}\widehat{\varphi}(\xi)$ and $\varepsilon_r := \frac{1}{2} - \frac{2}{r}$. The right-hand side of the above inequality equals

$$\begin{aligned}&\int\int dt d\xi |\xi|^{2\varepsilon_r} \left| \int e^{it(\xi-\eta)^2} \widehat{f}(\xi-\eta) e^{-it\eta^2} \overline{\widehat{f}(-\eta)} d\eta \right|^2 \\ &= \int\int dt d\xi |\xi|^{2\varepsilon_r} \left(\int e^{it(\xi-\eta_1)^2} \widehat{f}(\xi-\eta_1) e^{-it\eta_1^2} \overline{\widehat{f}(-\eta_1)} d\eta_1 \right) \\ &\quad \times \overline{\left(\int e^{it(\xi-\eta_2)^2} \widehat{f}(\xi-\eta_2) e^{-it\eta_2^2} \overline{\widehat{f}(-\eta_2)} d\eta_2 \right)} \\ &= \int\int dt d\xi |\xi|^{2\varepsilon_r} \int e^{2it\xi(\eta_2-\eta_1)} \widehat{f}(\xi-\eta_1) \overline{\widehat{f}(-\eta_1)}\, \overline{\widehat{f}(\xi-\eta_2)} \widehat{f}(-\eta_2) d\eta_1 d\eta_2 \\ &= \int d\xi |\xi|^{2\varepsilon_r} \int \delta\left(2\xi(\eta_1-\eta_2)\right) \widehat{f}(\xi-\eta_1) \overline{\widehat{f}(-\eta_1)}\, \overline{\widehat{f}(\xi-\eta_2)} \widehat{f}(-\eta_2) d\eta_1 d\eta_2 \\ &= \int d\xi d\eta_1 |\xi|^{2\varepsilon_r - 1} |\widehat{f}(\xi-\eta_1)|^2 |\widehat{f}(-\eta_1)|^2,\end{aligned}$$

which can be controlled by

$$\| |\widehat{f}|^2 \|_{L^{\frac{p'}{2}}} \cdot \left\| |\widehat{f}|^2 * |\cdot|^{2\varepsilon_r - 1} \right\|_{L^{\frac{p'}{p'-2}}} \le \|\widehat{f}\|^4_{L^{p'}}$$

by Hölder's inequality and the Hardy-Littlewood-Sobolev theorem. This completes the proof. □

Using interpolation between Lemma 4 and the linear Strichartz estimate for L^2-data, we obtain the following inequality.

Theorem 5. *Suppose that $(\frac{1}{r}, \frac{1}{q}) \in \widehat{T}_3$. Then there is a positive constant $C_1 > 0$ such that we have*

$$\|e^{it\partial_x^2} f\|_{L^q_{I_T} L^r} \le C_1 \|f\|_{\widehat{L}^p} \tag{9}$$

for any $T > 0$ and $f \in \widehat{L}^p$.

Another key estimate is the inhomogeneous Strichartz estimate for Hölder exponents which are not necessarily admissible.

Theorem 6. *Suppose that* $(\frac{1}{r}, \frac{1}{q}) \in \widehat{T}_1$, $(\frac{1}{\rho}, \frac{1}{\gamma}) \in \widehat{T}_2$ *and*

$$\frac{2}{\gamma} + \frac{1}{\rho} = 2 + \frac{2}{q} + \frac{1}{r}. \tag{10}$$

Then, there is a constant $C_2 > 0$ *such that the estimate*

$$\left\| \int_0^t e^{i(t-\tau)\partial_x^2} F(\cdot,\tau) d\tau \right\|_{L^q_{I_T} L^r_x} \leq C_2 \|F\|_{L^\gamma_{I_T} L^\rho_x} \tag{11}$$

is valid for any $T > 0$ *and* $F \in L^\gamma_{I_T} L^\rho_x$.

Proof. See [7, Theorem 2.1]. □

Proposition 7. *We get the following inequalities:*

(i) *Let* $q > \max\left(\frac{2(\alpha+1)}{\alpha-1}, \frac{2(\alpha+1)(\alpha-1)}{\alpha+3}, \alpha-1\right)$. *Then, there exists* $C_3 > 0$ *such that for any* $T > 0$ *we have*

$$\begin{aligned} &\left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w_1, w_2)(\cdot,\tau) d\tau \right\|_{L^q_{I_T} L^{\alpha+1}} \\ &\leq C_3 \Bigg[T^{\frac{5-\alpha}{4}} \|v\|^{\alpha-1}_{L^{\frac{4(\alpha+1)}{\alpha-1}}_{I_T} L^{\alpha+1}} \|w_1 - w_2\|_{L^q_{I_T} L^{\alpha+1}} \\ &\quad + T^{1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{q}} \left\{ \|w_1\|^{\alpha-1}_{L^q_{I_T} L^{\alpha+1}} + \|w_2\|^{\alpha-1}_{L^q_{I_T} L^{\alpha+1}} \right\} \times \|w_1 - w_2\|_{L^q_{I_T} L^{\alpha+1}} \Bigg] \end{aligned} \tag{12}$$

for all $v \in L^{\frac{4(\alpha+1)}{\alpha-1}}_{I_T} L^{\alpha+1}$, $w_1, w_2 \in L^q_{I_T} L^{\alpha+1}$.

(ii) *Let* $3 \leq \alpha \leq \frac{11}{3}$ *and* $q > \max\{\frac{8}{\alpha-1}, \frac{8(\alpha-1)}{9-\alpha}\}$. *Then, there exists* $C_3 > 0$ *such that for any* $T > 0$ *we have*

$$\begin{aligned} \left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w_1, w_2)(\cdot,\tau) d\tau \right\|_{L^q_{I_T} L^4} &\leq C_3 \Bigg[T^{\frac{5-\alpha}{4}} \|v\|^{\alpha-1}_{L^8_{I_T} L^4} \|w_1 - w_2\|_{L^q_{I_T} L^4} \\ &+ \ T^{\frac{9-\alpha}{8}-\frac{\alpha-1}{q}} \left\{ \|w_1\|^{\alpha-1}_{L^q_{I_T} L^4} + \|w_2\|^{\alpha-1}_{L^q_{I_T} L^4} \right\} \times \|w_1 - w_2\|_{L^q_{I_T} L^4} \Bigg] \end{aligned} \tag{13}$$

for all $v \in L^8_{I_T} L^4$, $w_1, w_2 \in L^q_{I_T} L^4$.

(iii) *Let* $\alpha > \frac{11}{3}$ *and* $q > \max\left(\frac{24\alpha}{11(\alpha-1)}, \frac{24\alpha(\alpha-1)}{13\alpha+11}\right)$. *Then, there exists* $C_3 > 0$ *such that for any* $T > 0$ *we have*

$$\begin{aligned} &\left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w_1, w_2)(\cdot,\tau) d\tau \right\|_{L^q_{I_T} L^{\frac{12}{11}\alpha}} \\ &\leq C_3 \Bigg[T^{\frac{5-\alpha}{4}} \|v\|^{\alpha-1}_{L^{\frac{24\alpha}{6\alpha-11}}_{I_T} L^{\frac{12}{11}\alpha}} \|w_1 - w_2\|_{L^q_{I_T} L^{\frac{12}{11}\alpha}} \\ &\quad + T^{\frac{13\alpha+11}{24\alpha}-\frac{\alpha-1}{q}} \left\{ \|w_1\|^{\alpha-1}_{L^q_{I_T} L^{\frac{12}{11}\alpha}} + \|w_2\|^{\alpha-1}_{L^q_{I_T} L^{\frac{12}{11}\alpha}} \right\} \times \|w_1 - w_2\|_{L^q_{I_T} L^{\frac{12}{11}\alpha}} \Bigg] \end{aligned} \tag{14}$$

for all $v \in L_{I_T}^{\frac{24\alpha}{6\alpha-11}} L^{\frac{12}{11}\alpha}$, $w_1, w_2 \in L_{I_T}^q L^{\frac{12}{11}\alpha}$.

Proof. All the inequalities are proven in the same way. We first use Theorem 6 to obtain

$$\left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w_1, w_2)(\cdot, \tau) d\tau \right\|_{L_{I_T}^q L^{\alpha+1}} \leq C \|\tilde{G}(v, w_1, w_2)\|_{L_{I_T}^A L^{\frac{\alpha+1}{\alpha}}}$$

for $1 < \alpha < 5$ and

$$\left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w_1, w_2)(\cdot, \tau) d\tau \right\|_{L_{I_T}^q L^{r_\alpha}} \leq C \|\tilde{G}(v, w_1, w_2)\|_{L_{I_T}^B L^{\frac{r_\alpha}{\alpha}}}$$

for $\alpha \in [\frac{11}{3}, 5)$, where

$$A = \left(1 + \frac{1}{q} - \frac{\alpha - 1}{2(\alpha+1)}\right)^{-1}, \quad B = \left(1 + \frac{1}{q} - \frac{\alpha-1}{2r_\alpha}\right)^{-1}.$$

Then, by the inequality

$$|\tilde{G}(v, w_1, w_2)| \lesssim (|v|^{\alpha-1} + |w_1|^{\alpha-1} + |w_2|^{\alpha-1})|w_1 - w_2| \tag{15}$$

and Hölder's inequality, we get the inequalities (12)-(14). One can easily check that the above procedure is justified by carefully checking that the indices in the calculation are in an appropriate range under the assumptions of the proposition. □

To obtain global results we need the following.

Proposition 8. (i) *Let* $q > \frac{4\alpha(\alpha+1)}{3\alpha+5}$. *Then, there exists* $C_3 > 0$ *such that for any* $T > 0$ *we have*

$$\begin{aligned} &\left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w, 0)(\cdot, \tau) d\tau \right\|_{L_{I_T}^\infty L^2} \\ &\leq C_3 \left[T^{\frac{5-\alpha}{4} + \frac{\alpha-1}{4(\alpha+1)} - \frac{1}{q}} \|v\|_{L_{I_T}^{\frac{4(\alpha+1)}{\alpha-1}} L^{\alpha+1}}^{\alpha-1} \|w\|_{L_{I_T}^q L^{\alpha+1}} + T^{\frac{3\alpha+5}{4(\alpha+1)} - \frac{\alpha}{q}} \|w\|_{L_{I_T}^q L^{\alpha+1}}^\alpha \right] \end{aligned} \tag{16}$$

for all $v \in L_{I_T}^{\frac{4(\alpha+1)}{\alpha-1}} L^{\alpha+1}$ *and* $w \in L_{I_T}^q L^{\alpha+1}$.

(ii) *Let* $3 \leq \alpha < \frac{11}{3}$ *and* $q > \frac{8\alpha}{10-\alpha}$. *Then, there exists* $C_3 > 0$ *such that for any* $T > 0$ *we have*

$$\begin{aligned} \left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w, 0)(\cdot, \tau) d\tau \right\|_{L_{I_T}^\infty L^2} \leq C_3 \Big[& T^{\frac{11-2\alpha}{8} - \frac{1}{q}} \|v\|_{L_{I_T}^8 L^4}^{\alpha-1} \|w\|_{L_{I_T}^q L^4} \\ & + T^{\frac{10-\alpha}{8} - \frac{\alpha}{q}} \|w\|_{L_{I_T}^q L^4}^\alpha \Big] \end{aligned} \tag{17}$$

for all $v \in L_{I_T}^8 L^4$ *and* $w \in L_{I_T}^q L^4$.

(iii) *Let* $\alpha > \frac{11}{3}$ *and* $q > \frac{24\alpha}{19}$. *Then, there exists* $C_3 > 0$ *such that for any* $T > 0$ *we have*

$$\left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v,w,0)(\cdot,\tau)d\tau \right\|_{L^\infty_{I_T} L^2} \leq C_3 \left[T^{\frac{-6\alpha^2+36\alpha-11}{24\alpha} - \frac{1}{q}} \|v\|^{\alpha-1}_{L^{\frac{24\alpha}{6\alpha-11}}_{I_T} L^{\frac{12}{11}\alpha}} \|w\|_{L^q_{I_T} L^{\frac{12}{11}\alpha}} + T^{\frac{19}{24}-\frac{\alpha}{q}} \|w\|^{\alpha}_{L^q_{I_T} L^{\frac{12}{11}\alpha}} \right] \tag{18}$$

for all $v \in L^{\frac{24\alpha}{6\alpha-11}}_{I_T} L^{\frac{12}{11}\alpha}$ *and* $w \in L^q_{I_T} L^{\frac{12}{11}\alpha}$.

Proposition 9. *Suppose that* $v : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ *with* $\varphi(\cdot) := v(0,\cdot) \in L^2_x$ *solves*

$$iv_t + v_{xx} \pm |v|^{\alpha-1} v = 0$$

and that (q,r) *is an admissible pair. Then, there are positive constants* K_1, K_2 *depending only on* α *such that*

$$\|v\|_{L^q_{I_\delta} L^r} \leq K_1 \|\varphi\|_{L^2_x} \tag{19}$$

for any $\delta \in [0, (K_2\|\varphi\|_{L^2_x})^{-\frac{4(\alpha-1)}{5-\alpha}}]$, *provided that*

(i) $1 < \alpha < 5$ *and* $q = \frac{4(\alpha+1)}{\alpha-1}$, $r = \alpha + 1$,

(ii) $3 < \alpha < 5$ *and* $q = q_{\alpha,2}$, $r = r_\alpha$.

Proof. See [6], Lemma 2.4. □

We close the section with a useful result from interpolation theory.

Lemma 10. *Suppose that* $0 < q_0 < q_1 \leq \infty$, $0 < \theta < 1$ *and* $q > 0$ *is defined by*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then,

$$L^q \subset L^{q_0} + L^{q_1}.$$

Furthermore, for any $f \in L^q$ *and for any* $t > 0$ *one can find* $f_0^t \in L^{q_0}$ *and* $f_1^t \in L^{q_1}$ *with* $f = f_0^t + f_1^t$ *satisfying*

$$ct^{-\theta} \max\{\|f_0^t\|_{L^{q_0}}, t\|f_1^t\|_{L^{q_1}}\} \leq \|f\|_{L^q} \tag{20}$$

for some positive constant c *depending only on* q_0, q_1.

For a proof, see [9],[6].

3. Proof of Theorem 1

Consider the integral equation

$$u(t) = e^{it\partial_x^2} u_0 \pm i \int_0^t e^{i(t-\tau)\partial_x^2} (|u|^{\alpha-1}u) d\tau \tag{21}$$

and put

$$\mathcal{V}^{q,r} := C([0,T] : \widehat{L}^p) \cap L^q_{[0,T]} L^r \tag{22}$$

with $\frac{2}{q} + \frac{1}{r} = \frac{1}{p}$. It is enough to show that the $\mathcal{V}^{q,r}$-norm of the right-hand side of (21) can be controlled by

$$\|u_0\|_{\widehat{L}^p} + T^\beta \|u\|^\alpha_{L^q_{[0,T]} L^r}$$

for some $\beta > 0$ and (q, r) (see Proposition 1.1 and a sketch of its proof in [4]).

We first prove (i). Let $2 > p > \max(\frac{4}{3}, \frac{\alpha-1}{2}, \frac{\alpha+1}{\alpha})$ and $q = q_{\alpha,p}, r = r_\alpha$. Then the $\mathcal{V}^{q,r}$-norm of the linear part is bounded from above by $\|u_0\|_{\widehat{L}^p}$ by the trivial equality $\|e^{it\partial_x^2}u_0\|_{\widehat{L}^p} = \|u_0\|_{\widehat{L}^p}$ and the generalized Fefferman-Stein inequality since $(\frac{1}{r_\alpha}, \frac{1}{q_{\alpha,p}}) \in \widehat{T}_3$. For the nonlinear part, it is not difficult to check that $q_{\alpha,p}$ satisfies the assumption of Proposition 7. Thus we have

$$\left\| \int_0^t e^{i(t-\tau)\partial_x^2}(|u|^{\alpha-1}u)d\tau \right\|_{L^{q_{\alpha,p}}_{[0,T]}L^{r_\alpha}} \le CT^\beta \|u\|^\alpha_{L^{q_{\alpha,p}}_{[0,T]}L^{r_\alpha}},$$

where β is the exponent of T in the second term on the right-hand side of (12)-(14). Moreover, by the duality of Lemma 3, we have

$$\sup_{t\in I_T} \left\| \int_0^t e^{i(t-\tau)\partial_x^2}(|u|^{\alpha-1}u)d\tau \right\|_{\widehat{L}^p} \le C\|u^\alpha\|_{L^B_{I_T}L^{\frac{r_\alpha}{\alpha}}}, \quad B^{-1} = 1 + \frac{1}{q_{\alpha,p}} - \frac{\alpha-1}{2r_\alpha},$$

since $\frac{2}{B'} + \frac{1}{(\frac{r_\alpha}{\alpha})'} = \frac{1}{p'}$. Therefore, the $C([0,T] : \widehat{L}^p)$-norm of the nonlinear part is smaller than $CT^\beta\|u\|^\alpha_{L^{q_{\alpha,p}}_{[0,T]}L^{r_\alpha}}$ by Hölder's inequality. This completes the proof of (i).

The proof of (ii) proceeds similarly. We put $r = \alpha + 1$ for any α. Obviously, we have $\|e^{it\partial_x^2}u_0\|_{\mathcal{V}^{q,\alpha+1}} \le C\|u_0\|_{\widehat{L}^p}$ by Lemma 3. For the $L^q_{[0,T]}L^r$-norm of the nonlinear part we get an upper bound $T^{1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{q}}\|u\|^\alpha_{L^q_{I_T}L^{\alpha+1}}$ in the same way. Finally, the duality of (9) yields

(23)
$$\sup_{t\in I_T} \left\| \int_0^t e^{i(t-\tau)\partial_x^2}(|u|^{\alpha-1}u)d\tau \right\|_{\widehat{L}^p} \le C\|u^\alpha\|_{L^A_{I_T}L^{\frac{\alpha+1}{\alpha}}}, \quad A^{-1} = 1 + \frac{1}{q} - \frac{\alpha-1}{2(\alpha+1)},$$

since $\frac{2}{A'} + \frac{1}{(\frac{\alpha+1}{\alpha})'} = \frac{1}{p'}$ and the assumption $\frac{1}{2} > \frac{1}{p} = \frac{2}{q} + \frac{1}{\alpha+1} > \max(\frac{\alpha-1}{2(\alpha+1)}, \frac{1}{\alpha+1})$ implies $(\frac{1}{\alpha+1}, \frac{1}{A'}) \in \widehat{T}_3$. Hence we once again obtain an upper bound

$$T^{1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{q}}\|u\|^\alpha_{L^q_{[0,T]}L^{\alpha+1}}$$

for (23), which completes the proof of (ii). □

4. Proof of Theorem 2

In this section, we give the proof of our global result. We only prove the case of the plus sign in ($\mathbf{NLS}^\alpha$) and construct a solution in $t > 0$. We mainly prove (ii); the assertion (i) can be proven in a similar manner (see the last part of the section). In what follows we assume that the initial data u_0 satisfies $u_0 \in \widehat{L}^p(\mathbb{R}) \setminus L^2(\mathbb{R})$. Let p_0 be such that $p < p_0 < \alpha + 1$. We write

$$\text{(24)} \qquad \frac{1}{p'} = \frac{1}{p'_\theta} := \frac{1-\theta}{p'_0} + \frac{\theta}{2}, \quad 0 < \theta < 1.$$

4.1. **Interpolation of u between L^2 and $\widehat{L}^{p_0}$.** Our first step of the proof is to decompose the initial data u_0 in $L^2 + \widehat{L}^{p_0}$ so that the size of the $\widehat{L}^{p_0}$-data can be controlled by any arbitrary small quantity:

Step 1. There exist a positive constant C_0 and sequences $(\varphi_N)_{N>1} \subseteq L^2_x$ and $(\psi_N)_{N>1} \subseteq \widehat{L}^{p_0}_x$ indexed by $(1,\infty)$ such that for any $N > 1$ we have

$$u_0 = \varphi_N + \psi_N \tag{25}$$

and

$$C_0^{-1} N^{\frac{1-\theta}{\theta}} \le \|\varphi_N\|_{L^2} \le C_0 N^{\frac{1-\theta}{\theta}}, \quad \|\psi_N\|_{\widehat{L}^{p_0}} \le C_0 \frac{1}{N}. \tag{26}$$

Proof of Step 1. By the assumption, $\hat{u}_0 \in L^{p'_\theta}$. We apply Lemma 10 with $f = \hat{u}_0, q_0 = p'_0, q_1 = 2$. Then, there exist c_0, $(v_0^t)_{t>0} \subseteq L^2$ and $(w_0^t)_{t>0} \subseteq L^{p'_0}$ such that $\hat{u}_0 = v_0^t + w_0^t$ and

$$ct^{-\theta} \max\{\|w_0^t\|_{L^{p'_0}}, t\|v_0^t\|_{L^2}\} \le \|\hat{u}_0\|_{L^{p'_\theta}} \tag{27}$$

for all $t > 0$. Note that $\|v_0^t\|_{L^2} \to \infty$ as $t \to 0$, since $\hat{u}_0 \notin L^2$. Therefore, we can choose $t_N > 0$ for any $N > 1$ such that

$$\|v_0^{t_N}\|_{L^2} = N^{\frac{1-\theta}{\theta}}.$$

Let us define

$$\varphi'_N := v_0^{t_N}, \ \psi'_N := w_0^{t_N}.$$

Then, we get by (20),

$$t_N^{1-\theta} \le \|\hat{u}_0\|_{L^{p'_\theta}} N^{-\frac{1-\theta}{\theta}}. \tag{28}$$

Thus, by (20) again, we have

$$\|\psi'_N\|_{L^{p'_0}} \le t_N^\theta \|\hat{u}_0\|_{L^{p'_\theta}} \le \|\hat{u}_0\|_{L^{p'_\theta}}^{\frac{1}{1-\theta}} \cdot N^{-1}. \tag{29}$$

Now, if we put

$$\widehat{\varphi}_N(\xi) := \varphi'_N(\xi), \ \widehat{\psi}_N(\xi) := \psi'_N(\xi),$$

we get the wanted decomposition. □

4.2. **Local existence to a time $T \sim N^{-\frac{4(\alpha-1)}{5-\alpha}\cdot\frac{1-\theta}{\theta}}$.** Let Q be such that

$$\frac{2}{Q} + \frac{1}{\alpha+1} = \frac{1}{p_0}.$$

Moreover, we introduce some constants. We first fix $M > 1$ satisfying

$$M > \max\{K_2, (3^{\frac{1}{\alpha-1}} C_3^{\frac{1}{\alpha-1}} K_1)^{\frac{5-\alpha}{4}}, 2^{-1} \cdot 4^{\frac{\alpha}{\sigma_\alpha}} C_0^{\frac{\alpha-1}{\sigma_\alpha}-1} C_1^{\frac{\alpha-1}{\sigma_\alpha}} C_3^{\frac{1}{\sigma_\alpha}}\},$$

where $C_0, C_1, C_3, K_i \, (i = 1,2)$ are constants in (26), (7), (12), and Proposition 9 respectively and

$$\sigma_\alpha := \frac{4(\alpha-1)}{5-\alpha}\left(1 - \frac{\alpha-1}{2(\alpha+1)} - \frac{\alpha-1}{Q}\right) > 0. \tag{30}$$

Now for $N > 1$ we define δ_N by

$$\delta_N := \left(M(2C_0 N^{\frac{1-\theta}{\theta}})\right)^{-\frac{4(\alpha-1)}{5-\alpha}}.$$

The purpose of Step 2 is to prove the existence of a solution on $[0, \delta_N]$.

Step 2. For any $N > 1$ there exists a unique solution u of $(\mathbf{NLS}^\alpha)$ of the form

$$u = v + w \in L_{I_{\delta_N}}^{\frac{4(\alpha+1)}{\alpha-1}} L^{\alpha+1} + L_{I_{\delta_N}}^{Q} L^{\alpha+1}.$$

Moreover, there is $C > 0$ such that we have

$$\|w(t) - e^{it\partial_x^2}\psi_0^N\|_{L_x^2} \leq CN^{-1+\{-\frac{(\alpha-1)^2}{(5-\alpha)(\alpha+1)}+\frac{4(\alpha-1)}{(5-\alpha)Q}\}\frac{1-\theta}{\theta}} \quad \text{uniformly in } N \tag{31}$$

for any $t \in [0, \delta_N]$.

Proof of Step 2. Fix $N > 1$. Instead of (26), we construct a solution of $(\mathbf{NLS}^\alpha)$ under the weaker assumption

$$(2C_0)^{-1}N^{\frac{1-\theta}{\theta}} \leq \|\varphi_0^N\|_{L_x^2} \leq (2C_0)N^{\frac{1-\theta}{\theta}}, \quad \|\hat{\psi}_N\|_{L^{p_0'}} \leq C_0\frac{1}{N} \tag{32}$$

for later use in the next step. The solution u can be expressed as $u = w + v$, where v, w satisfy

$$\begin{cases} iv_t + v_{xx} + |v|^{\alpha-1}v = 0, & t \in \mathbb{R}, \\ v(0,x) = \varphi_0^N(x), & x \in \mathbb{R}, \end{cases} \tag{33}$$

and

$$\begin{cases} iw_t + w_{xx} + \tilde{G}(v,w,0) = 0, & t \in \mathbb{R}, \\ w(0,x) = \psi_0^N(x), & x \in \mathbb{R}. \end{cases} \tag{34}$$

It is well-known that v is defined globally (see [10]) and

$$\|v(t,\cdot)\|_{L_x^2} = \|\varphi_0^N\|_{L_x^2}, \tag{35}$$

for all $t \geq 0$. Therefore, we will prove that w is well-defined on $[0, \delta_N]$. Since $M > K_2$, we have

$$\|v\|_{L_{I_{\delta_N}}^{\frac{4(\alpha+1)}{\alpha-1}} L^{\alpha+1}} \leq K_1\|\varphi_0^N\|_{L_x^2} \tag{36}$$

by Proposition 9. We use the fixed point theorem in the complete metric space $\mathcal{V}_N$ defined by

$$\mathcal{V}_N := \left\{ w \in L_{I_{\delta_N}}^Q L^{\alpha+1} \,:\, \|w\|_{L_{I_{\delta_N}}^Q L^{\alpha+1}} \leq \frac{3C_0C_1}{N} \right\}, \tag{37}$$

where C_1 is the constant in (7). We prove:

Claim. Define

$$(Tw)(t) := e^{it\partial_x^2}\psi_0^N + i\int_0^t e^{i(t-\tau)\partial_x^2}\tilde{G}(v,w,0)d\tau.$$

Then, the operator $T : \mathcal{V}_N \to \mathcal{V}_N$ is well-defined and is a contraction mapping.

□

Proof of Claim. Let $w \in L^Q_{I_{\delta_N}} L^{\alpha+1}$. Then, by (12) we have

$$\begin{aligned}\|Tw\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} &\leq \|e^{it\partial_x^2}\psi_0^N\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} + \left\| \int_0^t e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w, 0) d\tau \right\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \\ &\leq \|e^{it\partial_x^2}\psi_0^N\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} + C_3(\delta_N)^{\frac{5-\alpha}{4}} \|v\|^{\alpha-1}_{L^{\frac{4(\alpha+1)}{\alpha-1}}_{I_{\delta_N}} L^{\alpha+1}} \|w\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \\ &\quad + C_3(\delta_N)^{1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{Q}} \|w\|^{\alpha}_{L^Q_{I_{\delta_N}} L^{\alpha+1}}.\end{aligned}$$

For the first term of the right-hand side we have

$$\|e^{it\partial_x^2}\psi_0^N\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \leq C_1 \|\widehat{\psi_0^N}\|_{L^{p_0'}} \leq \frac{C_0C_1}{N}$$

using (7) and (26) .

The second term is bounded above by

$$\begin{aligned}& C_3 \left(\{M(2C_0N^{\frac{1-\theta}{\theta}})\}^{-\frac{4(\alpha-1)}{5-\alpha}} \right)^{\frac{5-\alpha}{4}} \times \left(K_1\|\varphi_0^N\|\right)^{\alpha-1} \times \frac{3C_0C_1}{N} \\ \leq\ & 3C_3K_1^{\alpha-1}M^{\frac{4(\alpha-1)}{5-\alpha}} \times \frac{C_0C_1}{N} \\ \leq\ & \frac{C_0C_1}{N},\end{aligned}$$

where the last inequality follows from the definition of M.

Similarly, we have

$$\begin{aligned}\text{The third term} &\leq C_3 \left\{ M(2C_0N^{\frac{1-\theta}{\theta}}) \right\}^{-\frac{4(\alpha-1)}{5-\alpha}\cdot(1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{Q})} \times \left(\frac{3C_0C_1}{N}\right)^{\alpha} \\ &\leq C_3M^{-\sigma_\alpha}(2C_0)^{-\sigma_\alpha} \cdot 3^\alpha C_0^{\alpha-1}C_1^{\alpha-1} \times \frac{C_0C_1}{N} \\ &\leq \frac{C_0C_1}{N}.\end{aligned}$$

Therefore, we obtain

$$\|Tw\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \leq \frac{C_0C_1}{N} + \frac{C_0C_1}{N} + \frac{C_0C_1}{N} \leq \frac{3C_0C_1}{N},$$

which implies $Tw \in \mathcal{V}_N$.

In a similar manner, we can prove that T is a contraction mapping. Let $w_1, w_2 \in \mathcal{V}_N$. Then, we have

$$\begin{aligned}
\|Tw_1 - Tw_2\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \le C_3 \Bigg[&(\delta_N)^{\frac{5-\alpha}{4}} \|v\|^{\alpha-1}_{L^{\frac{4(\alpha+1)}{\alpha-1}}_{I_{\delta_N}} L^{\alpha+1}} \\
&+ (\delta_N)^{1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{Q}} \|w_1\|^{\alpha-1}_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \\
&+ (\delta_N)^{1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{Q}} \|w_2\|^{\alpha-1}_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \Bigg] \times \|w_1 - w_2\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \\
\le C_3 \Bigg[&K_1^{\alpha-1} M^{-\frac{4(\alpha-1)}{5-\alpha}} + 2 \\
&\times \{M(2C_0 N^{\frac{1-\theta}{\theta}})\}^{-\frac{4(\alpha-1)}{5-\alpha}(1-\frac{\alpha-1}{2(\alpha+1)}-\frac{\alpha-1}{Q})} \\
&\times \left(\frac{3C_0C_1}{N}\right)^{\alpha-1} \Bigg] \times \|w_1 - w_2\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} \\
\le &\left(\frac{1}{3} + 2 \times \frac{1}{4}\right) \|w_1 - w_2\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}} = \frac{5}{6} \|w_1 - w_2\|_{L^Q_{I_{\delta_N}} L^{\alpha+1}}.
\end{aligned}$$

Thus, the claim is proved and we have a solution of ($\mathbf{NLS}^\alpha$) on $[0, \delta_N]$ by using Duhamel's principle and the fixed point theorem. One can also prove (31) in a similar way, using (16) instead of (12). □

4.3. **Global existence.**

Step 3. One can extend $u(t)$, $t \in [0, \delta_N]$ obtained in the previous step to the time T_N defined by

$$T_N := ck_0\delta_N = cN^{1-\left\{\frac{2(2\alpha^2-\alpha-5)}{(5-\alpha)(\alpha+1)}+\frac{4(\alpha-1)}{(5-\alpha)Q}\right\}\cdot\frac{1-\theta}{\theta}}, \tag{38}$$

where $c > 0$ is a constant independent of N. Moreover, if α, p satisfy (6), u can be extended globally by taking Q sufficiently large.

Proof of Step 3. We write

$$u(\delta_N, x) = \varphi_1^N(x) + \psi_1^N(x) \tag{39}$$

with

$$\varphi_1^N(x) := v(\delta_N, x) + \int_0^{\delta_N} e^{i(t-\tau)\partial_x^2} \tilde{G}(v, w, 0)(\cdot, \tau) d\tau \tag{40}$$

and

$$\psi_1^N(x) := e^{i\delta_N \partial_x^2} \psi_0^N. \tag{41}$$

Obviously, we have

$$\|\widehat{\psi_1^N}\|_{L^{p_0'}} = \|e^{i\delta_N \xi^2} \widehat{\psi_0^N}(\xi)\|_{L^{p_0'}_\xi} = \|\widehat{\psi_0^N}\|_{L^{p_0'}} \le \frac{C_0}{N}.$$

Therefore, if the estimate

$$(2C_0)^{-1} N^{\frac{1-\theta}{\theta}} \le \|\varphi_1^N\|_{L^2_x} \le (2C_0) N^{\frac{1-\theta}{\theta}} \tag{42}$$

is valid, we get the decomposition of $u(\delta_N,\cdot)$ as in Step 1, and then we obtain the solution of

$$\begin{cases} iu_t + u_{xx} + |u|^{\alpha-1}u = 0, & t > \delta_N, \\ u(\delta_N, x) = \varphi_1^N(x) + \psi_1^N(x), & x \in \mathbb{R}, \end{cases}$$

on $[\delta_N, 2\delta_N]$, repeating the calculation of Step 2. In this way, we can construct a solution of ($\mathbf{NLS}^\alpha$) to the time $2\delta_N, 3\delta_N, \cdots, k_0\delta_N$ inductively as far as

$$(2C_0)^{-1}N^{\frac{1-\theta}{\theta}} \le \|\varphi_k^N\|_{L_x^2} \le (2C_0)N^{\frac{1-\theta}{\theta}}, \quad \forall k \le k_0, \tag{43}$$

where

$$\varphi_k^N(x) := v(k\delta_N, x) + \int_0^{k\delta_N} e^{i(t-\tau)\partial_x^2}\tilde{G}(v,w,0)(\cdot,\tau)d\tau$$

and

$$\psi_k^N(x) := e^{ik\delta_N\partial_x^2}\psi_0^N.$$

We seek the largest k_0 for which (43) holds. By (31) and the conservation law (35), if

$$C_0N^{\frac{1-\theta}{\theta}} + Ck_0N^{-1+\{-\frac{(\alpha-1)^2}{(5-\alpha)(\alpha+1)}+\frac{4(\alpha-1)}{(5-\alpha)Q}\}\frac{1-\theta}{\theta}} \le 2C_0N^{\frac{1-\theta}{\theta}} \tag{44}$$

and

$$C_0^{-1}N^{\frac{1-\theta}{\theta}} - Ck_0N^{-1+\{-\frac{(\alpha-1)^2}{(5-\alpha)(\alpha+1)}+\frac{4(\alpha-1)}{(5-\alpha)Q}\}\frac{1-\theta}{\theta}} \ge (2C_0)^{-1}N^{\frac{1-\theta}{\theta}}, \tag{45}$$

we obtain (43). Solving (44), (45), we have

$$k_0 \le CN^{1+\{\frac{2(\alpha+3)}{(5-\alpha)(\alpha+1)}-\frac{4(\alpha-1)}{(5-\alpha)Q}\}\frac{1-\theta}{\theta}}.$$

So, we can extend the solution of ($\mathbf{NLS}^\alpha$) to the time

$$T_N := ck_0\delta_N = cN^{1-\left\{\frac{2(2\alpha^2-\alpha-5)}{(5-\alpha)(\alpha+1)}+\frac{4(\alpha-1)}{(5-\alpha)Q}\right\}\cdot\frac{1-\theta}{\theta}}.$$

Note that N can be taken arbitrarily large, which means that if the exponent of the right-hand side of (46) is strictly positive, T_N also can be arbitrarily large. Therefore, if

$$1 - \left\{\frac{2(2\alpha^2-\alpha-5)}{(5-\alpha)(\alpha+1)} + \frac{4(\alpha-1)}{(5-\alpha)Q}\right\}\cdot\frac{1-\theta}{\theta} > 0, \tag{46}$$

we extend a solution of ($\mathbf{NLS}^\alpha$) globally. Since Q can be a sufficiently large number (accordingly, p_0 becomes close to $\alpha+1$), if we solve

$$1 - \left\{\frac{2(2\alpha^2-\alpha-5)}{(5-\alpha)(\alpha+1)}\right\}\cdot\frac{1-\theta}{\theta} > 0, \tag{47}$$

and (24) with $p_0 = \alpha + 1$, we get the assertion of Theorem 2 (ii). □

Remark 11. Noting that $L^Q_{[-T,T]}L^{\alpha+1} \subseteq L^{\frac{4(\alpha+1)}{\alpha-1}}_{[-T,T]}L^{\alpha+1}$ when Q is large, we can easily prove the uniqueness assertion using Proposition 7.

Sketch of Proof of (i). By interpolating u_0 between $(\frac{1}{r_\alpha}, \frac{1}{q_{\alpha,2}}) \in \overline{BC}$ and $(\frac{1}{r_\alpha}, \frac{1}{q_\alpha^{\min}})$, we can prove the assertion (i) in the similar manner: Let $u_0 \in \widehat{L}^p$ with $q_\alpha^{\min} < q_{\alpha,p} < q_{\alpha,2}$. We take $\tilde{q}$ so that $q_\alpha^{\min} < \tilde{q} < q_{\alpha,p}$ and write

$$\frac{1}{p'} = \frac{1-\theta}{\tilde{p}'} + \frac{\theta}{2}, \quad 0 < \theta < 1, \tag{48}$$

where $\tilde{p}$ is defined by $\frac{2}{\tilde{q}} + \frac{1}{r_\alpha} = \frac{1}{\tilde{p}}$. We can split the initial data in the same way as the proof of Step 1:

$$u_0 = \varphi_N + \psi_N, \quad \text{with} \quad \|\varphi_N\|_{L^2} \sim N^{\frac{1-\theta}{\theta}}, \quad \|\psi_N\|_{\widehat{L}^{\tilde{p}}} \lesssim N^{-1}, \quad N > 1. \tag{49}$$

Then we can construct a local solution to the Cauchy problem to a time $T \sim N^{-\frac{4(\alpha-1)}{5-\alpha)}\cdot\frac{1-\theta}{\theta}}$:

Step $2'$. For any $N > 1$ there exists a unique solution u of ($\mathbf{NLS}^\alpha$) of the form

$$u = v + w \in L^{q_\alpha,2}_{I_{\delta_N}} L^{r_\alpha} + L^{\tilde{q}}_{I_{\delta_N}} L^{r_\alpha}.$$

Moreover, there is $C > 0$ such that we have

$$\|w(t) - e^{it\partial_x^2}\psi_0^N\|_{L^2_x} \le \begin{cases} CN^{-1+(\frac{4}{\tilde{q}}-\frac{\alpha-1}{\alpha+1})\cdot\frac{\alpha-1}{5-\alpha}\cdot\frac{1-\theta}{\theta}} & \text{when} \quad \alpha \in (1,3), \\ CN^{-1+(\frac{4}{\tilde{q}}-\frac{1}{2})\cdot\frac{\alpha-1}{5-\alpha}\cdot\frac{1-\theta}{\theta}} & \text{when} \quad \alpha \in [3, \frac{11}{3}], \\ CN^{-1+(\frac{4}{\tilde{q}}+\frac{11}{6\alpha}-1)\cdot\frac{\alpha-1}{5-\alpha}\cdot\frac{1-\theta}{\theta}} & \text{when} \quad \alpha \in [\frac{11}{3}, 5) \end{cases} \tag{50}$$

for any $t \in [0, \delta_N]$ (note that we can apply Proposition 8, since $\tilde{q} > q_\alpha^{\min}$). Then, proceeding as in the proof of Step 3, we can extend the solution in Step $2'$ up to the time T_N which is defined by

$$T_N := cN^{\beta(\alpha,\theta)} := \begin{cases} cN^{1+(\frac{2(-2\alpha^2+\alpha+5)}{\alpha+1}-\frac{4(\alpha-1)}{\tilde{q}})\cdot\frac{1}{5-\alpha}\cdot\frac{1-\theta}{\theta}} & \text{when} \quad \alpha \in (1,3), \\ cN^{1+(\frac{17-9\alpha}{2}-\frac{4(\alpha-1)}{\tilde{q}})\cdot\frac{1}{5-\alpha}\cdot\frac{1-\theta}{\theta}} & \text{when} \quad \alpha \in [3, \frac{11}{3}], \\ cN^{1+(\frac{-24\alpha^2+37\alpha+11}{6\alpha}-\frac{4(\alpha-1)}{\tilde{q}})\cdot\frac{1}{5-\alpha}\cdot\frac{1-\theta}{\theta}} & \text{when} \quad \alpha \in [\frac{11}{3}, 5). \end{cases} \tag{51}$$

Finally, letting $\tilde{q} \downarrow q_\alpha^{\min}$ and solving $\beta(\alpha,\theta) > 0$ with (48), we obtain the assertion (i). ☐

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Department of Applied Mathematics, School of Science and Engineering, Waseda University, Tokyo, Japan

Department of Applied Mathematics, School of Science and Engineering, Waseda University, Tokyo, Japan