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A GENERALIZATION OF PILLEN'S THEOREM FOR PRINCIPAL SERIES MODULES

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Abstract. Let G be a connected, semisimple and simply connected algebraic group defined and split over the finite field of order p. Pillen proved in 1997 that the highest weight vectors of some Weyl G-modules generate the principal series modules as submodules for the corresponding finite Chevalley groups. This result is generalized in this paper.

1. Introduction and notation

Let G be a connected, semisimple and simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$ which is defined and split over the finite field $\mathbb{F}_p$ of order p. Let $G(n)$ be the corresponding finite Chevalley group consisting of the set of fixed points of the n-th Frobenius map F^n, and set $q = p^n$.

In representation theory of finite groups of Lie type (in the defining characteristic), it is interesting to investigate induced $G(n)$-modules from one-dimensional modules for a Borel subgroup of $G(n)$, which are called principal series modules. These modules have been studied by a lot of people, such as Richen, Carter, Lusztig, Jantzen, Sawada, Pillen and so on (for example, see [2], [6], [10], [11], [12]). Here we pay attention to the fact that Pillen has given a necessary and sufficient condition for the highest weight vector of a Weyl module to generate a principal series module:

Theorem 1.1 ([10, Theorem 1.2])**.** *Suppose that $q > 2h-1$ where h is the Coxeter number of G and suppose that λ is a q-restricted weight. Then the highest weight vector of the Weyl module $V((q-1)\rho+\lambda)$ generates $M_n(\lambda)$ as a $kG(n)$-submodule if and only if $\langle \lambda, \alpha^\vee\rangle > 0$ for all simple roots α.*

The purpose of this paper is to generalize this result. More concretely, since the latter condition $\langle \lambda, \alpha^\vee\rangle > 0$ is somewhat strong, we would like to obtain a similar statement without the condition. The result is given in Theorem 2.1, where we see that for any q-restricted weight λ, the highest weight vector of $V((q-1)\rho+\lambda)$ generates a $kG(n)$-submodule which is a direct summand of $M_n(\lambda)$, and its indecomposable summands are completely determined. The argument is essentially similar to Pillen's original proof, but, actually, there is an error in the proof of Lemma 1.5 in [10]. However, it is not so difficult to modify it. Moreover, we can also weaken the assumption on q from $q > 2h-1$ to $q > h+1$.

Let T be a split maximal torus of G. Let B be an F^n-stable Borel subgroup of G containing T, and let B^+ be the opposite Borel subgroup. By $B^+(n)$ we denote

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the corresponding finite subgroup of B^+. Let $W = N_G(T)/T$ be the Weyl group of G, and let w_0 be the longest element of W. Let $X = X(T)$ be the character group of T, and let Φ be a root system with respect to G and T. Let Φ^+ be the set of positive roots which corresponds to the Borel subgroup B^+, and let Δ be the set of simple roots. By ρ we denote the half sum of all positive roots. The Weyl group W acts on X in a natural way, and by using this we also can define another action (called dot action) as $w \cdot \lambda = w(\lambda + \rho) - \rho$. We can define a W-invariant inner product $\langle \cdot, \cdot \rangle$ on the Euclidean space $\mathbb{E} = X \otimes_{\mathbb{Z}} \mathbb{R}$. For each root $\alpha \in \Phi$, we call $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ a coroot of α. Set

$$X^+ = \{\lambda \in X | \langle \lambda, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta\},$$
$$X_n = \{\lambda \in X^+ | \langle \lambda, \alpha^\vee \rangle < q, \forall \alpha \in \Delta\}.$$

The elements of X^+ and X_n are called dominant weights and q-restricted weights respectively. If α_0 is the highest short root, we call $h = \langle \rho, \alpha_0^\vee \rangle + 1$ the Coxeter number. The dual basis (relative to the inner product) of the basis $\{\alpha^\vee | \alpha \in \Delta\}$ is denoted by $\{\omega_\alpha | \alpha \in \Delta\}$ (namely, $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for any $\alpha, \beta \in \Delta$), and these elements are called fundamental weights. The sum of all fundamental weights is equal to ρ. We define an order relation on X: $\lambda \leq \mu$ if $\mu - \lambda \in \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha$.

By G-modules we mean finite-dimensional rational left G-modules. Let k_λ be the one-dimensional T-module with weight λ which can be regarded as the B- or B^+-module naturally. For each $\lambda \in X^+$, let $L(\lambda)$ be the simple G-module of highest weight λ, and let $V(\lambda)$ and $H^0(\lambda) = \mathrm{Ind}_B^G k_\lambda$ be the Weyl and the dual Weyl modules of highest weight λ respectively. If $\lambda = (q-1)\rho$, we write St_n instead of $L((q-1)\rho)$ and call it the (n-th) Steinberg module. Moreover, for each $\lambda \in X$, let $M_n(\lambda)$ be the principal series module $\mathrm{Ind}_{B^+(n)}^{G(n)} k_\lambda = kG(n) \otimes_{kB^+(n)} k_\lambda$.

2. Main result

Before stating our main theorem, we have to introduce further notation.

For a subset $I \subseteq \Delta$, let ρ_I denote the sum of all ω_α with $\alpha \in I$. For $\lambda \in X_n$, set

$$I_0(\lambda) = \{\alpha \in \Delta | \langle \lambda, \alpha^\vee \rangle = 0\}$$

and

$$I_{q-1}(\lambda) = \{\alpha \in \Delta | \langle \lambda, \alpha^\vee \rangle = q - 1\}.$$

Then [6, 4.6 (1)] says that

$$M_n(\lambda)/\mathrm{rad} M_n(\lambda) \cong \bigoplus_{J \subseteq I_0(\lambda)} \bigoplus_{J' \subseteq I_{q-1}(\lambda)} L(\lambda + (q-1)\rho_J - (q-1)\rho_{J'}).$$

On the other hand, it is well known that the induced module $\mathrm{Ind}_{U^+(n)}^{G(n)} k$ can be decomposed as

$$\mathrm{Ind}_{U^+(n)}^{G(n)} k = \bigoplus_{\lambda \in X_n} Y(\lambda),$$

where each indecomposable summand $Y(\lambda)$ has a simple head $L(\lambda)$ (see [12, Section 3]). Since the induced $B^+(n)$-module $\mathrm{Ind}_{U^+(n)}^{B^+(n)} k$ is semisimple, it follows that $M_n(\lambda)$ is a direct summand of $\mathrm{Ind}_{U^+(n)}^{G(n)} k$, and we can write

$$M_n(\lambda) = \bigoplus_{J \subseteq I_0(\lambda)} \bigoplus_{J' \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_J - (q-1)\rho_{J'}).$$

For a subset $I \subseteq \Delta$, we denote its complement by I^c. Let G_I be the Levi subgroup relative to I, and let $L_I(\lambda)$, $V_I(\lambda)$ and $H_I^0(\lambda)$ be the simple, the Weyl and the dual Weyl G_I-modules with highest weight λ respectively. Let $G_I(n)$ be the corresponding finite group of G_I and set $M_{n,I}(\lambda) = \mathrm{Ind}_{B_I^+(n)}^{G_I(n)} k_\lambda$, where $B_I^+ = G_I \cap B^+$ and $B_I^+(n)$ is the corresponding finite group of B_I^+. As in the case of G, we write $Y_I(\mu)$ for the indecomposable summand of $M_{n,I}(\lambda)$ with simple head $L_I(\mu)$.

Now we state our main theorem.

Theorem 2.1. *Let $\lambda \in X_n$, and suppose that $q > \langle \rho_{I_0(\lambda)^c}, \alpha_0^\vee \rangle + 2$. Then the $G(n)$-submodule generated by the highest weight vector of $V((q-1)\rho + \lambda)$ is isomorphic to*

$$\bigoplus_{J \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_{I_0(\lambda)} - (q-1)\rho_J).$$

Note that the theorem holds for all $\lambda \in X_n$ if $q > h+1$ and that the above direct sum is isomorphic to $M_n(\lambda)$ if $I_0(\lambda)$ is empty. Therefore, this theorem is a generalization of Theorem 1.1.

The proof will involve a series of lemmas.

Lemma 2.2 ([10, Lemma 1.3])**.** *Let $\lambda \in X_n$ and $\nu \in X^+$. Let m_λ be a generator of the $G(n)$-module $M_n(\lambda)$ and let v_ν be the highest weight vector of $V(\nu)$. Then $M_n(\lambda) \otimes V(\nu)$ contains $M_n(\lambda + \nu)$ as the $G(n)$-submodule generated by $m_\lambda \otimes v_\nu$.*

Lemma 2.3 ([10, Lemma 1.4])**.** *Let $\lambda, \mu \in X_n$. If the simple $G(n)$-module $L(\mu)$ is a composition factor of $\mathrm{soc}_{G(n)}(\mathrm{St}_n \otimes V(\lambda))$, then $\mu \geq (q-1)\rho + \lambda$.*

For the next lemmas, we have to introduce some notation. Let G_n be the (scheme-theoretic) kernel of the n-th Frobenius map. For $\lambda \in X$, $\widehat{L}_n(\lambda)$ denotes the simple G_nT-module with highest weight λ, and $\widehat{Q}_n(\lambda)$ denotes the injective hull (= projective cover) of $\widehat{L}_n(\lambda)$. Let $\widehat{Z}_n(\lambda)$ be the coinduced G_nT-module with highest weight λ (see [7, II, 9.1 (2)]). If H is a group and if V and L are H-modules with L simple, then we denote by $[V : L]_H$ the multiplicity of L in the composition factors of V.

The following two lemmas are generalizations of Lemmas 1.5 and 1.6 in [10]. Actually, Pillen's proof of Lemma 1.5 is not complete. (Indeed, he asserts there that $2\rho \geq (q-1)\nu \geq 0$ implies $2(h-1) \geq (q-1)\langle \nu, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, but this is clearly not true.) However, the modification is not so difficult, and moreover the modified proof requires a weaker assumption on q than $q > 2h-1$.

Lemma 2.4. *Let $I \subseteq \Delta$, and suppose that $q > \langle \rho_I, \alpha_0^\vee \rangle + 2$. If $\mu \in X_n$ satisfies $\mu \geq (q-1)\rho + w_0\rho_I$, then we have*

$$[M_n(\rho_I) : L(\mu)]_{G(n)} = \begin{cases} 1 & \text{if } \mu = (q-1)\rho + w_0\rho_I, \\ 0 & \text{if } \mu > (q-1)\rho + w_0\rho_I. \end{cases}$$

Proof. We have

$$\begin{aligned}[M_n(\rho_I) : L(\mu)]_{G(n)} &\leq \sum_{\nu \in X} [\widehat{Z}_n(\rho_I + \nu + (q-1)\rho) : \widehat{L}_n(q\nu + \mu)]_{G_nT} \\ &= \sum_{\nu \in X} [\widehat{Z}_n((q-1)\rho + \rho_I - (q-1)\nu) : \widehat{L}_n(\mu)]_{G_nT} \\ &= \sum_{\nu \in X} [\widehat{Q}_n(\mu) : \widehat{Z}_n((q-1)\rho + \rho_I - (q-1)\nu)]_{G_nT},\end{aligned}$$

where $[\widehat{Q}_n(\mu) : \widehat{Z}_n(\nu)]_{G_nT}$ is the multiplicity of $\widehat{Z}_n(\nu)$ in the $\widehat{Z}_n$-filtration of $\widehat{Q}_n(\mu)$ (see [7, Proposition II, 11.4]). The first inequality follows from [9, Lemma 4.3 (3)] and [5, 3.2], and the second and the last equalities follow from [4, 2.4 (7)] and Brauer-Humphreys reciprocity. Suppose that $\widehat{L}_n(\mu)$ appears in the composition factors of $\widehat{Z}_n((q-1)\rho + \rho_I - (q-1)\nu)$. Then we see from [4, Satz 3.2 (1)] that $\widehat{L}_n(\mu)$ appears also in those of $\widehat{Z}_n(q\rho + w \cdot (\rho_I - \rho - (q-1)\nu))$ for all $w \in W$. Then we have

$$\text{(A)} \qquad \mu \uparrow q\rho + w \cdot (\rho_I - \rho - (q-1)\nu) \uparrow 2(q-1)\rho + w_0\mu$$

(see [7, Lemma II, 9.16.a] or [4, Satz 3.3]), where $\uparrow$ is the order relation on X defined in [7, II, Chapter 6]. These inequalities and the assumption on μ imply

$$\begin{aligned} w\rho_I - \rho_I &\leq -(q-1)\rho - w_0\mu + w\rho_I \leq (q-1)w\nu \\ &\leq (q-1)\rho - \mu + w\rho_I \leq w\rho_I - w_0\rho_I. \end{aligned}$$

We shall show that $\nu = 0$, and then μ must be equal to $(q-1)\rho + w_0\rho_I$ by taking $w = 1$ in the last inequalities. Without loss of generality, we may assume that the root system Φ is indecomposable. Then we know that at most two root lengths occur in Φ and that all roots of a given length are W-conjugate (see [3, Lemma 10.4 C]). Let $\alpha \in \Delta$ be any simple root, and let β_0 be the highest root with the same length as α. Then there is an element $w \in W$ such that $\alpha = w\beta_0$. Moreover, we know that $\langle \lambda, \beta_0^\vee \rangle \leq \langle \lambda, \alpha_0^\vee \rangle$ for all $\lambda \in X^+$. Now we have

$$\begin{aligned} \langle w\rho_I - \rho_I, \beta_0^\vee \rangle &= \langle \rho_I, \alpha^\vee \rangle - \langle \rho_I, \beta_0^\vee \rangle \\ &> -(q-2), \end{aligned}$$

$$\begin{aligned} \langle w\rho_I - w_0\rho_I, \beta_0^\vee \rangle &= \langle \rho_I, \alpha^\vee \rangle + \langle \rho_I, \beta_0^\vee \rangle \\ &< q-1, \end{aligned}$$

and $\langle (q-1)w\nu, \beta_0^\vee \rangle = (q-1)\langle \nu, \alpha^\vee \rangle$. The fact that $\langle \gamma, \beta_0^\vee \rangle \geq 0$ for all $\gamma \in \Delta$ shows $-(q-2) < (q-1)\langle \nu, \alpha^\vee \rangle < q-1$; hence $\langle \nu, \alpha^\vee \rangle = 0$. Since $\alpha \in \Delta$ is arbitrary, we have $\nu = 0$.

It remains to check that $[M_n(\rho_I) : L((q-1)\rho + w_0\rho_I)]_{G(n)} = 1$. Now we know that the multiplicity is less than or equal to the multiplicity $[\widehat{Q}_n((q-1)\rho + w_0\rho_I) : \widehat{Z}_n((q-1)\rho + \rho_I)]_{G_nT}$, which is at most one since $(q-1)\rho + \rho_I$ is the highest weight of $\widehat{Q}_n((q-1)\rho + w_0\rho_I)$ with multiplicity one. On the other hand, by [6, 4.6 (2)], $L((q-1)\rho + w_0\rho_I)$ appears in the composition factors of the socle of $M_n(\rho_I)$, and the lemma follows. □

Remark. Even if q does not satisfy the assumption, the lemma holds if the pair (ν, μ) in (A) can be only $(0, (q-1)\rho + w_0\rho_I)$. But unfortunately, the lemma does not always hold. For example, if $q = 3$, $I = \Delta$ and $\mu = (q-1)\rho + w_0\rho_I = \rho$, then

$[M_n(\rho):L(\mu)]_{G(n)} \geq 2$ since $L(\rho)$ appears both in the head and in the socle of the nonsimple module $M_n(\rho)$. In types A_2, B_2 and G_2, we can check by computing the pairs (ν,μ) directly that the lemma always holds unless $q=2$ or 3.

Lemma 2.5. *Let $I \subseteq \Delta$, and suppose that $q > \langle \rho_I, \alpha_0^\vee\rangle + 2$. Then the $G(n)$-submodule generated by the highest weight vector of $V((q-1)\rho+\rho_I)$ is isomorphic to $Y(\rho_I+(q-1)\rho_{I^c})$.*

Proof. Let v be the highest weight vector (with weight $(q-1)\rho+\rho_I$) of the tensor product $\mathrm{St_n}\otimes V(\rho_I)$. Note that this tensor product has a Weyl filtration (see [8, Theorem 1]) and that v generates a G-submodule isomorphic to $V((q-1)\rho+\rho_I)$ since the Weyl module is the only one which occurs on the bottom of any Weyl filtration of it (see [1, 1.2]).

Consider the $kG(n)$-homomorphism $f: M_n(\rho_I)\to \mathrm{St}_n\otimes V(\rho_I)$ obtained from the natural $kB^+(n)$-homomorphism

$$k_{\rho_I}=k_{(q-1)\rho+\rho_I}\to \mathrm{St}_n\otimes V(\rho_I),\quad 1\mapsto v$$

through the Frobenius reciprocity

$$\mathrm{Hom}_{B^+(n)}(k_{(q-1)\rho+\rho_I},\mathrm{St}_n\otimes V(\rho_I))\cong \mathrm{Hom}_{G(n)}(M_n(\rho_I),\mathrm{St}_n\otimes V(\rho_I)).$$

Note that $\mathrm{Im}(f)\neq 0$. Now Lemmas 2.3 and 2.4 imply that the $G(n)$-socle of $\mathrm{Im}(f)$ is isomorphic to $L((q-1)\rho+w_0\rho_I)$. Let $\tilde{f}$ be the composite map of the canonical inclusion $Y(\rho_I+(q-1)\rho_{I^c})\subseteq M_n(\rho_I)$ and $f: M_n(\rho_I)\to \mathrm{St}_n\otimes V(\rho_I)$. By [12, Theorem 3.10], the socle of $Y(\rho_I+(q-1)\rho_{I^c})$ is $L((q-1)\rho+w_0\rho_I)$. Recall that $M_n(\rho)$ has $L((q-1)\rho+w_0\rho_I)$ as a composition factor only in the socle, whose multiplicity is one. Hence $\tilde{f}$ must be injective, since it takes the socle of $Y(\rho_I+(q-1)\rho_{I^c})$ into the socle of $\mathrm{Im}(f)$. On the other hand, since $\mathrm{Im}(\tilde{f})=\mathrm{Im}(f)$ is the $kG(n)$-submodule in $\mathrm{St}_n\otimes V(\rho_I)$ generated by v, it follows that $\mathrm{Im}(\tilde{f})\subseteq V((q-1)\rho+\rho_I)$. □

Remark. Since the socle of $M_n(\rho_I)$ is multiplicity-free, the above map f is always injective on $Y(\rho_I+(q-1)\rho_{I^c})$ and zero on any other summand for any indecomposable decomposition of $M_n(\rho_I)$.

Now we turn to prove our main theorem.

Proof of Theorem 2.1. We decompose $M_n(\rho_{I_0(\lambda)^c})$ as

$$M_n(\rho_{I_0(\lambda)^c})=\bigoplus_{J\subseteq I_0(\lambda)} Y(\rho_{I_0(\lambda)^c}+(q-1)\rho_J)$$

and consider the $kG(n)$-homomorphism

$$f\otimes id: M_n(\rho_{I_0(\lambda)^c})\otimes V(\lambda-\rho_{I_0(\lambda)^c})\to V((q-1)\rho+\rho_{I_0(\lambda)^c})\otimes V(\lambda-\rho_{I_0(\lambda)^c}),$$

where f is the map $M_n(\rho_{I_0(\lambda)^c})\to \mathrm{St}_n\otimes V(\rho_{I_0(\lambda)^c})$ constructed in the proof of Lemma 2.5 (note that $\mathrm{Im}(f)\subseteq V((q-1)\rho+\rho_{I_0(\lambda)^c})$). Let v be the highest weight vector of $V(\lambda-\rho_{I_0(\lambda)^c})$ and let $m\in k_{\rho_{I_0(\lambda)^c}}$ be a generator of the $kG(n)$-module $M_n(\rho_{I_0(\lambda)^c})$. Observe that $f(m)\otimes v$ generates the G-submodule $V((q-1)\rho+\lambda)$. Therefore, it suffices to prove that the resulting $kG(n)$-module in the theorem is the image of the composite map of the embedding $\varphi: M_n(\lambda)\to M_n(\rho_{I_0(\lambda)^c})\otimes V(\lambda-\rho_{I_0(\lambda)^c})$ in Lemma 2.2 and $f\otimes id$.

Consider the $kG_{I_0(\lambda)}(n)$-module embedding

$$\varphi_{I_0(\lambda)} : M_{n,I_0(\lambda)}(\lambda) \to M_{n,I_0(\lambda)}(\rho_{I_0(\lambda)^c}) \otimes V_{I_0(\lambda)}(\lambda - \rho_{I_0(\lambda)^c})$$

which is analogous to φ. By the argument in [7, II, 1.18], we see that the one-dimensional T-module of weight $\lambda - \rho_{I_0(\lambda)^c}$ can be uniquely extended to a $G_{I_0(\lambda)}$-module and then it is trivial as a module for the derived subgroup $[G_{I_0(\lambda)}, G_{I_0(\lambda)}]$ of $G_{I_0(\lambda)}$. Now [7, Proposition I, 6.13 (1)] shows that the induced module $H^0_{I_0(\lambda)}(\lambda - \rho_{I_0(\lambda)^c})$, and hence $V_{I_0(\lambda)}(\lambda - \rho_{I_0(\lambda)^c})$, is the one-dimensional $G_{I_0(\lambda)}$-module $k_{\lambda - \rho_{I_0(\lambda)^c}}$. By comparison of dimensions, we see that $\varphi_{I_0(\lambda)}$ is bijective and it takes the summand $Y_{I_0(\lambda)}(\lambda + (q-1)\rho_J)$ onto

$$Y_{I_0(\lambda)}(\rho_{I_0(\lambda)^c} + (q-1)\rho_J) \otimes V_{I_0(\lambda)}(\lambda - \rho_{I_0(\lambda)^c})$$

for any $J \subseteq \Delta$. We denote this restriction map by $\varphi_{I_0(\lambda),J}$ (this is independent of the choice of indecomposable decompositions of various $M_n(\cdot)$'s thanks to multiplicity-free property of the socle). Since the highest weight vector v of $V(\lambda - \rho_{I_0(\lambda)^c})$ generates the $G_{I_0(\lambda)}$-submodule $kv = V_{I_0(\lambda)}(\lambda - \rho_{I_0(\lambda)^c})$, we can regard $\varphi_{I_0(\lambda)}$ and $\varphi_{I_0(\lambda),J}$ as injective homomorphisms

$$\varphi_{I_0(\lambda)} : M_{n,I_0(\lambda)}(\lambda) \to M_{n,I_0(\lambda)}(\rho_{I_0(\lambda)^c}) \otimes V(\lambda - \rho_{I_0(\lambda)^c}),$$

$$\varphi_{I_0(\lambda),J} : Y_{I_0(\lambda)}(\lambda + (q-1)\rho_J) \to Y_{I_0(\lambda)}(\rho_{I_0(\lambda)^c} + (q-1)\rho_J) \otimes V(\lambda - \rho_{I_0(\lambda)^c}).$$

Now we apply the Harish-Chandra induction $\mathrm{HCInd}^{G(n)}_{G_{I_0(\lambda)}(n)}$ to these maps. Clearly we have

$$\begin{aligned}\varphi &= \mathrm{HCInd}(\varphi_{I_0(\lambda)}) = \mathrm{HCInd}\Big(\bigoplus_{J \subseteq I_0(\lambda)} \varphi_{I_0(\lambda),J}\Big) \\ &= \bigoplus_{J \subseteq I_0(\lambda)} \mathrm{HCInd}(\varphi_{I_0(\lambda),J}).\end{aligned}$$

We shall claim that

$$\text{(B)}\quad \mathrm{HCInd}^{G(n)}_{G_{I_0(\lambda)}(n)} Y_{I_0(\lambda)}(\lambda + (q-1)\rho_J) = \bigoplus_{J' \subseteq I_{q-1}(\lambda)} Y(\lambda + (q-1)\rho_J - (q-1)\rho_{J'})$$

and

$$\text{(C)}\qquad \mathrm{HCInd}^{G(n)}_{G_{I_0(\lambda)}(n)} Y_{I_0(\lambda)}(\rho_{I_0(\lambda)^c} + (q-1)\rho_J) = Y(\rho_{I_0(\lambda)^c} + (q-1)\rho_J).$$

Now since $q \neq 2$, (C) is the special case of $\lambda = \rho_{I_0(\lambda)^c}$ in (B). For a subset $J' \subseteq I_{q-1}(\lambda)$, by Frobenius reciprocity we have

$$\begin{aligned}&\mathrm{Hom}_{G(n)}(\mathrm{HCInd}^{G(n)}_{G_{I_0(\lambda)}(n)} Y_{I_0(\lambda)}(\lambda + (q-1)\rho_J), L(\lambda + (q-1)\rho_J - (q-1)\rho_{J'})) \\ &\cong \mathrm{Hom}_{P^+_{I_0(\lambda)}(n)}(Y_{I_0(\lambda)}(\lambda + (q-1)\rho_J), L(\lambda + (q-1)\rho_J - (q-1)\rho_{J'})),\end{aligned}$$

where $P^+_{I_0(\lambda)}$ is the parabolic subgroup of G which corresponds to $I_0(\lambda)$ and contains B^+ and $P^+_{I_0(\lambda)}(n)$ is the corresponding finite group. The G-module $L(\lambda + (q-1)\rho_J - (q-1)\rho_{J'})$ has a $P^+_{I_0(\lambda)}$-submodule isomorphic to $L_{I_0(\lambda)}(\lambda + (q-1)\rho_J - (q-1)\rho_{J'})$, which is isomorphic to $L_{I_0(\lambda)}(\lambda + (q-1)\rho_J)$ as a $kP^+_{I_0(\lambda)}(n)$-submodule. Hence the right-hand side of the last formula is nonzero, and so the direct sum $\bigoplus_{J'\subseteq I_{q-1}(\lambda)} L(\lambda+(q-1)\rho_J-(q-1)\rho_{J'})$ must be contained in the head of $\mathrm{HCInd}_{G_{I_0(\lambda)}(n)}^{G(n)} Y_{I_0(\lambda)}(\lambda+(q-1)\rho_J)$. However, the $kG(n)$-module

$$\bigoplus_{J\subseteq I_0(\lambda)} \bigoplus_{J'\subseteq I_{q-1}(\lambda)} L(\lambda+(q-1)\rho_J-(q-1)\rho_{J'})$$

fills the head of

$$\begin{aligned} M_n(\lambda) &= \mathrm{HCInd}_{G_{I_0(\lambda)}(n)}^{G(n)} M_{n,I_0(\lambda)}(\lambda) \\ &= \bigoplus_{J\subseteq I_0(\lambda)} \mathrm{HCInd}_{G_{I_0(\lambda)}(n)}^{G(n)} Y_{I_0(\lambda)}(\lambda+(q-1)\rho_J); \end{aligned}$$

hence the equality must hold in (B).

Therefore, φ maps $\bigoplus_{J'\subseteq I_{q-1}(\lambda)} Y(\lambda+(q-1)\rho_J-(q-1)\rho_{J'})$ into the tensor product $Y(\rho_{I_0(\lambda)^c}+(q-1)\rho_J)\otimes V(\lambda-\rho_{I_0(\lambda)^c})$ injectively. Moreover, by Lemma 2.5, the restriction of $f\otimes id$ on the tensor product is injective for $J=I_0(\lambda)$ and zero otherwise. Since the image of the composite map $(f\otimes id)\circ\varphi$ is the $kG(n)$-submodule generated by the highest weight vector, the result follows. □

Even if $q\leq\langle\rho_{I_0(\lambda)^c},\alpha_0^\vee\rangle+2$, when $q\neq 2$ and $I\subseteq\Delta$ satisfies Lemma 2.4, it is shown similarly that Theorem 2.1 holds. Hence the remark of Lemma 2.4 says that in types A_2, B_2 and G_2, Theorem 2.1 holds for all $\lambda\in X_n$ unless $q=2$ or 3. Though the approach in this paper does not cover all q and λ, the theorem is expected to hold in general:

Conjecture 2.6. *If $\lambda\in X_n$, the $G(n)$-submodule generated by the highest weight vector of $V((q-1)\rho+\lambda)$ is isomorphic to*

$$\bigoplus_{J\subseteq I_{q-1}(\lambda)} Y(\lambda+(q-1)\rho_{I_0(\lambda)}-(q-1)\rho_J).$$

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