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DISPERSIVE WAVE ESTIMATES ON 3D HYPERBOLIC SPACE

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ABSTRACT. Stimulated by a recent paper of J.-Ph. Anker and V. Pierfelice, we sharpen some dispersive estimates that arose in our previous work on nonlinear waves on 3D hyperbolic space.

1. INTRODUCTION

In [3], the authors studied semilinear wave equations of the form

$$\partial_t^2 u - \Delta u = a|u|^b, \tag{1.1}$$

on $\mathbb{R} \times M$, where $M = \mathcal{H}^3$ is 3D hyperbolic space, and Δ is the Laplace-Beltrami operator on M. They established global existence of solutions with small initial data (measured in certain L^2-Sobolev spaces), provided $b \geq 5/3$. A key ingredient in the analysis was a family of dispersive estimates on

$$R(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}, \tag{1.2}$$

namely

$$\|R(t)f\|_{H^{1-s,q}} \leq \frac{C}{\psi_s(t)}\|f\|_{L^{q'}}, \tag{1.3}$$

for

$$s \in (0,2), \quad q = \frac{2}{1-s/2}, \quad q' = \frac{2}{1+s/2}, \tag{1.4}$$

and

$$\psi_s(t) = |t|^{s/2} + |t|^{3/2}, \tag{1.5}$$

for $1 \leq s < 2$, while [3] obtained (1.3) with

$$\psi_s(t) = |t|^{s/2} + |t|^{3s/2}, \tag{1.6}$$

for $0 < s \leq 1$. These dispersive estimates are equivalent to estimates on 3D Euclidean space $M = \mathbb{R}^3$ for small $|t|$, but they are stronger than Euclidean estimates for large $|t|$. They yield Strichartz estimates, which in turn imply the small data global solvablity stated above.

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A recent paper [1] of J.-Ph. Anker and V. Pierfelice improves these results in several ways. First, they show that (1.3) holds, for the parameters (1.4), with $\psi_s(t)$ given by (1.5), for all $s \in (0,2)$, when $M = \mathcal{H}^3$. This gives rise to a larger family of Strichartz estimates, from which they deduce global solvability of (1.1), for small initial data, whenever $b > 1$. Furthermore, they get optimal results for $M = \mathcal{H}^n$, n-dimensional hyperbolic space, for all $n \geq 2$, including global solvability of (1.1) for small data, whenever $b > 1$.

Our purpose here is to show how the argument in [3] can be sharpened to obtain (1.3), with the parameters (1.4) and with $\psi_s(t)$ given by (1.5), for all $s \in (0,2)$, thus rederiving this sharp dispersive estimate of [1], in the 3-dimensional case. Since our approach is quite different from that of [1], this second proof should be of interest to the PDE community.

This paper proceeds as follows. In §2 we recall results of [3], leading to the dispersive estimates established there, and set the stage for the material of §3, in which we present the desired sharpening.

2. First estimates on $R(t)$

In general, the Laplace-Beltrami operator on n-dimensional hyperbolic space $\mathcal{H}^n$ has

$$\operatorname{Spec}(-\Delta) = [a^2, \infty), \quad a = \frac{n-1}{2}. \tag{2.1}$$

Hence

$$L = \Delta + a^2 \Longrightarrow \operatorname{Spec}(-L) = [0, \infty). \tag{2.2}$$

For $n = 3,\ a = 1$, we write

$$R(t) = R_0(t) + R_1(t), \tag{2.3}$$

where

$$R_0(t) = \frac{\sin t\sqrt{-L}}{\sqrt{-L}}. \tag{2.4}$$

For L and Δ related by (2.2), one has

$$\cos t\sqrt{-\Delta} = \cos t\sqrt{-L} - at \int_0^t \frac{J_1(a\sqrt{t^2 - s^2})}{\sqrt{t^2 - s^2}} \cos s\sqrt{-L}\, ds \tag{2.5}$$

(cf. [3], (2.10)), and integrating this yields

$$R_1(t) = -a \int_0^t s\, \frac{J_1(a\sqrt{t^2 - s^2})}{\sqrt{t^2 - s^2}}\, R_0(s)\, ds. \tag{2.6}$$

The usefulness of these formulas arises from the relatively simple nature of $R_0(t)$. In case $n = 3$ (which we take from here on), we have, for $t > 0$,

$$R_0(t)\delta_y(x) = \frac{\delta(t - r)}{4\pi \sinh t}, \quad r = \operatorname{dist}(x, y). \tag{2.7}$$

Then (2.6)–(2.7) yield (with y-dependence suppressed)

$$R_1(t)\delta_y(x) = \Phi(t, x) = -\frac{r}{4\pi \sinh r}\, \frac{J_1(\sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}}\, \chi_{\{r \leq t\}}. \tag{2.8}$$

We mention that $J_1(\zeta)/\zeta$ is an entire function of ζ^2, satisfying

$$\Big|\frac{J_1(\zeta)}{\zeta}\Big| \le C\langle\zeta\rangle^{-3/2}, \quad \zeta \in [0,\infty), \tag{2.9}$$

and

$$\frac{J_1(\zeta)}{\zeta} \sim C\zeta^{-3/2}\cos\Big(\zeta - \frac{3\pi}{4}\Big) + \cdots, \quad \zeta \to +\infty. \tag{2.10}$$

Here and below,

$$\langle x\rangle = (1+|x|^2)^{1/2}. \tag{2.11}$$

We estimate $R_0(t)f$ by interpolating an $L^2 - L^2$ estimate with (a replacement for) an $L^1 - L^\infty$ estimate, as follows. In view of (2.1),

$$\|R(t)f\|_{H^{1,2}} \le C\|f\|_{L^2}. \tag{2.12}$$

Also,

$$\|\sqrt{-L}R_0(t)f\|_{L^2} \le \|f\|_{L^2}, \tag{2.13}$$

but

$$\|R_0(t)f\|_{L^2} = |t|\cdot\Big\|\frac{\sin t\sqrt{-L}}{t\sqrt{-L}}f\Big\|_{L^2} \le C|t|\cdot\|f\|_{L^2}, \tag{2.14}$$

so we have

$$\|R_0(t)f\|_{H^{1.2}} \le C\langle t\rangle\|f\|_{L^2}. \tag{2.15}$$

Meanwhile, an examination of (2.7) yields

$$\|R_0(t)f\|_{\mathfrak{h}^{-1,\infty}} \le \frac{C}{\sinh|t|}\|f\|_{L^1}, \tag{2.16}$$

where

$$\mathfrak{h}^{-1,\infty}(M) = (1-\Delta)^{1/2}\mathrm{bmo}(M), \tag{2.17}$$

and bmo is the "local" version of BMO, studied for manifolds with bounded geometry in [5]. An interpolation argument, a variant of one in [2] established in [5], gives

$$\|R_0(t)f\|_{H^{1-s,q}} \le C\frac{\langle t\rangle^{1-s/2}}{(\sinh|t|)^{s/2}}\|f\|_{L^{q'}} \tag{2.18}$$

when (1.4) holds. Such an estimate also appears in [4]. (For more on this, see the remark following (3.25) in [3].)

It remains to estimate $R_1(t)f$. For this it is convenient to take a smooth cutoff $\psi_1 \in C_0^\infty(\mathbb{R})$, such that $\psi_1(\tau) = 1$ for $|\tau| \le 1/3$, 0 for $|\tau| \ge 2/3$, set $\psi_2 = 1 - \psi_1$, and write

$$\Phi(t,x) = \Phi_1(t,x) + \Phi_2(t,x), \quad \Phi_j(t,x) = \Phi(t,x)\psi_j\Big(\frac{r}{t}\Big), \tag{2.19}$$

corresponding to

$$\begin{aligned} R_1(t) &= R_{11}(t) + R_{12}(t), \\ R_{1j}(t) &= -\int_0^t s\,\frac{J_1(\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}}\,\psi_j\Big(\frac{s}{t}\Big)R_0(s)\,ds, \end{aligned} \tag{2.20}$$

in view of (2.7). The estimates

$$\|\Phi_1(t,\cdot)\|_{L^q} \le C_q\langle t\rangle^{-3/2}, \quad \forall\, q > 2, \tag{2.21}$$

and

$$\|\Phi_2(t,\cdot)\|_{L^q} \le C_q(t \vee t^{3/q})e^{-(1-2/q)t/3}, \quad \forall\, q > 2, \tag{2.22}$$

are readily established from the formula (2.8) and the estimate (2.9); cf. (3.29)–(3.30) of [3].

The estimates (2.21)–(2.22) imply that

$$\|R_1(t)f\|_{L^q} \le C_q\langle t\rangle^{-3/2}\|f\|_{L^1}, \quad \forall\, q > 2. \tag{2.23}$$

Since $R_1(t)$ is selfadjoint, duality gives

$$\|R_1(t)f\|_{L^\infty} \le C_p\langle t\rangle^{-3/2}\|f\|_{L^p}, \quad \forall\, p \in [1,2). \tag{2.24}$$

Then interpolation gives

$$\|R_1(t)f\|_{L^q} \le C_{pq}\langle t\rangle^{-3/2}\|f\|_{L^p}, \quad p \in [1,2),\ q > 2, \tag{2.25}$$

and in particular

$$\|R_1(t)f\|_{L^q} \le C_q\langle t\rangle^{-3/2}\|f\|_{L^{q'}}, \quad q > 2. \tag{2.26}$$

This gives the following complement to (2.18):

$$\|R_1(t)f\|_{H^{1-s,q}} \le C_q\langle t\rangle^{-3/2}\|f\|_{L^{q'}}, \tag{2.27}$$

in turn implying (1.3), when (1.4) holds, as long as $1 - s \le 0$, i.e., $1 \le s < 2$. But (2.26) does not imply (2.27) for $s \in (0,1)$. In [3], the authors introduced an interpolation argument at this point, leading to the weaker estimate with $\psi_s(t)$ given by (1.6) for $s \in (0,1)$. Our remaining task is to prove (2.27), with parameters as in (1.4), for $s \in (0,1)$, and $|t| \ge 1$. We take this up in §3.

3. Improved estimates on $R_1(t)$

Here we will estimate $R_{1j}(t)f$ separately, for $j = 1,2$. Throughout this section, we assume $|t| \ge 1$. Supplementing (2.21), an examination of the radial derivative of $\Phi_1(t,x)$, using (2.8), (2.9), and (2.19), gives, parallel to (3.29) of [3],

$$\begin{aligned}\int_{\mathcal{H}^3} |\nabla\Phi_1(t,x)|^q\,dV(x) &= c_q\int_0^{2t/3}\Big|\partial_r\Big(\frac{r\psi_1(r/t)}{\sinh r}\,\frac{J_1(\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}}\Big)\Big|^q\sinh^2 r\,dr \\ &\le C_q\sup_{\lambda\ge t/2}\langle\lambda\rangle^{-3q/2}\int_0^{2t/3}\langle r\rangle^q e^{(2-q)r}\,dr \\ &\le C_q\langle t\rangle^{-3q/2},\end{aligned} \tag{3.1}$$

provided $q > 2$. Together with (2.21), this gives

$$\|\Phi_1(t,\cdot)\|_{H^{1,q}} \le C_q\langle t\rangle^{-3/2}, \quad \forall\, q > 2; \tag{3.2}$$

hence

$$\|R_{11}(t)f\|_{H^{1,q}} \le C_q\langle t\rangle^{-3/2}\|f\|_{L^1}, \quad \forall\, q > 2. \tag{3.3}$$

Duality and selfadjointness of $R_{11}(t)$ give

$$\|R_{11}(t)f\|_{L^\infty} \le C_p\langle t\rangle^{-3/2}\|f\|_{H^{-1,p}}, \quad \forall\, p \in (1,2). \tag{3.4}$$

To proceed, note that the operators $R_{1j}(t)$ all commute with $\Lambda = (1-\Delta)^{1/2}$. Set

$$S_{11}(t) = \Lambda R_{11}(t) = R_{11}(t)\Lambda. \tag{3.5}$$

We have

$$\begin{aligned} \|S_{11}(t)f\|_{L^q} &\le C_q\langle t\rangle^{-3/2}\|f\|_{L^1}, \quad \forall\, q\in(2,\infty), \\ \|S_{11}(t)g\|_{L^\infty} &\le C_p\langle t\rangle^{-3/2}\|g\|_{L^p}, \quad \forall\, p\in(1,2). \end{aligned} \tag{3.6}$$

Elementary interpolation gives

$$\|S_{11}(t)f\|_{L^q} \le C_{pq}\langle t\rangle^{-3/2}\|f\|_{L^p}, \quad 1<p<2<q<\infty. \tag{3.7}$$

Take $p=q'$. We get

$$\|R_{11}(t)f\|_{H^{1,q}} \le C_q\langle t\rangle^{-3/2}\|f\|_{L^{q'}}, \tag{3.8}$$

which is stronger than

$$\|R_{11}(t)f\|_{H^{1-s,q}} \le C\langle t\rangle^{-3/2}\|f\|_{L^{q'}}, \tag{3.9}$$

with parameters as in (1.4).

It remains to estimate $R_{12}(t)$. From (2.15), (2.20), and (2.9), we have

$$\|R_{1j}(t)f\|_{H^{1,2}} \le C\langle t\rangle^3\|f\|_{L^2}. \tag{3.10}$$

Also, inspection gives $\|\Phi_2(t,\cdot)\|_{L^\infty}\le Ce^{-\alpha t}$, with $\alpha>0$; hence

$$\|R_{12}(t)f\|_{L^\infty} \le Ce^{-\alpha t}\|f\|_{L^1}. \tag{3.11}$$

Now, an argument weaker than that needed to pass from (2.15)–(2.16) to (2.18) applies, to pass from (3.10)–(3.11) to

$$\|R_{12}(t)f\|_{H^{1-s,q}} \le Ce^{-\beta t}\|f\|_{L^{q'}}, \tag{3.12}$$

with s,q,q' as in (1.4) and $\beta=\beta(s)>0$.

Putting together (2.18), (3.9), and (3.12), we have the desired result, which we state formally.

Theorem 3.1. *Given s,q,q' as in* (1.4)*, $M=\mathcal{H}^3$, we have the dispersive estimate*

$$\|R(t)f\|_{H^{1-s,q}} \le \frac{C}{|t|^{s/2}+|t|^{3/2}}\|f\|_{L^{q'}}. \tag{3.13}$$

Remark. A parallel argument gives

$$\|\cos t\sqrt{-\Delta}\, f\|_{H^{-s,q}} \le \frac{C}{|t|^{s/2}+|t|^{3/2}}\|f\|_{L^{q'}}, \tag{3.14}$$

for such parameters. Applying the result

$$(-\Delta)^{\sigma/2} : H^{\tau,p}(M) \longrightarrow H^{\tau-\sigma,p}(M), \tag{3.15}$$

valid for $\tau,\sigma\in\mathbb{R}$, $p\in(1,\infty)$, when M is hyperbolic space, to (3.13) and (3.14) yields

$$\|e^{it\sqrt{-\Delta}}f\|_{H^{\sigma-s,q}} \le \frac{C}{|t|^{s/2}+|t|^{3/2}}\|f\|_{H^{\sigma,q'}}, \tag{3.16}$$

for s,q,q' as in Theorem 3.1, $\sigma\in\mathbb{R}$.

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