PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 11, November 2012, Pages 4017–4025
S 0002-9939(2012)11648-X
Article electronically published on March 28, 2012

ON RIGIDITY OF GRADIENT KÄHLER-RICCI SOLITONS WITH HARMONIC BOCHNER TENSOR

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(Communicated by Lei Ni)

Abstract. In this paper, we prove that complete gradient steady Kähler-Ricci solitons with harmonic Bochner tensor are necessarily Kähler-Ricci flat, i.e., Calabi-Yau, and that complete gradient shrinking (or expanding) Kähler-Ricci solitons with harmonic Bochner tensor must be isometric to a quotient of $N^k \times \mathbb{C}^{n-k}$, where N is a Kähler-Einstein manifold with positive (or negative) scalar curvature.

1. Introduction

A complete Riemannian manifold (M, g_{ij}) is called a Ricci soliton if there is a vector field X and a constant λ such that

$$R_{ij} + \frac{1}{2}(\nabla_i X_j + \nabla_j X_i) = \frac{\lambda}{2} g_{ij},$$

where $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$ corresponds to a shrinking, steady or expanding soliton respectively. Moreover, a Ricci soliton is called a gradient Ricci soliton if the vector field is a gradient vector field, i.e. $X = \nabla f$ for some smooth function f on M. In this case, the Ricci soliton equation above becomes

$$R_{ij} + \nabla_i \nabla_j f = \frac{\lambda}{2} g_{ij}.$$

Since R. Hamilton [18] introduced the concept of Ricci solitons in the mid 1980's, the study of Ricci solitons has attracted a lot of attention. Ricci solitons, as self-similar solutions to Hamilton's Ricci flow, are natural generalizations of Einstein metrics. Since they often arise as the dilation limits of the singularities of the Ricci flow, Ricci solitons play an important role in the singularity analysis of the Ricci flow.

In the past decade, a significant number of results have been obtained in classifying or understanding the geometry of shrinking solitons. In particular, the classification of gradient shrinkers is known in dimensions 2 and 3, and assuming locally conformally flatness in all dimensions $n \geq 4$; see, e.g., [23], [22], [7], [16], [25], [27], [12], [21], [20], etc. Recently, a rigidity result for Bach flat gradient shrinkers has also been obtained by H.-D. Cao and the first author [9]. We refer the reader to a recent survey paper of H.-D. Cao [5] on Ricci shrinkers.

Regarding steady solitons, it is well-known that compact ones (as well as compact expanding solitons) must be Einstein; see, e.g., [11] for a proof. However, much less

Received by the editors May 12, 2011.
2010 *Mathematics Subject Classification.* Primary 53C44, 53C55.
Key words and phrases. Kähler-Ricci soliton, Bochner tensor, rigidity.

is known for noncompact steady Ricci solitons. In dimension $n = 2$, R. Hamilton showed that any 2-dimensional steady soliton is isometric to the Cigar soliton, up to scaling. For $n = 3$, G. Perelman [23] conjectured that any 3-dimensional complete (κ-noncollapsed) steady soliton with positive sectional curvature is isometric to the Bryant soliton, which is the unique rotationally symmetric example on $\mathbb{R}^3$. Very recently, H.-D. Cao and the first author [8] made the first important progress on this problem. They showed that any nonflat n-dimensional, $n \geq 3$, complete locally conformally flat steady Ricci soliton is isometric to the Bryant soliton (for $n \geq 4$, Catino and Mantegazza [13] independently proved this result by using a different method). Subsequently, their work has been used by S. Brendle [1] in classifying 3-dimensional steady Ricci solitons satisfying certain asymptotic conditions, and X. X. Chen and Y. Wang [15] in classifying 4-dimensional half-conformally flat steady solitons, respectively. Most recently, H.-D. Cao, G. Catino, C. Mantegazza, L. Mazzieri and the first author [6] have proved that complete Bach-flat gradient steady solitons with positive Ricci curvature such that the scalar curvature attains a maximum at an interior point are also isometric to the Bryant soliton.

For more thorough discussions and results in Ricci solitons, the reader can look at the survey [4] of H.-D. Cao and the references therein. From now on, we will focus on Kähler-Ricci solitons.

Definition 1.1. An n-dimensional Kähler manifold $(M^n, g_{i\bar{j}})$ is called a gradient Kähler-Ricci soliton if there is a real-valued smooth function f satisfying the soliton equation

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \lambda g_{i\bar{j}},$$

for some constant $\lambda \in \mathbb{R}$ and such that ∇f is a holomorphic vector field, i.e. $\nabla_i \nabla_j f = 0$.

In [10], H.-D. Cao and R. Hamilton observed that complete noncompact gradient steady Kähler-Ricci solitons with positive Ricci curvature such that the scalar curvature attains the maximum must be Stein (and diffeomorphic to $\mathbb{R}^{2n}$). Later, under the same assumptions, A. Chau and L.-F. Tam [14], and R. Bryant [3] independently proved that such steady Kähler-Ricci solitons are actually biholomorphic to $\mathbb{C}^n$. Moreover, Chau and Tam showed that complete noncompact expanding Kähler-Ricci solitons with nonnegative Ricci curvature are also biholomorphic to $\mathbb{C}^n$.

To state our results, let us first recall that on Kähler manifolds there is a similar notion to the Weyl tensor, called the Bochner tensor. The Bochner tensor $W_{i\bar{j}k\bar{l}}$ is defined by

$$\begin{aligned} W_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} &- \frac{R}{(n+1)(n+2)}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \\ &+ \frac{1}{(n+2)}(R_{i\bar{j}}g_{k\bar{l}} + R_{k\bar{l}}g_{i\bar{j}} + R_{i\bar{l}}g_{k\bar{j}} + R_{k\bar{j}}g_{i\bar{l}}), \end{aligned}$$

where $R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$. We also denote the divergence of the Bochner tensor by

$$\begin{aligned} C_{i\bar{j}k} &= g^{l\bar{q}} \nabla_l W_{i\bar{j}k\bar{q}} \\ &= \frac{n}{n+2} \nabla_i R_{k\bar{j}} - \frac{n}{(n+1)(n+2)}(g_{k\bar{j}} \nabla_i R + g_{i\bar{j}} \nabla_k R). \end{aligned}$$

Definition 1.2. A Kähler manifold M^n is said to have harmonic Bochner tensor if $C_{i\bar{j}k} = 0$, i.e.,

$$\nabla_i R_{k\bar{j}} = \frac{1}{n+1}(g_{k\bar{j}}\nabla_i R + g_{i\bar{j}}\nabla_k R).$$

Very recently, by using a similar argument as in the paper [8] of H.-D. Cao and the first author, Y. Su and K. Zhang [26] have shown that any complete noncompact gradient Kähler-Ricci soliton with vanishing Bochner tensor is necessarily Kähler-Einstein, and hence a quotient of the corresponding complex space form.

In this paper we investigate gradient Kähler-Ricci solitons with harmonic Bochner tensor, and extend the classification results of Su and Zhang. Our main results are:

Theorem 1.3. *Any complete gradient steady Kähler-Ricci soliton with harmonic Bochner tensor must be Kähler-Ricci flat (i.e., Calabi-Yau).*

Theorem 1.4. *Any complete gradient shrinking (or expanding) Kähler-Ricci soliton with harmonic Bochner tensor must be isometric to the quotient of $N^k \times \mathbb{C}^{n-k}$, where N^k is a k-dimensional Kähler-Einstein manifold with positive (or negative) scalar curvature.*

Remark 1.5. It is known that a compact Kähler manifold with vanishing Bochner tensor (also called Bochner-Kähler or Bochner-flat) is necessarily a compact quotient of $M_c^k \times M_{-c}^{n-k}$, where M_c^k and M_{-c}^{n-k} denote the complex space forms of constant holomorphic sectional curvature c and $-c$ respectively (cf., e.g., Corollary 4.17 in [2]). It follows immediately that any compact Kähler-Ricci soliton with vanishing Bochner tensor must be a quotient of a complex space form.

Remark 1.6. In the Riemannian case, by using a rigidity result of Petersen and Wylie [24], Fernández-López and García-Río [17], and Munteanu and Sesum [20] proved that Ricci shrinkers with harmonic Weyl tensor must be rigid, i.e., a quotient of the product of an Einstein manifold and $\mathbb{R}^k$.

2. Proof of the main theorems

Let $(M^n, g_{i\bar{j}}, f)$ be a gradient Kähler-Ricci soliton, i.e.

$$R_{i\bar{j}} + \nabla_i\nabla_{\bar{j}} f = \lambda g_{i\bar{j}} \quad \text{and} \quad \nabla_i\nabla_j f = 0.$$

It is well-known that the following basic identities hold (see e.g. [11]).

Lemma 2.1. *On a gradient Kähler-Ricci soliton, we have*

$$R + |\nabla f|^2 - \lambda f = Const, \tag{2.1}$$

$$R + \Delta f = n\lambda, \tag{2.2}$$

$$\nabla_i R_{k\bar{j}} = R_{i\bar{j}k\bar{l}}\nabla_l f, \tag{2.3}$$

and

$$\nabla_i R = R_{i\bar{j}}\nabla_j f. \tag{2.4}$$

From now on, we assume that $(M^n, g_{i\bar{j}}, f)$ is a gradient Kähler-Ricci soliton with harmonic Bochner tensor so that

$$\nabla_i R_{k\bar{j}} = \frac{1}{n+1}(\nabla_i R g_{k\bar{j}} + \nabla_k R g_{i\bar{j}}). \tag{2.5}$$

Lemma 2.2. *We have*

$$\begin{aligned}&\lambda R_{i\bar{j}}-R_{i\bar{j}k\bar{l}}R_{\bar{k}l}\\ &=\frac{1}{n+1}\Big[\frac{1}{n+1}\nabla_kR\nabla_{\bar{k}}fg_{i\bar{j}}+(\lambda R-|Rc|^2)g_{i\bar{j}}-\frac{n}{n+1}\nabla_iR\nabla_{\bar{j}}f\\ &\qquad+\lambda R_{i\bar{j}}-R_{i\bar{k}}R_{k\bar{j}}\Big]\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}&2(n+1)\lambda\nabla_iR-2R\nabla_iR-2R_{i\bar{j}}\nabla_jR\\ &=-\frac{1}{n+1}\nabla_iR|\nabla f|^2-\frac{1}{n+1}\nabla_kR\nabla_{\bar{k}}f\nabla_if.\end{aligned}\tag{2.7}$$

Proof. On the one hand, by differentiating (2.4), we obtain

$$\Delta R=\nabla_k\nabla_{\bar{k}}R=\nabla_kR\nabla_{\bar{k}}f+R_{k\bar{l}}\nabla_{\bar{k}}\nabla_lf.$$

From (2.5), we get

$$\begin{aligned}\nabla_k\nabla_{\bar{k}}R_{i\bar{j}}&=\frac{1}{n+1}(\Delta Rg_{i\bar{j}}+\nabla_i\nabla_{\bar{j}}R)\\ &=\frac{1}{n+1}(\nabla_kR\nabla_{\bar{k}}fg_{i\bar{j}}+R_{k\bar{l}}\nabla_{\bar{k}}\nabla_lfg_{i\bar{j}}+\nabla_iR_{k\bar{j}}\nabla_{\bar{k}}f+R_{k\bar{j}}\nabla_i\nabla_{\bar{k}}f)\\ &=\frac{1}{n+1}\Big[\nabla_kR\nabla_{\bar{k}}fg_{i\bar{j}}+(\lambda R-|Rc|^2)g_{i\bar{j}}+\frac{1}{n+1}\nabla_iR\nabla_{\bar{j}}f\\ &\qquad+\frac{1}{n+1}\nabla_kR\nabla_{\bar{k}}fg_{i\bar{j}}+\lambda R_{i\bar{j}}-R_{i\bar{k}}R_{k\bar{j}}\Big].\end{aligned}\tag{2.8}$$

On the other hand, by differentiating (2.3), we have

$$\begin{aligned}\nabla_k\nabla_{\bar{k}}R_{i\bar{j}}&=\nabla_iR_{\bar{j}l}\nabla_{\bar{l}}f+R_{i\bar{j}k\bar{l}}\nabla_{\bar{k}}\nabla_lf\\ &=\nabla_kR_{i\bar{j}}\nabla_{\bar{k}}f+R_{i\bar{j}k\bar{l}}\nabla_{\bar{k}}\nabla_lf\\ &=\nabla_kR_{i\bar{j}}\nabla_{\bar{k}}f+\lambda R_{i\bar{j}}-R_{i\bar{j}k\bar{l}}R_{\bar{k}l}.\end{aligned}$$

Now, by plugging in formula (2.8), we obtain (2.6).

Next, by taking the divergence on both sides of (2.6), we get

$$\begin{aligned}&\lambda\nabla_iR-(\nabla_iR_{k\bar{l}})R_{\bar{k}l}-R_{i\bar{j}k\bar{l}}\nabla_jR_{\bar{k}l}\\ &=\frac{1}{n+1}\Big[\frac{1}{n+1}\nabla_i\nabla_kR\nabla_{\bar{k}}f+\frac{1}{n+1}\nabla_kR\nabla_i\nabla_{\bar{k}}f+\lambda\nabla_iR-\nabla_i|Rc|^2\\ &\qquad-\frac{n}{n+1}\nabla_j\nabla_iR\nabla_{\bar{j}}f-\frac{n}{n+1}\nabla_iR\Delta f+\lambda\nabla_iR-(\nabla_jR_{i\bar{k}})R_{k\bar{j}}-R_{i\bar{k}}\nabla_kR\Big]\\ &=\frac{1}{n+1}\Big[\frac{1}{n+1}\nabla_iR_{k\bar{l}}\nabla_lf\nabla_{\bar{k}}f+\frac{\lambda}{n+1}\nabla_iR-\frac{1}{n+1}R_{i\bar{k}}\nabla_kR+\lambda\nabla_iR-2R_{k\bar{l}}\nabla_iR_{\bar{k}l}\\ &\qquad-\frac{n}{n+1}\nabla_iR_{j\bar{k}}\nabla_kf\nabla_{\bar{j}}f-\frac{\lambda n^2}{n+1}\nabla_iR+\frac{n}{n+1}R\nabla_iR+\lambda\nabla_iR\\ &\qquad-R_{k\bar{j}}\nabla_iR_{j\bar{k}}-R_{i\bar{k}}\nabla_kR\Big].\end{aligned}$$

That is,

$$\begin{aligned}
&\lambda\nabla_i R-(\nabla_i R_{k\bar l})R_{\bar k l}-R_{i\bar j k\bar l}\nabla_j R_{\bar k l}\\
&=\frac{1}{n+1}[-\frac{n-1}{(n+1)^2}\nabla_i R|\nabla f|^2-\frac{n-1}{(n+1)^2}\nabla_k R\nabla_{\bar k}f\nabla_i f\\
&\qquad+(3-n)\lambda\nabla_i R-(1+\frac{1}{n+1})R_{i\bar k}\nabla_k R-3R_{k\bar l}\nabla_i R_{\bar k l}+\frac{n}{n+1}R\nabla_i R].
\end{aligned}$$

But,

$$\begin{aligned}
R_{l\bar k}\nabla_i R_{k\bar l}&=\frac{1}{n+1}R_{l\bar k}(\nabla_i R g_{k\bar l}+\nabla_k R g_{i\bar l})\\
&=\frac{1}{n+1}R\nabla_i R+\frac{1}{n+1}R_{i\bar j}\nabla_j R
\end{aligned}$$

and

$$\begin{aligned}
R_{i\bar j k\bar l}\nabla_j R_{l\bar k}&=\frac{1}{n+1}R_{i\bar j k\bar l}(\nabla_j R g_{l\bar k}+\nabla_l R g_{j\bar k})\\
&=\frac{1}{n+1}R_{i\bar j}\nabla_j R+\frac{1}{n+1}R_{i\bar l}\nabla_l R\\
&=\frac{2}{n+1}R_{i\bar j}\nabla_j R.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\lambda\nabla_i R-\frac{1}{n+1}R\nabla_i R-\frac{3}{n+1}R_{i\bar j}\nabla_j R\\
&=\lambda\nabla_i R-(\nabla_i R_{k\bar l})R_{\bar k l}-R_{i\bar j k\bar l}\nabla_j R_{\bar k l}\\
&=\frac{1}{n+1}[-\frac{n-1}{(n+1)^2}\nabla_i R|\nabla f|^2-\frac{n-1}{(n+1)^2}\nabla_k R\nabla_{\bar k}f\nabla_i f\\
&\qquad+(3-n)\lambda\nabla_i R-(1+\frac{1}{n+1})R_{i\bar k}\nabla_k R-3R_{k\bar l}\nabla_i R_{\bar k l}+\frac{n}{n+1}R\nabla_i R].
\end{aligned}$$

Therefore, formula (2.7) follows easily. □

Now, suppose that $\nabla f\neq 0$ at some point p. Then we may choose an orthonormal frame $\{e_1,e_2,\cdots,e_n\}$ of holomorphic vector fields at p such that e_1 is parallel to ∇f. Therefore, we have $|\nabla_1 f|=|\nabla f|$ and $\nabla_k f=0$ for $k=2,\cdots,n$.

Lemma 2.3. *Suppose $\nabla f\neq 0$ at p. Then, under the frame $\{e_1,e_2,\cdots,e_n\}$ chosen above, we have*

$$R_{k\bar 1}=R_{1\bar k}=0\quad \textit{for}\quad k\geq 2.$$

Proof. From (2.3) and (2.5), we have at p,

$$R_{i\bar j k\bar 1}\nabla_1 f=\frac{1}{n+1}(\nabla_i R g_{k\bar j}+\nabla_k R g_{i\bar j})=\frac{1}{n+1}(R_{i\bar 1}g_{k\bar j}+R_{k\bar 1}g_{i\bar j})\nabla_1 f.$$

It follows that

$$R_{i\bar j k\bar 1}=\frac{1}{n+1}(R_{i\bar 1}g_{k\bar j}+R_{k\bar 1}g_{i\bar j}).$$

In particular, for $k\geq 2$, we have that

$$R_{1\bar 1 k\bar 1}=\frac{1}{n+1}R_{k\bar 1}\quad\text{and}\quad R_{1\bar k 1\bar 1}=0.$$

However, on the other hand, it is easy to see that

$$R_{1\bar{1}k\bar{1}} = \overline{R_{\bar{1}1\bar{k}1}} = \overline{R_{1\bar{k}1\bar{1}}} = 0.$$

Therefore, $R_{k\bar{1}} = R_{1\bar{k}} = 0$ for $k \geq 2$. □

Lemma 2.3 tells us that ∇f is an eigenvector of the Ricci curvature tensor. Thus we may choose another orthonormal frame $\{w_1 = e_1, w_2, \cdots, w_n\}$ at p such that $|\nabla_1 f| = |\nabla f|$ and the Ricci curvature tensor is diagonalized at p, i.e.

$$R_{i\bar{j}} = R_{i\bar{i}}\delta_{ij}.$$

Proposition 2.4. *Suppose that $\nabla f \neq 0$ at p. Then under the orthonormal frame $\{w_1, w_2, \cdots, w_n\}$ chosen above, we have the following identities at p:*

$$n\lambda R_{1\bar{1}} - RR_{1\bar{1}} = \lambda R - |Rc|^2 - \frac{n-1}{n+1}R_{1\bar{1}}|\nabla f|^2 \tag{2.9}$$

and

$$(n+1)\lambda R_{1\bar{1}} - RR_{1\bar{1}} - R_{1\bar{1}}^2 = -\frac{1}{n+1}R_{1\bar{1}}|\nabla f|^2. \tag{2.10}$$

Proof. In (2.6), setting $i = j = 1$, we have

$$\begin{aligned}
&\lambda R_{1\bar{1}} - \frac{1}{n+1}R_{1\bar{1}}^2 - \frac{1}{n+1}RR_{1\bar{1}} \\
&= \lambda R_{1\bar{1}} - \frac{2}{n+1}R_{1\bar{1}}^2 - \frac{1}{n+1}R_{1\bar{1}}(R - R_{1\bar{1}}) \\
&= \lambda R_{1\bar{1}} - \frac{2}{n+1}R_{1\bar{1}}^2 - \frac{1}{n+1}R_{1\bar{1}}\sum_{k=2}^{n} R_{k\bar{k}} \\
&= \lambda R_{1\bar{1}} - R_{1\bar{1}1\bar{1}}R_{1\bar{1}} - \sum_{k=2}^{n} R_{1\bar{1}k\bar{k}}R_{k\bar{k}} \\
&= \lambda R_{1\bar{1}} - \sum_{k=1}^{n} R_{1\bar{1}k\bar{k}}R_{k\bar{k}} \\
&= \frac{1}{n+1}[\frac{1}{n+1}R_{1\bar{1}}|\nabla f|^2 + \lambda R - |Rc|^2 - \frac{n}{n+1}R_{1\bar{1}}|\nabla f|^2 + \lambda R_{1\bar{1}} - R_{1\bar{1}}^2].
\end{aligned}$$

Thus, formula (2.9) follows immediately.

Next, by setting $i = 1$ in (2.7) and dividing both sides of the equation by $\nabla_1 f$, we get (2.10). □

Proposition 2.5. *At a point p where $\nabla f \neq 0$, we have either*

$$Rc(\nabla f, \nabla f) = 0$$

or

$$Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2.$$

Proof. Since at point p, $\nabla f \neq 0$, formula (2.10) implies that in a neighborhood of p we have

$$\left[(n+1)\lambda - R - \frac{R_{j\bar{i}}\nabla_i f\nabla_{\bar{j}} f}{|\nabla f|^2} + \frac{1}{n+1}|\nabla f|^2\right]\frac{R_{j\bar{i}}\nabla_i f\nabla_{\bar{j}} f}{|\nabla f|^2} = 0. \tag{2.11}$$

Therefore, there are two possibilities:

I) $R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f = 0$ at p

or

II) $R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f \neq 0$ at p. In this case, near p we have

$$-(n+1)\lambda + R + \frac{R_{j\bar{i}}\nabla_i f \nabla_{\bar{j}} f}{|\nabla f|^2} - \frac{1}{n+1}|\nabla f|^2 = 0.$$

Taking the covariant derivative on both sides gives us

$$\begin{aligned}
0 &= \nabla_k R + \frac{1}{|\nabla f|^2}(\nabla_i f \nabla_{\bar{j}} f \nabla_k R_{j\bar{i}} + R_{j\bar{i}}\nabla_i f \nabla_k \nabla_{\bar{j}} f) - \frac{\nabla_j f \nabla_k \nabla_{\bar{j}} f}{|\nabla f|^4} R_{l\bar{i}} \nabla_i f \nabla_{\bar{l}} f \\
&- \frac{1}{n+1}(\nabla_j f \nabla_k \nabla_{\bar{j}} f) \\
&= \nabla_k R + \frac{1}{(n+1)|\nabla f|^2}\nabla_i f \nabla_{\bar{j}} f(\nabla_k R g_{j\bar{i}} + \nabla_j R g_{k\bar{i}}) + \frac{1}{|\nabla f|^2}(\lambda \nabla_k R - R_{k\bar{j}} \nabla_j R) \\
&- \frac{\lambda \nabla_k f - \nabla_k R}{|\nabla f|^4}\nabla_i R \nabla_{\bar{i}} f - \frac{1}{n+1}(\lambda \nabla_k f - \nabla_k R).
\end{aligned}$$

Evaluating the identity above at p under the orthonormal frame $\{w_1, w_2, \cdots, w_n\}$ yields

$$\begin{aligned}
0 = R_{1\bar{1}} &+ \frac{2}{(n+1)|\nabla f|^2} R_{1\bar{1}}|\nabla f|^2 + \frac{1}{|\nabla f|^2}(\lambda R_{1\bar{1}} - R_{1\bar{1}}^2) \\
&\qquad - \frac{\lambda - R_{1\bar{1}}}{|\nabla f|^4} R_{1\bar{1}}|\nabla f|^2 - \frac{1}{n+1}(\lambda - R_{1\bar{1}}) \\
= &\ \frac{n+4}{n+1}R_{1\bar{1}} - \frac{1}{n+1}\lambda.
\end{aligned}$$

Thus, we have $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2$ whenever $Rc(\nabla f, \nabla f) \neq 0$. □

Now we are ready to prove the main theorems.

First, we may assume that $f \neq Const$, for otherwise we get that M is Kähler-Einstein from the soliton equation.

Proof of Theorem 1.3. For steady Kähler-Ricci solitons, we have $\lambda = 0$.

From Proposition 2.2, we know that $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2 = 0$ whenever $Rc(\nabla f, \nabla f) \neq 0$, which is a contradiction. Therefore, we always have $Rc(\nabla f, \nabla f) = 0$. Then (2.9) implies that $Rc = 0$ in the set $\{p \in M | \nabla f(p) \neq 0\}$. On the other hand, by the soliton equation, it is easy to see that we also have $Rc = 0$ in the interior of the set $\{p \in M | \nabla f(p) = 0\}$. Thus the steady soliton M must be Kähler-Ricci flat. □

Proof of Theorem 1.4. For shrinking and expanding Kähler-Ricci solitons, we have $\lambda \neq 0$.

In this case, from Proposition 2.2 and the continuity of $\frac{Rc(\nabla f, \nabla f)}{|\nabla f|^2}$, we conclude that in each component of the open set $A = \{p \in M | \nabla f(p) \neq 0\}$, we have either $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2$ or $Rc(\nabla f, \nabla f) = 0$.

If $Rc(\nabla f, \nabla f) = \frac{\lambda}{n+4}|\nabla f|^2$ in some component Ω of A, then at any point $p \in \Omega$ we have $R_{1\bar{1}} = \frac{\lambda}{n+4}$ and $\nabla R(p) = \frac{\lambda}{n+4}\nabla f(p)$ from formula (2.4). Therefore, we have $\nabla R = \frac{\lambda}{n+4}\nabla f$ in Ω. It then follows that $R = \frac{\lambda}{n+4}f + C$ in Ω. Thus (2.10) implies that $|\nabla f|^2 = \frac{n+1}{n+4}\lambda f + C$ in Ω. Since $R + |\nabla f|^2 - \lambda f = C_0$, we have $f = C_1$ in Ω, which contradicts the fact that $\nabla f \neq 0$ in Ω.

Therefore, we must have $Rc(\nabla f, \nabla f) = 0$ in A. Since $f = Const$ in the interior of $M \backslash A$, we have $Rc(\nabla f, \nabla f) = 0$ on the whole manifold M. It follows that $\nabla R = 0$ on M. Then (2.5) implies that the Ricci curvature tensor is parallel on M. Therefore, by the de-Rahm decomposition theorem, the universal cover of M is isometric to $N^{n-1} \times \mathbb{C}$, where N is again an $(n-1)$-dimensional Kähler-Ricci soliton with harmonic Bochner tensor. Thus by induction, we can finally get that M is isometric to a quotient of the product of a Kähler-Einstein manifold and the complex Euclidean space.

This finishes the proof. $\square$

Acknowledgments

The authors would like to thank their advisor, Professor Huai-Dong Cao, for suggesting this problem and for stimulating discussions. We are grateful for his constant encouragement and support. We would also like to thank the referee for many helpful suggestions and comments.

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