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A NOTE ON THE ALMOST-SCHUR LEMMA ON 4-DIMENSIONAL RIEMANNIAN CLOSED MANIFOLDS

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Abstract. In this short paper, we prove a type of the almost-Schur lemma, introduced by De Lellis-Topping, on 4-dimensional Riemannian closed manifolds assuming no conditions on the Ricci tensor or the scalar curvature.

1. Introduction

Let (M, g) be an n(≥ 3)-dimensinal closed Riemannian manifold. We say that (M, g) is an Einstein manifold if its traceless Ricci tensor is identically zero:

$$Ric_g - \frac{R_g}{n} g = 0\,,$$

where Ric_g and R_g denote the Ricci tensor and the scalar curvature, respectively, of the metric g. The Schur lemma states that every Einstein manifold of dimension $n \geq 3$ has constant scalar curvature. In [2], De Lellis and P. Topping ask to what extent the scalar curvature is constant if the traceless Ricci tensor is assumed to be small rather than identically zero. They proved the following result.

Theorem 1.1 (Almost-Schur Lemma [2])**.** *Let (M, g) be an $n(\geq 3)$-dimensinal closed Riemannian manifold with nonnegative Ricci tensor. Then*

$$\int_M |Ric_g - \frac{\overline{R}_g}{n} g|^2 dv_g \leq \frac{n^2}{(n-2)^2} \int_M |Ric_g - \frac{R_g}{n} g|^2 dv_g\,, \tag{1.1}$$

where $\overline{R}_g = vol_g^{-1} \int_M R_g dv_g$ is the average of the scalar curvature R_g of g.

In [2] they also showed that the constant in inequality (1.1) is optimal and the nonnegativity of the Ricci tensor cannot be removed in general: when $n \geq 5$ they gave examples of metrics on $\mathbb{S}^n$ which make the ratio of the left hand side of (1.1) to the right hand side of (1.1) arbitrarily large; when $n = 3$, they found manifolds which makes the same ratio arbitrarily large. However, in the case of dimension $n = 4$, they left it open. Then, Y. Ge and G. Wang, in [1], showed that Theorem 1.1 holds under the condition of nonnegativity of the scalar curvature for dimension $n = 4$. They proved the following theorem.

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Theorem 1.2 (Almost-Schur Lemma [2])**.** *Let (M,g) be a 4-dimensinal closed Riemannian manifold with nonnegative scalar curvature. Then*

$$\int_M |Ric_g - \frac{\overline{R}_g}{4} g|^2 dv_g \leq 4 \int_M |Ric_g - \frac{R_g}{4} g|^2 dv_g \,, \tag{1.2}$$

where $\overline{R}_g = vol_g(M)^{-1} \int_M R_g dv_g$ is the average of the scalar curvature R_g of g. Moreover, equality in (1.2) holds if and only if (M,g) is an Einstein manifold.

In this note we will prove a type of the almost-Schur lemma in dimension $n=4$ assuming no conditions on the Ricci tensor or the scalar curvature. Our main result is the following.

Theorem 1.3. *Let (M,g) be a 4-dimensinal closed Riemannian manifold. Then*

$$\int_M |Ric_g - \frac{\overline{R}_g}{4} g|^2 dv_g \leq 4 \int_M |Ric_g - \frac{R_g}{4} g|^2 dv_g + 9\lambda_g^2 - \frac{\overline{R}_g}{4} \int_M R_g dv_g \,, \tag{1.3}$$

where $\overline{R}_g = vol_g(M)^{-1} \int_M R_g dv_g$ is the average of the scalar curvature R_g of g and λ_g is the Yamabe invariant. Moreover, equality in (1.3) holds if and only if there exists a metric $g_1 \in [g]$ such that (M,g_1) is an Einstein manifold.

We will note that if the Yamabe invariant λ_g is nonnegative, then

$$9\lambda_g^2 - \frac{\overline{R}_g}{4} \int_M R_g dv_g \leq 0 \,.$$

Hence, as an immediate consequence of Theorem 1.3 we obtain the following corollary.

Corollary 1.4. *Let (M,g) be a 4-dimensinal closed Riemannian manifold with nonnegative Yamabe invariant. Then*

$$\int_M |Ric_g - \frac{\overline{R}_g}{4} g|^2 dv_g \leq 4 \int_M |Ric_g - \frac{R_g}{4} g|^2 dv_g \,, \tag{1.4}$$

where $\overline{R}_g = vol_g(M)^{-1} \int_M R_g dv_g$ is the average of the scalar curvature R_g of g. Moreover, equality in (1.4) holds if and only if (M,g) is an Einstein manifold.

2. Proof of Theorem 1.3

Proof. Let

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g \right)$$

be the Schouten tensor of g. For an integer k with $1 \leq k \leq n$ let σ_k be the k-th elementary symmetric function in $\mathbb{R}^n$. The k-scalar curvature is

$$\sigma_k(g) := \sigma_k(\Lambda_g) \,,$$

where Λ_g is the set of eigenvalues of the matrix $g^{-1} \cdot S_g$. In particular, $\sigma_1(g) = trS_g$ and $\sigma_2 = \frac{1}{2}((trS_g)^2 - |S_g|^2)$. Here, the Yamabe invariant is defined by

$$\lambda_g = \inf_{\overline{g} \in [g]} \frac{\int_M \sigma_1(\overline{g}) dv_{\overline{g}}}{vol_{\overline{g}}(M)^{\frac{n-2}{n}}} \,,$$

where $[g]$ denotes the conformal class of the metric g. Note that the definition of the Yamabe invariant is different from the standard one by a multiple factor $\frac{1}{2(n-1)}$. We can find the following identities:

$$\sigma_1(g) = \frac{R_g}{2(n-1)}\,,$$
$$\sigma_2(g) = \frac{1}{2(n-2)^2}\left(-|Ric_g|^2 + \frac{n}{4(n-1)}R_g^2\right)\,,$$
$$|Ric_g - \frac{R_g}{4}g|^2 = |Ric_g|^2 - \frac{R_g^2}{4}$$

and

$$|Ric_g - \frac{\overline{R}_g}{4}g|^2 = |Ric_g|^2 - \frac{\overline{R}_g}{2}R_g + \frac{\overline{R}_g^2}{4}\,.$$

Hence, we see that inequality (1.3) is equivalent to the following:

$$\frac{8}{3}\int_M \sigma_2(g)dv_g \leq \lambda_g^2\,. \tag{2.1}$$

The proof of (2.1) closely follows an argument given by Gursky in [3] and Ge-Wang in [1]. Let g_1 be a solution of the Yamabe problem. Thus the scalar curvature, and hence $\sigma_1(g_1)$, is constant. First, we have that for any 4×4 symmetric matrix A, the

$$(\sigma_1(A))^2 \geq \frac{8}{3}\sigma_2(A)$$

equality holds if and only if the matrix is a multiple of the identity one. Now the following calculations lead to

$$\frac{8}{3}vol_{g_1}(M)\int_M \sigma_2(g_1)dv_g \leq vol_{g_1}\int_M (\sigma_1(g_1))^2 dv_g = \left(\int_M \sigma_1(g_1)dv_g\right)^2\,,$$

since $\sigma_1(g_1)$ is a constant. Therefore,

$$\frac{8}{3}\int_M \sigma_2(g_1)dv_{g_1} \leq \left(\frac{1}{(vol_{g_1}(M))^{\frac{1}{2}}}\int_M \sigma_1(g_1)dv_{g_1}\right)^2 = \lambda_g^2\,,$$

since g_1 is a Yamabe solution. Now, in the case of dimension $n = 4$, it is well-known that $\int_M \sigma_2(g)dv_g$ is constant in any given conformal class. Hence,

$$\frac{8}{3}\int_M \sigma_2(g)dv_g = \frac{8}{3}\int_M \sigma_2(g_1)dv_{g_1} \leq \lambda_g^2\,,$$

and this proves the inequality (2.1). Finally, observe that if the equality holds in (1.3), then

$$\frac{8}{3}\int_M \sigma_2(g_1)dv_{g_1} = \int_M (\sigma_1(g_1))^2 dv_{g_1}$$

and the Schouten tensor S_{g_1} is proportional to the metric g_1, i.e., g_1 is an Einstein metric. On the other hand, if there exists an Einstein metric $h \in [g]$, we find from [4] that each Yamabe metric g_1 is an Einstein metric. Hence, in this case we obtain that

$$\frac{8}{3}\int_M \sigma_2(g_1)dv_{g_1} = \lambda_g^2\,.$$

Hence,

$$\frac{8}{3}\int_M \sigma_2(g)dv_g = \frac{8}{3}\int_M \sigma_2(g_1)dv_{g_1} = \lambda_g^2,$$

and the equality holds in (1.3). $\square$

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