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ON THE GOOD-λ INEQUALITY FOR NONLINEAR POTENTIALS

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Abstract. This paper concerns an extension of the good-λ inequality for fractional integrals, due to B. Muckenhoupt and R. Wheeden. The classical result is refined in two aspects. Firstly, general nonlinear potentials are considered, and secondly, the constant in the inequality is proven to decay exponentially. As a consequence, the exponential integrability of the gradient of solutions to certain quasilinear elliptic equations is deduced. This in turn is a consequence of certain Morrey space embeddings which extend classical results for the Riesz potential. In addition, the good-λ inequality proved here provides an elementary proof of the result of Jawerth, Perez and Welland regarding the positive cone in certain weighted Triebel-Lizorkin spaces.

1. Introduction

For $n \geq 1$, $0 < \alpha < n$ and $0 < q \leq \infty$, we form the nonlinear potential $\mathbf{T}^q_\alpha(\mu)$ of a nonnegative measure μ, by

$$\mathbf{T}(\mu)(x) = \mathbf{T}^q_\alpha(\mu)(x) = \Big(\int_0^\infty \Big(\frac{\mu(B(x,r))}{r^{n-\alpha}}\Big)^q \frac{dr}{r}\Big)^{1/q}. \tag{1.1}$$

Whenever there is no confusion we will omit the indices in the definition of $\mathbf{T}$. In the case that $q = 1$, we recover (up to a constant) the classical Riesz potential $\mathbf{I}_\alpha(\mu)$, and if $q = \infty$, the resulting operator is the fractional maximal operator $\mathbf{M}_\alpha(\mu)$.

For general $0 < q < \infty$, $\mathbf{T}(\mu)$ can be written as a power of T. Wolff's potential and plays an important role in the study of Sobolev and Triebel-Lizorkin spaces [AH, HW, JPW], as well as in the theory of certain quasilinear equations [KM, DM2, DM3].

It is well known that although the potential operator $\mathbf{T}^q_\alpha(d\mu)$ is not in general pointwise comparable to the fractional maximal operator $\mathbf{M}_\alpha(d\mu)$, a useful substitute is the distributional *good-λ inequality* of Muckenhoupt and Wheeden [MW]. Here we consider an extension and sharpening of the result of [MW], along with applications to certain exponential integrability results. Our primary theorem is the following (for the definition of weak A_∞ weights, see Definition 1.4 below):

Theorem 1.1. *Let $0 < \alpha < n$ and $0 < q < \infty$, and suppose that σ is a weak A_∞ weight. Then there exist constants $\tau > 1$ and C, $c > 0$ depending on n, q, α and*

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the weak A_∞ character of σ, such that for each $\lambda > 0$ and $0 < \epsilon < 1$, it follows that

$$\begin{aligned} &|\{x \in \mathbf{R}^n \,:\, \mathbf{T}(\mu)(x) > \tau\lambda, \mathbf{M}_\alpha(\mu)(x) \le \epsilon\lambda\}|_\sigma \\ &\qquad\qquad \le Ce^{-c/\epsilon^q}|\{x \in \mathbf{R}^n \,:\, \mathbf{T}(\mu)(x) > \lambda\}|_\sigma. \end{aligned} \tag{1.2}$$

Here $|E|_\sigma$ denotes the σ measure of a set E.

The main observation of this paper is the growth of the display (1.2) in the parameter ϵ. Exploiting this improvement in the constant allows us to deduce certain exponential integrability results for nonlinear potentials (Theorem 1.2 below). In the classical inequality of [MW], one replaces the exponential constant in (1.2) with the polynomial factor $\epsilon^{n/(n-\alpha)}$. It is shown below that the inequality (1.2) is sharp in terms of the power of ϵ; see Proposition 2.2. The exact behavior of the parameter ϵ in such distributional inequalities is a topic which has attracted the interest of several authors; see e.g. Banuelos [B1], Chang, Wilson and Wolff [CWW] and the references therein.

To illustrate the principal ideas in the proof of Theorem 1.1, we present the proof first for dyadic operators in the unweighted case. In this setup, one can give a simple upper bound on the exponential constant in (1.2) explicitly; see Theorem 2.1 below.

Theorem 1.1 provides an elementary treatment of several useful inequalities in harmonic analysis and potential theory. Indeed, from integrating out the good-λ inequality in Theorem 1.1 in the standard fashion (see e.g. [MW, AH]), one readily recovers the inequality of Jawerth, Perez and Welland [JPW]. This states that for each weak A_∞ weight σ, and for indices $0 < p < \infty$, $0 < \alpha < n$ and $0 < q < \infty$, it follows that

$$\Big(\int_{\mathbf{R}^n} (\mathbf{T}^q_\alpha(\mu)(x))^p d\sigma\Big)^{1/p} \le C(\sigma, q, \alpha, p)\Big(\int_{\mathbf{R}^n} (\mathbf{M}_\alpha(\mu)(x))^p d\sigma\Big)^{1/p}, \tag{1.3}$$

for all positive measures μ (the case $q > 1$ is contained in the original result of [MW]).

The inequality (1.3) was first proved (for the usual Muckenhoupt A_∞ class) in [JPW] by means of local maximal functions. An alternative approach was found by Cohn and Verbitsky [CV], which employed techniques developed by Wolff [HW].

Furthermore, from Theorem 1.1 one can obtain the natural end point of (1.3) at $q = \infty$. This comes courtesy of the exponential nature of the constant in (1.2) and concerns the exponential integrability of $\mathbf{T}(\mu)$ when μ lies in a suitable local Morrey space. The following theorem extends to nonlinear potential operators certain well-known Morrey space embeddings for Riesz potentials due to D. Adams [A1], who in turn was building on the work of Stampacchia [St1].

Let us fix the notation that for a set $E \subset \mathbf{R}^n$, we define the restriction of a measure μ to E by $d\mu_E = \chi_E d\mu$.

Theorem 1.2. *Let σ be a weak A_∞ weight. There exist finite positive constants $C, c > 0$ depending on n, q, α and the weak A_∞ character of σ, so that*

$$\frac{1}{|2B|_\sigma}\int_{2B} \exp\Big(c\frac{\mathbf{T}^q_\alpha(\mu_B)(x)}{||\mathbf{M}_\alpha(\mu_B)||_{L^\infty(B)}}\Big)^q d\sigma \le C, \tag{1.4}$$

for all balls $B \subset \mathbf{R}^n$ so that $||\mathbf{M}_\alpha(\mu_B)||_{L^\infty(B)} < \infty$.

In display (1.4) and elsewhere, $2B$ denotes the concentric double of B. The exponent in (1.4) is sharp; see Remark 3.1. Theorem 1.2 follows from Theorem 1.1 and a simple localization and is carried out in Section 3. We remark that certain

exponential integrability results for linear integral operators can be obtained via the rearranged good-λ inequality introduced in Bagby and Kurtz [BK] and generalized by Vybíral [Vyb].

In addition, we offer an alternative proof of Theorem 1.2 that goes via a distributional inequality in the parameter τ appearing in (1.2). This may be of independent interest and is the content of the following result:

Theorem 1.3. *Let $0 < \alpha < n$ and $0 < q < \infty$, and suppose that σ is a weak A_∞ weight. Then there exist constants $c > 1$ and $0 < C < 1$, depending on n, q, α and the A_∞ character of σ, such that for each $\lambda > 0$ and $0 < \epsilon < 1$, it follows that*

$$\begin{aligned} &|\{x \in \mathbb{R}^n : (\mathbf{T}^q_\alpha(\mu))^q(x) > (1+c\epsilon)\lambda, (\mathcal{M}_\alpha(\mu))^q(x) \le \epsilon\lambda\}|_\sigma \\ &\qquad\qquad \le C|\{x \in \mathbb{R}^n : (\mathbf{T}^q_\alpha(\mu))^q(x) > \lambda\}|_\sigma. \end{aligned} \tag{1.5}$$

Before we describe further consequences of Theorem 1.1 in terms of regularity of solutions to certain elliptic equations, we turn to the definition of weak A_∞ weights:

Definition 1.4. A nonnegative measurable function σ is said to be a weak A_∞ weight if there exist constants $C_\sigma > 0$ and $\theta > 0$ so that for each cube Q and measurable set $E \subset Q$:

$$\frac{|E|_\sigma}{|2Q|_\sigma} \le C_\sigma \Big(\frac{|E|}{|Q|}\Big)^\theta. \tag{1.6}$$

A constant is said to depend on the weak A_∞ character of σ if it depends on θ and C_σ.

This class of weights was introduced by Sawyer [S1]. Note that weak A_∞ weights may vanish on open sets, and as a result are not necessarily doubling.

By combining Theorem 1.2 with the deep gradient estimates recently proved by Duzaar and Mingione [DM2, DM3] (see Theorem 3.3 below), one can obtain exponential integrability results for the gradient of certain quasilinear equations. For the sake of brevity, we will only consider a simple model equation, but the same results hold for quasilinear operators in the generality considered in [DM2]. For $1 < p < \infty$, let us define the p-Laplacian operator by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Let Ω be a bounded domain, and μ a finite nonnegative measure defined on Ω. We consider a class of very weak solutions[1] u (see Definition 3.2) of the equation

$$-\Delta_p u = \mu \text{ in } \Omega. \tag{1.7}$$

Theorem 1.5. *Suppose u is a very weak solution of* (1.7).

(i) *If $2 - 1/n < p \le 2$, then there exist constants c, $C > 0$ depending on n and p so that*

$$\fint_B \exp\Big(c\frac{|\nabla u|^{p-1}}{||\mathbf{M}_1(\mu_{2B})||_{L^\infty(2B)}}\Big)dx \le C\exp\Big(c\fint_B \frac{|\nabla u|\,dx}{||\mathbf{M}_1(\mu_{2B})||^{\frac{1}{p-1}}_{L^\infty(2B)}}\Big)^{p-1} \tag{1.8}$$

for all balls B so that $2B \subset \Omega$ and $||\mathbf{M}_1(\mu_{2B})||_{L^\infty(2B)} < \infty$.

[1]The reader with some experience with (1.7) can use any reasonable notion of solution here, for example *renormalized solutions* (see [DMMOP]) or *p-superharmonic solutions* (see [HKM]). We make the definition here to reduce technicalities.

(ii) *If $p \geq 2$, then there exist constants C, $c > 0$ depending on n and p so that*

$$\fint_B \exp\Big(c\frac{|\nabla u|^{\frac{p}{2}}}{||\mathbf{M}_1(\mu_{2B})||_{L^\infty(2B)}^{\frac{p}{2(p-1)}}}\Big)dx \leq C\exp\Big(c\fint_B \frac{|\nabla u|^{\frac{p}{2}}\,dx}{||\mathbf{M}_1(\mu_{2B})||_{L^\infty(2B)}^{\frac{p}{2(p-1)}}}\Big) \tag{1.9}$$

for all balls B so that $2B \subset \Omega$ and $||\mathbf{M}_1(\mu_{2B})||_{L^\infty(2B)} < \infty$.

One should compare Theorem 1.5 with the recent Lipschitz regularity results of Cianchi and Maz′ya [CM], and Duzaar and Mingione [DM1]. In these papers, the Lipschitz regularity of a solution u of (1.7) is proved if $d\mu = f(x)dx$ with $f \in L^{n,1}(\Omega)$. Here $L^{n,1}(\Omega)$ is the Lorentz space. In particular, the exponential integrability result Theorem 1.5 holds for all $f \in L^n(\Omega)$.

Using Theorem 1.2, one can also recover known exponential integrability results for the solution u of (1.7) from the estimates of Kilpeläinen and Malý [KM]. The following result (at least when $p \geq 2$) is contained in [Min07]:

Theorem 1.6. *Let $1 < p < n$, and suppose that $u \in W^{1,p-1}_{loc}(\Omega)$ is a positive solution of* (1.7)*. Then there exist constants $c, C > 0$, depending on n and p, so that for all balls $B = B(x,r)$ so that $2B \subset \Omega$:*

$$\fint_B \exp\Big(c\frac{u(x)}{||\mathbf{M}_p(\mu_{2B})||_{L^\infty(2B)}}\Big)dx \leq C\exp\Big(c\fint_B \frac{u}{||\mathbf{M}_p(\mu_{2B})||_{L^\infty(2B)}}dx\Big). \tag{1.10}$$

Additional applications of Theorem 1.2 to certain quasilinear elliptic equations will be presented in [JV]. In these latter applications it will be important that Theorem 1.2 holds for the full class of weak A_∞ weights.

The results of this paper are split into two sections. Firstly, in Section 2 we consider the good-λ inequalities. Subsequently, in Section 3, the results concerning exponential integrability and its consequences to quasilinear elliptic equations are proved.

2. The good-λ inequality

2.1. **The dyadic inequality.** In this section we will prove Theorem 1.1. To illustrate the idea of the proof in the simplest case we will work primarily with the dyadic analogue of the operator $\mathbf{T}$, which we introduce now. Let $\mathcal{Q}$ be the lattice of dyadic cubes in $\mathbf{R}^n$. At each level $k \in \mathbb{Z}$, the dyadic cubes of length 2^k (which we denote by $\mathcal{Q}_k$) are the collection of cubes in $\mathbf{R}^n$ which are the translations by $2^k\lambda$ for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{Z}^n$ of the cube $[0, 2^k)^n$. Then the dyadic lattice $\mathcal{Q}$ is the union of these collections $\mathcal{Q}_k$ over all integers $k \in \mathbb{Z}$.

Given $0 < \alpha < n$ and $0 < q < \infty$, we form the nonlinear homogeneous operator $\mathcal{T}$, acting on a nonnegative Borel measure μ, by

$$\mathcal{T}(\mu)(x) = \Big\{\sum_{x\in Q\in\mathcal{Q}} \Big(\frac{\mu(Q)}{\ell(Q)^{n-\alpha}}\Big)^q\Big\}^{1/q}. \tag{2.1}$$

In the case when $q = \infty$, we denote the resulting operator by $\mathcal{M}_\alpha(\mu)$, the dyadic fractional maximal operator. Our primary result for the dyadic operator is the following, which we state in the unweighted case:

Theorem 2.1. *Let $0 < \alpha < n$ and $0 < q < \infty$. Then there exists a positive constant C, depending on n, q, and α, such that for each $\lambda > 0$ and $0 < \epsilon < 1$, it*

follows that

$$\begin{aligned}|\{x \in \mathbf{R}^n \,:\, \mathcal{T}(\mu)(x) > 2\lambda,\, \mathcal{M}_\alpha(\mu)(x) \le \epsilon\lambda\}| \\ \le C2^{-\frac{\alpha}{\epsilon^q}(2^q-1)}|\{x \in \mathbf{R}^n \,:\, \mathcal{T}(\mu)(x) > \lambda\}|.\end{aligned} \tag{2.2}$$

Proof of Theorem 2.1. We will suppose μ has compact support. Once (2.2) has been established for such measures, the theorem follows by a routine approximation.

First denote $G_\lambda = \{x \,:\, \mathcal{T}(\mu)(x) > \lambda\}$. This is an open set by lower semicontinuity of $\mathcal{T}$. Let us decompose $G_\lambda = \bigcup_j Q_j$, where the Q_j are pairwise disjoint maximal dyadic (Whitney) cubes. Thus, if Q_j' is the dyadic parent cube of Q_j, then there is a point $z \in Q_j'$ so that $\mathcal{T}(\mu)(z) \le \lambda$.

It suffices to prove, for each j, that

$$|\{x \in Q_j \,:\, \mathcal{T}(\mu)(x) > 2\lambda,\ \mathcal{M}_\alpha(\mu)(x) \le \epsilon\lambda\}| \le Ce^{-c/\epsilon^q}|Q_j|, \tag{2.3}$$

for positive constants C and c, depending on n, q, and α. Let us fix such a Q_j, and let j_0 be such that $\ell(Q_j) = 2^{j_0}$. Without loss of generality we may suppose there exists $z \in Q_j$ such that $\mathcal{M}_\alpha(\mu)(z) \le \epsilon\lambda$. Let $S = \{x \in Q_j \,:\, \mathcal{T}(\mu)(x) > 2\lambda,\ \mathcal{M}_\alpha(\mu)(x) \le \epsilon\lambda\}$.

Let us first note that since Q_j is a maximal cube, there is a straightforward estimate on the tail of the potential:

$$\sum_{\substack{Q \supsetneq Q_j \\ Q \in \mathcal{Q}}} \Big(\frac{\mu(Q)}{\ell(Q)^{n-p}}\Big)^q \le \lambda^q. \tag{2.4}$$

The main technical novelty of our proof in comparison to the classical case is the following step, which uses the smallness of the fractional maximal operator somewhat more efficiently. For a fixed $k \in \mathbb{Z}$, we define $g_k(x)$:

$$g_k(x) = \sum_{\substack{x \in Q \\ Q \in \mathcal{Q}_k}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}}. \tag{2.5}$$

Note in particular that the sum in (2.5) is over those cubes of side length 2^k, so there is only one nonzero term in the sum, for any x. Then for any $\gamma > 0$, and $k \le j_0$, it follows that

$$|\{x \in Q_j \,:\, g_k(x) > \gamma\}| \le \frac{\epsilon\lambda}{\gamma} 2^{(k-j_0)\alpha}|Q_j|. \tag{2.6}$$

To see the estimate (2.6), note from Chebyshev's inequality:

$$\begin{aligned}|\{x \in Q_j \,:\, g_k(x) > \gamma\}| &\le \frac{1}{\gamma} \sum_{\substack{Q \in \mathcal{Q}_k \\ Q \subset Q_j}} \frac{\mu(Q)}{\ell(Q)^{n-\alpha}}|Q| \le \frac{1}{\gamma} 2^{k\alpha}\mu(Q_j) \\ &\le \frac{1}{\gamma}|Q_{j_0}|2^{(k-j_0)\alpha}\epsilon\lambda.\end{aligned}$$

In the last inequality we have used the estimate $\mathcal{M}_\alpha(\mu)(z) \le \epsilon\lambda$ for some $z \in Q_j$.

Let $m \in \mathbb{N}$ be chosen later. Then, note that if $x \in Q_j$ and $\mathcal{M}_\alpha(\mu)(x) \le \epsilon\lambda$, it follows from (2.4) that

$$\sum_{\substack{x \in Q, \\ Q \in \mathcal{Q}}} \Big(\frac{\mu(Q)}{\ell(Q)^{n-p}}\Big)^q \le \lambda^q + m(\epsilon\lambda)^q + \sum_{\substack{x \in Q,\, Q \in \mathcal{Q}, \\ \ell(Q) \le 2^{j_0-m}}} \Big(\frac{\mu(Q)}{\ell(Q)^{n-p}}\Big)^q. \tag{2.7}$$

Let $\beta < \alpha$, and note that

$$S \subset Q_j \cap E, \text{ where } E = \bigcup_{k \le j_0 - m}^{\infty} \{g_k \ge \lambda 2^{\beta(k-j_0+m)}((2^q - m\epsilon^q - 1)(1-2^{-q\beta}))^{1/q}\}.$$

This is a simple consequence of (2.7), together with $\mathcal{T}(f)(x) > 2\lambda$ for $x \in S$. We now use the estimate (2.6) to estimate $|S|$. Indeed,

$$\begin{aligned} |S| &\le \sum_{k \le j_0 - m}^{\infty} |Q_j \cap \{g_k \ge \lambda 2^{\beta(k-j_0+m)}[(2^q - m\epsilon^q - 1)(1-2^{-q\beta})]^{1/q}\}| \\ &\le \sum_{k \le j_0 - m}^{\infty} |Q_j| \frac{\epsilon}{[(1-2^{-q\beta})(2^q - m\epsilon^q - 1)]^{1/q}} 2^{\alpha(k-j_0)} 2^{\beta(j_0-m-k)}. \end{aligned}$$

In conclusion,

$$|S| \le 2^{-\alpha m} |Q_j| \frac{\epsilon}{[(1-2^{-q\beta})(2^q - m\epsilon^q - 1)]^{1/q}} \sum_{k=0}^{\infty} 2^{(\beta-\alpha)k}.$$

It remains to make a good choice of m. Let m be

$$m = \lfloor \frac{2^q - 1}{\epsilon^q} - 1 \rfloor.$$

Then, $|S| \le C 2^{-\alpha m} |Q_j|$, where $C = C(n, q, \alpha) > 0$. □

2.2. **On the sharpness of Theorem 2.1.** In this section we present an example to exhibit the sharpness of the good-λ inequality. The example concerns the linear case $q = 1$, but a simple adaptation works for the nonlinear potential. Let us denote $\mathcal{T}_\alpha^1 = \mathcal{I}_\alpha$, as is standard for the dyadic Riesz potential. Let us continue using the notation that $\mathcal{Q}$ be the lattice of dyadic cubes in $\mathbf{R}^n$. Our aim is to prove the following:

Proposition 2.2. *There are constants $c_1, c_2 > 0$, depending on α and n, so that for each $\epsilon > 0$, there exists $f \ge 0$ with*

$$|\{x : \mathcal{I}_\alpha(f)(x) > 2,\ \mathcal{M}_\alpha(f)(x) \le \epsilon\}| \ge c_1 e^{-c_2/\epsilon} |\{x : \mathcal{I}_\alpha(f)(x) > 1\}|. \tag{2.8}$$

Proof. Let $\epsilon > 0$. To begin the proof, let us fix a cube $P \in \mathcal{Q}$. Note that

$$\mathcal{I}_\alpha(\chi_P)(x) \approx \ell(P_x)^{\alpha - n} |P|,$$

where P_x is the smallest dyadic cube containing both P and x. Now, given $N \in \mathbb{N}$ and $\delta > 0$ to be chosen, let $Q^j = [0, 2^j)^n$ for $j = 0, \dots, N$, and define $f(x)$ by

$$f(x) = \delta \chi_{Q^0} + \sum_{j=1}^{N} \delta \frac{\ell(Q^j)^{n-\alpha}}{|Q^j \backslash Q^{j-1}|} \cdot \chi_{Q^j \backslash Q^{j-1}}(x). \tag{2.9}$$

Note that we can write

$$Q^j \backslash Q^{j-1} = \bigcup_{\ell=1}^{(2^n-1)2^{n(j-1)}} Q^{j,\ell},$$

where $Q^{j,\ell} \in \mathcal{Q}$, $\ell(Q^{j,\ell}) = 1$, for all $\ell = 1, \dots, (2^n - 1)2^{n(j-1)}$. It therefore follows that

$$\mathcal{I}_\alpha(f)(x) \approx \delta \ell(Q_x^0)^{\alpha - n} + \delta \sum_{j=1}^{N} \frac{\ell(Q^j)^{n-\alpha}}{|Q^j \backslash Q^{j-1}|} \sum_{\ell=1}^{(2^n-1)2^{n(j-1)}} \ell(Q_x^{j,\ell})^{\alpha - n}. \tag{2.10}$$

Let us define an auxiliary operator $\mathcal{A}$ by

$$\mathcal{A}(x) = \delta \ell(Q_x^0)^{\alpha - n} + \delta \sum_{j=1}^{N} \frac{\ell(Q^j)^{n-\alpha}}{|Q^j \backslash Q^{j-1}|} \sum_{\ell=1}^{(2^n-1)2^{n(j-1)}} \ell(Q_x^{j,\ell})^{\alpha - n}. \tag{2.11}$$

By (2.10) it suffices to prove that, for suitable $c_1, c_2 > 0$:

$$|\{x : \mathcal{A}(x) > 2, \mathcal{M}_\alpha(f)(x) < \epsilon\}| \geq c_1 e^{-c_2/\epsilon} |\{\mathcal{A}(x) > 1\}|. \tag{2.12}$$

First we claim that if $x \in Q^0$, then $\mathcal{A}(x) = (N+1)\delta$. To see the claim, note that by construction $Q_x^{j,\ell} = Q^j$, for all $j \geq 1$. Since $|Q^j \backslash Q^{j-1}| = (2^n - 1)2^{n(j-1)}$ it follows that

$$\mathcal{A}(x) = \delta + \delta \sum_{j=1}^{N} \frac{\ell(Q^j)^{n-\alpha}}{|Q^j \backslash Q^{j-1}|} (2^n - 1) 2^{n(j-1)} \ell(Q^j)^{\alpha - n} = \delta(N+1), \tag{2.13}$$

as required.

Our second claim concerns the decay of the potential in each annulus $Q^k \backslash Q^{k-1}$. We will see that the decay is approximately δ at each step. In particular we claim that if $x \in Q^k \backslash Q^{k-1}$, then

$$\mathcal{A}(x) = \delta \Big(\sum_{j=1}^{k} 2^{-j(n-\alpha)} + \sum_{j=1}^{k-1} 2^{-j\alpha} + \frac{2^n - 2 + 2^{-k\alpha}}{2^n - 1} + (N - k - 1) \Big). \tag{2.14}$$

To prove this claim, let us break up the sum as $\mathcal{A}(x) = I + II + III$. Here

$$I = \delta \ell(Q_x^0)^{\alpha - n} + \delta \sum_{j=1}^{k-1} \frac{\ell(Q^j)^{n-\alpha}}{|Q^j \backslash Q^{j-1}|} \sum_{\ell=1}^{(2^n-1)2^{n(j-1)}} \ell(Q_x^{j,\ell})^{\alpha - n},$$

$$II = \delta \frac{\ell(Q^k)^{n-\alpha}}{|Q^k \backslash Q^{k-1}|} \sum_{\ell=1}^{(2^n-1)2^{n(k-1)}} \ell(Q_x^{k,\ell})^{\alpha - n}, \text{ and}$$

$$III = \delta \sum_{j=k+1}^{N} \frac{\ell(Q^j)^{n-\alpha}}{|Q^j \backslash Q^{j-1}|} \sum_{\ell=1}^{(2^n-1)2^{n(j-1)}} \ell(Q_x^{j,\ell})^{\alpha - n}.$$

To estimate I, note that $Q_x^{j,\ell} = Q^k$ for any $j = 1, \dots, k-1$ and $Q_x^0 = Q^k$. Thus

$$I = \delta 2^{k(\alpha - n)} \sum_{j=0}^{k-1} 2^{j(n-\alpha)} = \delta \sum_{j=1}^{k} 2^{(\alpha - n)j}.$$

Now, we move onto estimate II. Let us first make some elementary observations. Note that there is one cube in $\{Q^{k,\ell}\}_\ell$, such that $\ell(Q_x^{k,\ell}) = 1$, and for each $j = 1, \dots, k-1$, there are $(2^n - 1)2^{n(j-1)} (= |Q^j \backslash Q^{j-1}|)$ cubes in the collection $\{Q^{k,\ell}\}_\ell$,

such that $\ell(Q_x^{k,\ell}) = 2^j$. When $j = k$, there are $(2^n - 2)2^{n(k-1)}$ cubes in $\{Q^{j,\ell}\}_\ell$ with $\ell(Q_x^{k,\ell}) = 2^k$. With this in mind we see that

$$II = \delta \frac{2^{-k\alpha+1}}{2^n - 1} + \delta \sum_{j=1}^{k-1} 2^{\alpha(j-k)} + \delta \frac{2^n - 2}{2^n - 1}.$$

Finally, to estimate III, note that whenever $j > k$, then $Q_x^{j,\ell} = Q^j$. Thus, similarly to (2.13), we see that

$$III = (N - k - 1)\delta, \text{ and hence (2.14) is proved.}$$

We now turn to the size estimates for the fractional maximal operator. If $x \in Q_0$,

$$\mathcal{M}_\alpha(f)(x) = \sup_{0 \le k \le N} \frac{\delta}{2^{k(n-\alpha)}} \sum_{j=0}^{k} 2^{j(n-\alpha)} \le \frac{\delta}{1 - 2^{\alpha - n}}. \tag{2.15}$$

Now, let us choose

$$\delta = \epsilon(1 - 2^{\alpha-n}) \text{ and } N = \left\lfloor \frac{2}{\delta} \right\rfloor.$$

It follows from (2.13), (2.14) and (2.15) that $\{\mathcal{A}(x) > 2,\ \mathcal{M}_\alpha(f)(x) \le \epsilon\} \supset Q^0$, and $\{\mathcal{A}(x) > 1\} \subset Q^{k_0}$ with $k_0 \approx 1/\epsilon$. The proposition follows. □

2.3. **The proof of Theorem 1.1.** In this subsection we show how the dyadic proof above can be modified to obtain Theorem 1.1.

Proof of Theorem 1.1. We will again assume that μ is compactly supported. It will be most convenient to work with the pointwise equivalent operator:

$$\mathbf{F}(\mu) = \Big(\sum_{j \in \mathbb{Z}} \Big(\frac{\mu(B(x, 2^j))}{2^{j(n-\alpha)}} \Big)^q \Big)^{1/q}.$$

Let us denote by $G_\lambda = \{x : \mathbf{F}(\mu)(x) > \lambda\}$. From the Whitney cube decomposition, one obtains disjoint dyadic cubes $\{Q_j\}$, so that

$$\begin{aligned} &(i) \bigcup_j Q_j = G_\lambda,\ (ii) \sum_j \chi_{2Q_j} \le C(n)\chi_{G_\lambda}, \text{ and} \\ &(iii)\ \frac{3}{2} < \frac{\operatorname{dist}(Q_j, \mathbf{R}^n)}{\operatorname{diam}(Q_j)} \le 6. \end{aligned} \tag{2.16}$$

Arguing as in (2.3), we fix such a cube Q_j with side length 2^{j_0}, and we may suppose there exists $z \in Q_j$ so that $\mathbf{M}_\alpha(\mu)(z) \le \epsilon\lambda$. Let us denote $S = \{x \in Q_j : \mathbf{F}(\mu)(x) > \tau\lambda,\ \mathbf{M}_\alpha(\mu)(x) \le \epsilon\lambda\}$. As in (2.4), we see by definition of the Whitney cube that there exists a constant $\tau > 1$ so that if $x \in Q_j$ is such that $M_\alpha(\mu)(x) \le \epsilon\lambda$, then

$$\sum_{j > j_0} \Big(\frac{\mu(B(x, 2^j))}{2^{j(n-\alpha)}} \Big)^q \le \Big(\frac{\tau}{2} \Big)^q \lambda^q. \tag{2.17}$$

Indeed, by the property (iii) of the Whitney decomposition, we find $y \notin Q_j$ so that $d(y, Q_j) \le 6\sqrt{n}\ell(Q_j)$ and $\mathbf{F}(\mu)(y) \le \lambda$. Therefore, with $x \in Q_j$ such that

$M_\alpha(\mu)(x) \le \epsilon\lambda$, it follows, for a constant $A > 0$ depending on dimension, that

$$\begin{aligned}\sum_{j>j_0}\Big(\frac{\mu(B(x,2^j))}{2^{j(n-\alpha)}}\Big)^q &\le \sum_{j_0+A\ge j>j_0}\Big(\frac{\mu(B(x,r))}{2^{j(n-\alpha)}}\Big)^q + \sum_{j>j_0+A}\Big(\frac{\mu(B(x,2^j))}{2^{j(n-\alpha)}}\Big)^q \\ &\le A\epsilon\lambda + C(n,q,\alpha)\sum_{j>j_0}\Big(\frac{\mu(B(y,2^j))}{2^{j(n-\alpha)}}\Big)^q \le \Big(\frac{\tau}{2}\Big)^q\lambda^q.\end{aligned} \tag{2.18}$$

Let us now, for $j \le j_0$, define

$$g_k(x) = 2^{j(\alpha-n)}\mu(B(x,2^j)).$$

Then, by Fubini's theorem and since $k \le j_0$, it follows that for any $z \in Q_j$:

$$\begin{aligned}\int_{Q_j} g_k(x)dx &= \int_{Q_j+B(0,2^k)} 2^{k(\alpha-n)}|B(y,2^k)|d\mu(y) \\ &= 2^{k\alpha}|(Q_j)+2^k|_\mu \le c(n)2^{(k-j_0)\alpha}|Q_j|\mathbf{M}_\alpha(\mu)(z).\end{aligned} \tag{2.19}$$

With z chosen as above, $\mathbf{M}_\alpha(\mu)(z) \le \epsilon\lambda$. Hence, as in the proof of (2.6):

$$|\{x \in Q_j \,:\, g_k > \gamma\}| \le \frac{c(n)}{\gamma}|Q_j|2^{k-j_0}\epsilon\lambda. \tag{2.20}$$

The rest of the proof therefore follows as in the dyadic case. Indeed, for $x \in Q_j$ so that $\mathbf{M}_\alpha(\mu)(x) \le \epsilon\lambda$:

$$\sum_{j\in\mathbb{Z}}\Big(\frac{\mu(B(x,2^j))}{2^{j(n-\alpha)}}\Big)^q \le \Big(\frac{\tau}{2}\Big)^q\lambda^q + m(\epsilon\lambda)^q + \sum_{j\le j_0-m}\Big(\frac{\mu(B(x,2^j))}{2^{j(n-\alpha)}}\Big)^q.$$

Let $\beta < \alpha$. If $x \in S$, then $\mathbf{F}(\mu)(x) > 2\lambda$, and hence

$$S \subset Q_j \cap E, \text{ where}$$

$$E = \bigcup_{k\le j_0-m}^{\infty}\{g_k \ge \lambda 2^{\beta(k-j_0+m)}((\tau^q - m\epsilon^q - (\tau/2)^q)(1-2^{-q\beta}))^{1/q}\}.$$

Using (2.20), one estimates

$$|S| \le 2^{-\alpha m}|Q_j|\frac{c(n)\epsilon}{[(1-2^{-q\beta})(\tau^q - m\epsilon^q - (\tau/2)^q)]^{1/q}}\sum_{k=0}^{\infty}2^{(\beta-\alpha)k}.$$

Now, let m be

$$m = \lfloor\frac{\tau^q(1-2^{-q})}{\epsilon^q} - 1\rfloor.$$

Then, $|S| \le C2^{-\alpha m}|Q_j|$, where $C = C(n,p,\alpha) > 0$. To complete the argument we recall a very standard trick. Thus far, we have asserted the existence constants τ, c, C, depending on n, q and α, such that

$$|\{x \in Q_j \,:\, \mathbf{F}(\mu)(x) > \tau\lambda,\ \mathbf{M}_\alpha(\mu)(x) \le \epsilon\lambda\}| \le Ce^{-c\lambda^q}|Q_j|,$$

for each Whitney cube Q_j. Let σ be a weak A_∞-weight. Then by the definition (1.6), it follows that

$$|\{x \in Q_j \,:\, \mathbf{F}(\mu)(x) > \tau\lambda,\ \mathbf{M}_\alpha(\mu)(x) \le \epsilon\lambda\}|_\sigma \le Ce^{-c\lambda^q}|2Q_j|_\sigma \tag{2.21}$$

for constants c, C now in addition depending on the weak A_∞ character. Using property (ii) of the Whitney decomposition (see (2.16)), we can sum (2.21) to obtain (1.2). □

2.4. **The proof of Theorem 1.3.**

Proof of Theorem 1.3. Once again we assume that μ has compact support. We split the set $\{(\mathbf{T}(\mu))^q(x) > \lambda\}$ into the Whitney cubes satisfying (2.16) as usual. Let us assume that cube Q_j contains x such that $\mathcal{M}_\alpha(\mu)^q(x) < \epsilon\lambda$. Let us put $r_0 = \ell(Q_j)$ and fix a point y such that $(\mathbf{T}(\mu))^q(x) \le \lambda$ and $|x-y| \le Ar_0$, with $A = A(n) > 0$. We write

$$\int_{r_0}^{\infty} \Big(\frac{\mu(B(x,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} \le \sum_{k=0}^{\infty} \int_{2^k r_0}^{2^{k+1} r_0} \Big(\frac{\mu(B(x,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r}. \tag{2.22}$$

First we observe that since $\mathcal{M}_\alpha(\mu)^q(x) < \epsilon\lambda$ we have

$$\int_{2^k r_0}^{2^{k+1} r_0} \Big(\frac{\mu(B(x,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} \le \epsilon\lambda$$

for any k. Next, we see that

$$\begin{aligned} \int_{2^k r_0}^{2^{k+1} r_0} \Big(\frac{\mu(B(x,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} &\le (1+A2^{-k})^{(n-\alpha)q} \int_{(2^k+A)r_0}^{(2^{k+1}+A)r_0} \Big(\frac{\mu(B(y,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} \\ &\le \int_{(2^k+A)r_0}^{(2^{k+1}+A)r_0} \Big(\frac{\mu(B(x,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} + s(k)\epsilon\lambda. \end{aligned} \tag{2.23}$$

Here $s(k) = (1+A2^{-k})^{(n-\alpha)q} - 1$ is a term with geometric decay in k. Summing the above, we obtain

$$\begin{aligned} \int_{r_0}^{\infty} \Big(\frac{\mu(B(x,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} &\le C_s\epsilon\lambda + \sum_{k=0}^{\infty} \int_{(2^k+A)r_0}^{(2^{k+1}+A)r_0} \Big(\frac{\mu(B(y,r)}{r^{n-\alpha}}\Big)^q \frac{dr}{r} \\ &\le C_s\epsilon\lambda + (\mathbf{T}(\mu))^q(y) \le (1+C_s\epsilon)\lambda. \end{aligned}$$

We denote $\mu^j = \mu\chi_{3*Q_j}$ and note that we have $\|\mu^j\| \le C_n(\epsilon\lambda)^{1/q} r_0^{n-\alpha}$ from the fractional maximal operator estimate. Now

$$\begin{aligned} (\mathbf{T}(\mu))^q(x) = \int_0^{r_0} \Big(\frac{\mu(B(x,r))}{r^{n-\alpha}}\Big)^q \frac{dr}{r} &\le \int_0^{r_0} \Big(\frac{\mu^j(B(x,r))}{r^{n-\alpha}}\Big)^q \frac{dr}{r} \\ &\le C_{n,q} r^{\alpha q} (M\mu^j(x))^q. \end{aligned} \tag{2.24}$$

Here M is the Hardy-Littewood maximal operator. From the classical weak type estimate for M, we conclude:

$$\begin{aligned} |\{(\mathbf{T}(\mu^j))^q > c'\epsilon\lambda\} \cap Q_j| &\le |\{Cr_0^\alpha M\mu^j(x) > c'(\epsilon\lambda)^{1/q}\}| \\ &\le \frac{C_n(\epsilon\lambda)^{1/q} r_0^{n-\alpha}}{c'(\epsilon\lambda)^{1/q} C_{n,q}^{-1} r_0^{-\alpha}} \le \frac{C_n}{c' C_{n,q}} |Q_j|. \end{aligned} \tag{2.25}$$

Now, fix σ to be a weak-A_∞ weight. It immediately follows from the definition (1.6) that

$$|\{(\mathbf{T}(\mu))^q > (1+c'\epsilon)\lambda\} \cap Q_j|_\sigma \le C(\sigma)\Big(\frac{C_n}{c' C_{n,q}}\Big)^\theta |2Q_j|_\sigma.$$

Now, by summation of this inequality, using the property (ii) from (2.16), we deduce

$$|\{(\mathbf{T}(\mu))^q > (1+c'\epsilon)\lambda\} \cap \{\mathbf{M}_\alpha \le \epsilon\lambda\}|_\sigma \le C_n C(\sigma)\Big(\frac{C_n}{c' C_{n,q}}\Big)^\theta |\{(\mathbf{T}(\mu))^q > \lambda\}|_\sigma.$$

Picking c' sufficiently large (in terms of α, q, n and the weak A_∞ character of σ) proves the theorem. □

3. Exponential integrability

We begin this section by deducing Theorem 1.2 from Theorem 1.1.

Proof of Theorem 1.2 *from Theorem* 1.1. Let $B = B(x_0, R)$, and without loss of generality let us assume that

$$||\mathbf{M}_\alpha(\mu_B)||_{L^\infty(B)} \leq 1. \tag{3.1}$$

Note that for $x \notin 2B$, it follows that

$$\mathbf{T}(\mu_B)(x) \leq ((n-\alpha)q)^{1/q} \frac{\mu(B)}{R^{n-\alpha}} \leq c(q).$$

Therefore, whenever $\lambda > c(q)$, it follows that

$$\{x \in \mathbf{R}^n \,:\, \mathbf{T}(\mu_B)(x) > \lambda\} \subset 2B.$$

For fixed $\lambda > c(q)$, letting $\epsilon = 1/\lambda$ in (1.2), we deduce that

$$|\{x \in 2B : \mathbf{T}(\mu_B) > 2\lambda\}|_\sigma \leq e^{-c\lambda^q} |\{x \in 2B : \mathbf{T}(\mu_B) > \lambda\}|_\sigma. \tag{3.2}$$

Here we are using the normalization (3.1). Therefore, for all $\lambda > c(q)$:

$$|\{x \in 2B : \mathbf{T}(\mu_B) > 2\lambda\}|_\sigma \leq Ce^{-c\lambda^q} |2B|_\sigma.$$

The theorem follows from integrating this inequality. □

Let us now turn to proving Theorem 1.2 as a result of Theorem 1.3.

Proof of Theorem 1.2 *from Theorem* 1.3. Assume that $\mathbf{M}_\alpha(\mu_B) \leq 1$ and proceed by induction. We first put $\lambda = c$ and we choose $\epsilon = 1/c$. This leads to an estimate

$$|\{x \in \mathbb{R}^n \,:\, (\mathbf{T}(\mu_B))^q(x) > 2\lambda\}|_\sigma \leq C|\{x \in \mathbb{R}^n \,:\, (\mathbf{T}(\mu_B))^q(x) > \lambda\}|_\sigma.$$

Now at level k, we put $\epsilon = 1/(2^k c)$. For $1 \leq l \leq 2^k$ we obtain the inequalities

$$\begin{aligned} &|\{x \in \mathbb{R}^n \,:(\mathbf{T}(\mu_B))^q(x) > (1 + l2^{-k})2^k\lambda\}|_\sigma \\ &\qquad \leq C|\{x \in \mathbb{R}^n \,:\, (\mathbf{T}(\mu_B))^q(x) > (1 + (l-1)2^{-k})2^k\lambda\}|_\sigma. \end{aligned} \tag{3.3}$$

This proves the required exponentional decay of the distribution function. □

Remark 3.1. To exhibit the sharpness of this inequality, let $\alpha = 1$, and consider $\mu = |x|^{-1}\chi_{B(0,1)}$. Then, one can readily compute

$$\mathbf{T}_1^q(\mu)(x) \approx (\log|x|)^{1/q}, \text{ for } x \in B(0,1).$$

Therefore, $\mathbf{T}_1^q(\mu)$ is exponentially integrable to the q-th power, but not to any power greater than q.

Let us now turn to Theorem 1.5. We will first define a suitable notion of solution:

Definition 3.2. Let Ω be a bounded open set. We say that u is a *very weak solution* of (1.7) if

$$u \in L^{\frac{(p-1)n}{n-p},\infty}_{\text{loc}}(\Omega),^2 \text{ and } |\nabla u| \in L^{\frac{(p-1)n}{n-1},\infty}_{\text{loc}}(\Omega),$$

and equation (1.7) holds in the sense of distributions.

[2]By definition, $f \in L^{s,\infty}_{\text{loc}}(\Omega)$ if for each compact set $K \subset \Omega$, there exists a constant $C_K > 0$ so that $\sup_{\lambda>0} \lambda|\{x \in K \,:\, |f(x)| > \lambda\}|^{1/s} \leq C_K$.

That such a solution exists for each finite measure μ is not obvious, and the result can be found in [DMMOP, HKM].[3] There are many open problems around the notions of solution to such quasilinear elliptic equations. Our results for the equation (1.7) are a result of the following recent result of Duzaar and Mingione [DM2].

Theorem 3.3 ([DM2], Theorems 1.1, 1.2 and Remark 1.1)**.** *Suppose that u is a very weak solution of*

$$-\Delta_p u = \mu \text{ in } \Omega$$

for a finite nonnegative measure μ. Then:

(i) *If $1 - 2/n < p < 2$, then there exists a constant $C = C(n,p) > 0$ so that for all balls $B(x,R) \subset \Omega$:*

$$|\nabla u(x)|^{p-1} \leq C\mathbf{I}_1(\mu\chi_{B(x,R)})(x) + C\Big(\int_{B(x,r)} |\nabla u| dz\Big)^{p-1}. \tag{3.4}$$

(ii) *If $p \geq 2$, then there exists a constant $C = C(n,p) > 0$ so that for all balls $B(x,R) \subset \Omega$:*

$$|\nabla u(x)|^{p-1} \leq C\mathbf{T}_1^{\frac{p}{2(p-1)}}(\mu\chi_{B(x,R)})(x) + C\Big(\int_{B(x,r)} |\nabla u|^{\frac{p}{2}} dz\Big)^{\frac{2(p-1)}{p}}. \tag{3.5}$$

Let us now use Theorem 3.3 and Theorem 1.2 to deduce Theorem 1.5.

Proof of Theorem 1.5. Let $B = B(x_0, R)$ be such that $2B \subset \Omega$. Without loss of generality, let us suppose that

$$||\mathbf{M}_\alpha(\mu_{2B})||_{L^\infty(2B)} = 1.$$

From Theorem 3.3, it follows that for all $x \in B(x_0, R)$:

$$|\nabla u(x)|^{p-1} \leq C\mathbf{T}(\mu\chi_{B(x_0,2R)})(x) + C\Big(\fint_{B(x_0,2R)} |\nabla u|^{\max(1,p/2)} dx\Big)^{\frac{p-1}{\max(1,p/2)}}. \tag{3.6}$$

Here $\mathbf{T}$ denotes the relevant operator appearing in Theorem 3.3. From Theorem 1.2:

$$\fint_{2B} \exp\Big(c\frac{\mathbf{T}(\mu\chi_{2B})(x)}{||\mathbf{M}_\alpha(\mu\chi_{2B})||_{L^\infty(2B)}}\Big)^q dx \leq C \tag{3.7}$$

with $q = 1$ if $p < 2$ and $q = p/(2(p-1))$ if $p > 2$. From (3.6) and (3.7) the result follows. □

Remark 3.4. The exponential integrability result (1.8) in the case when $p \leq 2$ is sharp. It seems reasonable to conjecture that one should also have integrability to the $p-1$ power in the bound (1.9). In lieu of Remark 3.1, this bound could only be achieved by potential estimates if the estimate (3.5) was improved to match (3.4). In light of [DM2, DM3], this would be a deep result.

[3] In [DMMOP] the stronger notion of *renormalized solution* is considered.

Proof of Theorem 1.6. The proof of Theorem 1.6 follows from combining Theorem 1.2 with the potential estimate of Kilpeläinen and Malý (see [KM], Theorem 1.6) in an analogous way to the proof of Theorem 1.5. We leave the details to the reader. □

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