PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 140, Number 12, December 2012, Pages 4075–4083
S 0002-9939(2012)11416-9
Article electronically published on April 6, 2012

LOG CANONICAL THRESHOLDS OF QUASI-ORDINARY HYPERSURFACE SINGULARITIES

NERO BUDUR, PEDRO D. GONZÁLEZ-PÉREZ, AND MANUEL GONZÁLEZ VILLA

(Communicated by Lev Borisov)

ABSTRACT. The log canonical thresholds of irreducible quasi-ordinary hypersurface singularities are computed using an explicit list of pole candidates for the motivic zeta function found by the last two authors.

1. Let $f \in \mathbb{C}[x_1, \dots, x_{d+1}]$ be a non-zero polynomial vanishing at the origin in $\mathbb{C}^{d+1}$. Denote by Z the zero locus of f in a small open neighborhood U of the origin. Consider a log resolution $\mu : Y \to U$ of Z that is an isomorphism above the complement of Z, and let E_i for $i \in J$ be the irreducible components of $\mu^{-1}(Z)$. Denote by a_i the order of vanishing of $f \circ \mu$ along E_i, and by k_i the order of vanishing of the determinant of the Jacobian of μ along E_i.

The *log canonical threshold of f at the origin* is defined as

$$\mathrm{lct}_0(f) := \min \left\{ \frac{k_i + 1}{a_i} \mid i \in J \right\}.$$

This is independent of the choice of log resolution. A polynomial f is *log canonical at* 0 if $\mathrm{lct}_0(f) = 1$. The definition of the log canonical threshold extends similarly to the case of a germ of a complex analytic function $f : (\mathbb{C}^{d+1}, 0) \to (\mathbb{C}, 0)$.

The log canonical threshold is an interesting local invariant of the singularities of Z (the smaller the log canonical threshold is, the worse the singularities of Z are) with connections with many other concepts; see [5, 15, 17]. For example, the log canonical threshold of f is the smallest number $c > 0$ such that $|f|^{-2c}$ is not locally integrable. It is also the smallest jumping number of f, the negative of the biggest root of the Bernstein-Sato polynomial of f, and in certain cases it is a spectral number of f. The log canonical threshold can be computed in terms of jet spaces of $\mathbb{C}^{d+1}$ and Z, [19]. Furthermore, the set of log canonical thresholds when d is fixed but f varies is known to satisfy the ascending chain condition, [6].

In this paper we give a formula for the log canonical threshold of an irreducible quasi-ordinary polynomial in terms of the associated characteristic exponents; see Theorem 3.1. This result generalizes the well-known case of plane curve singularities; see Example 3.4. Unlike the curve case, the log canonical threshold of a quasi-ordinary hypersurface can involve the second characteristic exponent, not only the first one.

Received by the editors May 23, 2011.

2010 *Mathematics Subject Classification.* Primary 14B05, 32S45.

Key words and phrases. Log canonical threshold, quasi-ordinary singularity.

The first author is supported by the NSA grant H98230-11-1-0169. The second and third authors are supported by MCI-Spain grant MTM2010-21740-C02.

2. A germ $(Z, 0)$ of an equidimensional complex analytic variety of dimension d is *quasi-ordinary* (q.o.) if there exists a finite projection $\pi : (Z, 0) \to (\mathbb{C}^d, 0)$ that is a local isomorphism outside a normal crossing divisor. If $(Z, 0)$ is a q.o. hypersurface, there is an embedding $(Z, 0) \subset (\mathbb{C}^{d+1}, 0)$ defined by an equation $f = 0$, where $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ is a *q.o. polynomial*, that is, a Weierstrass polynomial in y with discriminant $\Delta_y f$ of the form $\Delta_y f = x^\delta u$, for a unit u in the ring $\mathbb{C}\{x\}$ of convergent power series in the variables $x = (x_1, \dots, x_d)$ and $\delta \in \mathbb{Z}^d_{\geq 0}$. In these coordinates the projection π is the restriction of the projection

$$\mathbb{C}^{d+1} \to \mathbb{C}^d, \quad (x_1, \dots, x_d, y) \mapsto (x_1, \dots, x_d). \tag{1}$$

The Jung-Abhyankar theorem guarantees that the roots of a q.o. polynomial f, called *q.o. branches*, are fractional power series in the ring $\mathbb{C}\{x^{1/m}\}$, for some integer $m \geq 1$; see [1]. Denoting by K the field of fractions of $\mathbb{C}\{x_1, \dots, x_d\}$, if $\tau \in \mathbb{C}\{x_1^{1/m}, \dots, x_d^{1/m}\}$ is a q.o. branch, then the minimal polynomial $F \in K[y]$ of τ over K has coefficients in the ring $\mathbb{C}\{x_1, \dots, x_d\}$ and defines the q.o. hypersurface parametrized by τ.

In this paper we suppose that the germ $(Z, 0)$ is analytically irreducible, that is, the polynomial f is irreducible in $\mathbb{C}\{x_1, \dots, x_d\}[y]$. The geometry of an irreducible q.o. polynomial is often expressed in terms of the combinatorics of the corresponding *characteristic exponents*, which we recall next.

If $\alpha, \beta \in \mathbb{Q}^d$ we consider the preorder relation given by $\alpha \leq \beta$ if $\beta \in \alpha + \mathbb{Q}^d_{\geq 0}$. We also set $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. The notation $\alpha \nleq \beta$ means that the relation $\alpha \leq \beta$ does not hold. In $\mathbb{Q}^d \sqcup \{\infty\}$ we set that $\alpha < \infty$.

Proposition 2.1 ([9, Proposition 1.3])**.** *If $\zeta = \sum c_\lambda x^\lambda \in \mathbb{C}\{x^{1/m}\}$ is a q.o. branch, there exist unique vectors $\lambda_1, \dots, \lambda_g \in \mathbb{Q}^d_{\geq 0}$ such that $\lambda_1 \leq \cdots \leq \lambda_g$ and the three conditions below hold. We set $\lambda_0 = 0$, $\lambda_{g+1} = \infty$, and introduce the lattices $M_0 := \mathbb{Z}^d$, $M_j := M_{j-1} + \mathbb{Z}\lambda_j$, for $j = 1, \dots, g$.*

(i) *We have that $c_{\lambda_j} \neq 0$ for $j = 1, \dots, g$.*
(ii) *If $c_\lambda \neq 0$, then the vector λ belongs to the lattice M_j, where j is the unique integer such that $\lambda_j \leq \lambda$ and $\lambda_{j+1} \nleq \lambda$.*
(iii) *For $j = 1, \dots, g$, the vector λ_j does not belong to M_{j-1}.*

If $\zeta \in \mathbb{C}\{x^{1/m}\}$ is a fractional power series satisfying the three conditions above, then ζ is a q.o. branch.

Definition 2.2. The vectors $\lambda_1, \dots, \lambda_g$ in Proposition 2.1 are called the *characteristic exponents* of the q.o. branch ζ.

We also introduce some numerical invariants associated to the characteristic exponents. We denote by n_j the index $[M_{j-1} : M_j]$ for $j = 1, \dots, g$. We have that $e_0 := \deg_y f = n_1 \dots n_g$ (see [18]). We define inductively the integers e_j by the formula $e_{j-1} = n_j e_j$ for $j = 1, \dots, g$. We set $\ell_0 = 0$. If $1 \leq j \leq g$ we denote by ℓ_j the number of coordinates of λ_j which are different from zero.

We denote by $(\lambda_{j,1}, \dots, \lambda_{j,d})$ the coordinates of the characteristic exponent λ_j with respect to the canonical basis of $\mathbb{Q}^d$, and by $\geq_{\text{lex}}$ the lexicographic order. We assume in this paper that

$$(\lambda_{1,1}, \dots, \lambda_{g,1}) \geq_{\text{lex}} \cdots \geq_{\text{lex}} (\lambda_{1,d}, \dots, \lambda_{g,d}), \tag{2}$$

a condition which holds after a suitable permutation of the variables $x_1, \dots, x_d$.

The q.o. branch ζ is *normalized* if the inequalities (2) hold and if λ_1 is not of the form $(\lambda_{1,1}, 0, \dots, 0)$ with $\lambda_{1,1} < 1$. Lipman proved that if the q.o. branch is not normalized, then there exists a normalized q.o. branch ζ' parametrizing the same germ $(Z, 0)$ (see [9, Appendix]). Lipman and Gau studied q.o. singularities from a topological viewpoint. They proved that the *embedded topological type* of the hypersurface germ $(Z, 0) \subset (\mathbb{C}^{d+1}, 0)$ is classified by the characteristic exponents of a normalized q.o. branch ζ parametrizing $(Z, 0)$; see [9, 18].

3. We introduce the following numbers in terms of the characteristic exponents:

$$A_1 := \frac{1+\lambda_{1,1}}{e_0\lambda_{1,1}},\ A_2 := \frac{n_1(1+\lambda_{2,1})}{e_1(n_1(1+\lambda_{2,1})-1)},\ \text{ and } A_3 := \frac{1+\lambda_{2,\ell_1+1}}{e_1\lambda_{2,\ell_1+1}} \text{ if } \ell_1 < \ell_2.$$

With the above notation our main result is the following:

Theorem 3.1. *Let $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ be an irreducible quasi-ordinary polynomial. We assume that the associated characteristic exponents satisfy* (2)*. Then the log canonical threshold of f at the origin is equal to:*

$$(3)\qquad lct_0(f) = \begin{cases} \min\{1, A_1\} & \text{if } \lambda_{1,1} \neq \frac{1}{n_1}, \quad \text{or if } g = 1, \\ \min\{A_2, A_3\} & \text{if } \lambda_{1,1} = \frac{1}{n_1}, \quad g > 1 \quad \text{and} \quad \ell_1 < \ell_2, \\ A_2 & \text{if } \lambda_{1,1} = \frac{1}{n_1}, \quad g > 1 \quad \text{and} \quad \ell_1 = \ell_2. \end{cases}$$

The number $lct_0(f)$ is determined by the embedded topological type of the germ defined by $f = 0$ at the origin.

Corollary 3.2. *With the hypothesis of Theorem* 3.1*, a singular polynomial f is log canonical if and only if $g = 1$ and either $\lambda_{1,i} \in \{1, \frac{1}{2}\}$ or $\lambda_{1,i} = \frac{1}{n_1}$ for $1 \leq i \leq \ell_1$.*

Remark 3.3. Suppose that $\lambda_1 = (1/n_1, 0, ..., 0)$. By the *inversion formulae* of [18] the germ $(Z, 0)$ is parametrized by a normalized q.o. branch ζ' with characteristic exponents $\lambda'_i = (n_1(1 + \lambda_{i+1,1} - 1/n_1), \lambda_{i+1,2}, \dots, \lambda_{i+1,d})$ for $i = 1, \dots, g-1$, in particular $\lambda'_{i,1} > 1$. If f' is the quasi-ordinary polynomial defined by ζ' we get that $\mathrm{lct}_0(f) = \mathrm{lct}_0(f')$ since both are square-free and define the same germ.

Example 3.4. If $n = 2$ and $f \in \mathbb{C}\{x\}[y]$ defines a singular irreducible plane germ, then $\mathrm{lct}_0(f) = \frac{1+\lambda_1}{e_0\lambda_1}$. This example is well known; see [12]. The log canonical thresholds of plane curve singularities have been considered several times. For example, [16] gave an explicit formula for this invariant in the case of two branches and explained how to compute it for more branches. The case of transversal branches is treated with the help of adjoint ideals in [8]. The general non-reduced case is done in [3]. See also [2].

4. Notation. We introduce a sequence of vectors $\alpha_1, \dots, \alpha_g \in \mathbb{Q}^d_{\geq 0}$ in terms of the characteristic exponents $\lambda_1, \dots, \lambda_g$. We denote by $(\frac{q_1^{(j)}}{p_1^{(j)}}, \dots, \frac{q_d^{(j)}}{p_d^{(j)}})$ the coordinates of α_j in terms of the canonical basis of $\mathbb{Q}^d$, with $\gcd(q_i^{(j)}, p_i^{(j)}) = 1$. The coordinates of α_j are defined inductively by

$$\frac{q_i^{(1)}}{p_i^{(1)}} := \lambda_{1,i} \text{ and } \frac{q_i^{(j)}}{p_i^{(j)}} := p_i^{(1)} \cdots p_i^{(j-1)}(\lambda_{j,i} - \lambda_{j-1,i}).$$

The sequences $\{\lambda_j\}_{j=1}^g$ and $\{\alpha_j\}_{j=1}^g$ determine each other and by Proposition 2.1 we get that $p_i^{(j)}$ divides n_j for $1 \leq i \leq d$.

Definition 4.1. The following formulas define pairs of integers $(B_i^{(j)}, b_i^{(j)})$ for $1 \leq i \leq d$ and $1 \leq j \leq g$:

$$\begin{array}{rclrcl} b_i^{(1)} & := & p_i^{(1)} + q_i^{(1)}, & b_i^{(j)} & := & p_i^{(j)} b_i^{(j-1)} + q_i^{(j)}, \\ B_i^{(1)} & := & e_0 q_i^{(1)}, & B_i^{(j)} & := & p_i^{(j)} B_i^{(j-1)} + e_{j-1} q_i^{(j)}. \end{array}$$

Remark 4.2. Notice that $B_i^{(j)} = 0$ if and only if $\ell_j < i \leq d$ and in that case $b_i^{(j)} = 1$. We also have that $A_1 = \frac{b_1^{(1)}}{B_1^{(1)}}$, $A_2 = \frac{b_1^{(2)}}{B_1^{(2)}}$ and $A_3 = \frac{b_{\ell_1+1}^{(2)}}{B_{\ell_1+1}^{(2)}}$.

5. In this section we give some properties of the set of the quotients $\frac{b_i^{(j)}}{B_i^{(j)}}$.

The following formulas are useful in the discussion below. The first one is a consequence of Proposition 2.1:

$$0 = \ell_0 < \ell_1 \leq \cdots \leq \ell_g \leq d. \tag{4}$$

If $\ell_{j-1} < \ell_j$ we deduce from the inequalities (2) that

$$\lambda_{j,\ell_{j-1}+1} \geq \cdots \geq \lambda_{j,\ell_j} \text{ and } \lambda_{j,\ell_j+1} = \cdots = \lambda_{j,d} = 0. \tag{5}$$

Lemma 5.1. *We have the following inequalities for $1 \leq k \leq g$ and $\ell_{k-1} < i \leq \ell_k$:*

$$\frac{b_{\ell_{k-1}+1}^{(k)}}{B_{\ell_{k-1}+1}^{(k)}} \leq \frac{b_i^{(k)}}{B_i^{(k)}}, \tag{6}$$

$$\frac{1}{e_{k-1}} < \frac{b_i^{(k)}}{B_i^{(k)}}; \tag{7}$$

in addition, if $\frac{q_i^{(k)}}{p_i^{(k)}} > \frac{1}{n_k}$, then we have

$$\frac{b_i^{(k)}}{B_i^{(k)}} \leq \frac{1}{e_k}; \tag{8}$$

if $k < g$ and $\frac{q_i^{(k)}}{p_i^{(k)}} = \frac{1}{n_k}$, then we have

$$\frac{1}{e_k} < \frac{b_i^{(k+1)}}{B_i^{(k+1)}} < \frac{1}{e_{k+1}}; \tag{9}$$

and if $k < g$ and $\frac{q_{\ell_{k-1}+1}^{(k)}}{p_{\ell_{k-1}+1}^{(k)}} = \frac{1}{n_k}$, then we have

$$\frac{b_{\ell_{k-1}+1}^{(k+1)}}{B_{\ell_{k-1}+1}^{(k+1)}} \leq \frac{b_i^{(k+1)}}{B_i^{(k+1)}}. \tag{10}$$

Proof. Notice first that if $\ell_{k-1} < i \leq \ell_k$, then $q_i^{(j)} = 0$ for $1 \leq j < k$; hence we obtain that $b_i^{(k)} = p_i^{(k)} + q_i^{(k)}$ and $B_i^{(k)} = e_{k-1} q_i^{(k)}$. We deduce (6) from (5) and the definitions. We get (7) from the definitions and the inequality

$$\frac{1}{e_{k-1}} < \frac{1}{e_{k-1}} \left(1 + \frac{1}{\frac{q_i^{(k)}}{p_i^{(k)}}} \right) = \frac{1}{e_k} \left(\frac{1}{n_k} + \frac{1}{n_k \frac{q_i^{(k)}}{p_i^{(k)}}} \right) = \frac{b_i^{(k)}}{B_i^{(k)}}. \tag{11}$$

If in addition $\frac{q_i^{(k)}}{p_i^{(k)}} > \frac{1}{n_k}$, then we get that $\frac{q_i^{(k)}}{p_i^{(k)}} \geq \frac{2}{n_k}$. Then we deduce the inequality (8) from the expression for $\frac{b_i^{(k)}}{B_i^{(k)}}$ given at formula (11) by using that $n_k \geq 2$.

If in addition $k < g$ and $\frac{q_i^{(k)}}{p_i^{(k)}} = \frac{1}{n_k}$, we get from the definitions that

$$\frac{b_i^{(k+1)}}{B_i^{(k+1)}} = \frac{1}{e_k}\left(1 + \frac{1}{n_k + \frac{q_i^{(k+1)}}{p_i^{(k+1)}}}\right). \tag{12}$$

This implies that $\frac{1}{e_k} < \frac{b_i^{(k+1)}}{B_i^{(k+1)}}$. By formula (12) and the inequalities $\frac{q_i^{(k+1)}}{p_i^{(k+1)}} \geq \frac{1}{n_{k+1}}$, $n_k, n_{k+1} \geq 2$ and $e_k = n_{k+1}e_{k+1}$, we deduce that

$$\frac{b_i^{(k+1)}}{B_i^{(k+1)}} = \frac{1}{e_{k+1}}\left(\frac{1}{n_{k+1}} + \frac{1}{n_{k+1}(n_k + \frac{q_i^{(k+1)}}{p_i^{(k+1)}})}\right) \leq \frac{1}{e_{k+1}}\left(\frac{1}{n_{k+1}} + \frac{1}{n_{k+1}n_k + 1}\right) < \frac{1}{e_{k+1}}.$$

This proves that inequality (9) holds.

Finally, notice that $\frac{1}{n_k} \leq \frac{q_i^{(k)}}{p_i^{(k)}} \leq \frac{q_{\ell_{k-1}+1}^{(k)}}{p_{\ell_{k-1}+1}^{(k)}}$ by formula (5) and the definitions. If $\frac{q_{\ell_{k-1}+1}^{(k)}}{p_{\ell_{k-1}+1}^{(k)}} = \frac{1}{n_k}$, it follows that $\frac{q_i^{(k)}}{p_i^{(k)}} = \frac{1}{n_k}$. We deduce from this and formula (12) that (10) holds. □

It is easy to see from the inductive definition of the pairs $(b_i^{(k)}, B_i^{(k)})$ that

$$\frac{b_i^{(k)}}{B_i^{(k)}} \leq \frac{b_i^{(k+1)}}{B_i^{(k+1)}} \quad \Leftrightarrow \quad q_i^{(k+1)} e_k b_i^{(k)} \leq q_i^{(k+1)} B_i^{(k)}, \tag{13}$$

for $\ell_{k-1} < i \leq \ell_g$ and $1 \leq k < g$.

Lemma 5.2. *If $\ell_{k-1} < i \leq \ell_k$ and $\frac{q_i^{(k)}}{p_i^{(k)}} > \frac{1}{n_k}$, then the following inequality holds:*

$$\frac{b_i^{(k)}}{B_i^{(k)}} \leq \frac{b_i^{(j)}}{B_i^{(j)}} \quad \textit{for } 1 \leq k \leq j \leq g. \tag{14}$$

Proof. We set $R_j := B_i^{(j)} - e_j b_i^{(j)}$. By the equivalence (13) it is enough to prove that the inequality $R_j \geq 0$ holds for $k \leq j \leq g-1$. We prove this by induction.

For $j = k$ we have the equivalences

$$e_k b_i^{(k)} \leq B_i^{(k)} \Leftrightarrow e_k(p_i^{(k)} + q_i^{(k)}) \leq e_{k-1} q_i^{(k)} \Leftrightarrow p_i^{(k)} + q_i^{(k)} \leq n_k q_i^{(k)} \Leftrightarrow \frac{1}{n_k - 1} \leq \frac{q_i^{(k)}}{p_i^{(k)}}.$$

We deduce that the inequality

$$R_k = B_i^{(k)} - e_k b_i^{(k)} \geq 0 \tag{15}$$

holds since $n_k \geq 2$ and $\frac{q_i^{(k)}}{p_i^{(k)}} \geq \frac{2}{n_k}$ by hypothesis.

Assume that $k < j$ and $R_{j-1} \geq 0$. Using that $e_{j-1} = n_j e_j$, we get the following inequalities:

$$B_i^{(j-1)} - e_j b_i^{(j-1)} \geq B_i^{(j-1)} - e_{j-1} b_i^{(j-1)} = R_{j-1} \geq 0 \tag{16}$$

and

$$R_j = p_i^{(j)}\left(B_i^{(j-1)} - e_j b_i^{(j-1)}\right) + e_j q_i^{(j)}(n_j - 1) \overset{(16)}{\geq} e_j q_i^{(j)}(n_j - 1) \geq 0.$$

This completes the proof. ☐

Lemma 5.3. *If $\ell_{k-1} < i \leq \ell_k$ and if $\frac{q_i^{(k)}}{p_i^{(k)}} = \frac{1}{n_k}$, then the following inequality holds:*

$$\frac{b_i^{(k+1)}}{B_i^{(k+1)}} \leq \frac{b_i^{(j)}}{B_i^{(j)}} \quad \text{for } 1 \leq k \leq j \leq g. \tag{17}$$

Proof. To compare $\frac{b_i^{(k+1)}}{B_i^{(k+1)}}$ and $\frac{b_i^{(k)}}{B_i^{(k)}}$ we use the expressions (12) and (11).

By (13) it is enough to prove that $R_j := B_i^{(j)} - e_j b_i^{(j)} \geq 0$ for $k < j < g$. We prove this by induction on j. The inequality $R_{k+1} \geq 0$ is equivalent to

$$e_{k+1}(p_i^{(k+1)} b_i^{(k)} + q_i^{(k+1)}) \leq p_i^{(k+1)} B_i^{(k)} + e_k q_i^{(k+1)}. \tag{18}$$

By hypothesis we have $B_i^{(k)} = e_{k-1}$ and $b_i^{(k)} = 1 + n_k$; hence (18) holds since $e_{k-1} = n_k e_k = n_k n_{k+1} e_{k+1}$ and $n_k, n_{k+1} \geq 2$.

If $k+1 < j < g$, then we deduce from the induction hypothesis that $R_j \geq 0$ as in Lemma 5.2. ☐

We set

$$\mathcal{B} := \{1\} \cup \{\frac{b_i^{(j)}}{B_i^{(j)}} \mid 1 \leq i \leq \ell_j \text{ and } 1 \leq j \leq g\} \subset \mathbb{Q}_{\geq 0}. \tag{19}$$

Proposition 5.4. *The minimum of the set $\mathcal{B}$ is the number defined by the right-hand side of formula* (3).

Proof. We deal first with the case $\frac{q_1^{(1)}}{p_1^{(1)}} > \frac{1}{n_1}$.

If $1 \leq i \leq \ell_1$ and $\frac{q_i^{(1)}}{p_i^{(1)}} > \frac{1}{n_1}$ we get the following inequalities for $1 \leq j \leq g$:

$$A_1 = \frac{b_1^{(1)}}{B_1^{(1)}} \overset{(6)}{\leq} \frac{b_i^{(1)}}{B_i^{(1)}} \overset{(14)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}.$$

If $1 < i \leq \ell_1$ and $\frac{q_i^{(1)}}{p_i^{(1)}} = \frac{1}{n_1}$ we obtain that

$$A_1 = \frac{b_1^{(1)}}{B_1^{(1)}} \overset{(8)}{\leq} \frac{1}{e_1} \overset{(9)}{<} \frac{b_i^{(2)}}{B_i^{(2)}} \overset{(17)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}, \text{ for } 1 \leq j \leq g.$$

Suppose now that $1 < k \leq j \leq g$ and $\ell_{k-1} < i \leq \ell_k$. We have

$$A_1 = \frac{b_1^{(1)}}{B_1^{(1)}} \overset{(8)}{\leq} \frac{1}{e_1} \overset{k>1}{\leq} \frac{1}{e_{k-1}} \overset{(7)}{<} \frac{b_{\ell_{k-1}+1}^{(k)}}{B_{\ell_{k-1}+1}^{(k)}} \overset{(6)}{\leq} \frac{b_i^{(k)}}{B_i^{(k)}} \overset{(14)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}.$$

Formula (14) in the line above only applies if $\frac{q_i^{(k)}}{p_i^{(k)}} > \frac{1}{n_k}$. Otherwise $\frac{q_i^{(k)}}{p_i^{(k)}} = \frac{1}{n_k}$ and we use that

$$\frac{1}{e_1} \overset{k>1}{<} \frac{1}{e_k} \overset{(7)}{<} \frac{b_i^{(k+1)}}{B_i^{(k+1)}} \overset{(17)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}.$$

This finishes the proof in the case $\frac{q_1^{(1)}}{p_1^{(1)}} > \frac{1}{n_1}$.

We suppose now that $\frac{q_1^{(1)}}{p_1^{(1)}} = \frac{1}{n_1}$. By (5) it follows that $\frac{q_i^{(1)}}{p_i^{(1)}} = \frac{1}{n_1}$ for $1 \leq i \leq \ell_1$. We get the inequalities for $1 \leq i \leq \ell_1$ and $1 \leq j \leq g$,

$$A_2 = \frac{b_1^{(2)}}{B_1^{(2)}} \overset{(10)}{\leq} \frac{b_i^{(2)}}{B_i^{(2)}} \overset{(17)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}. \tag{20}$$

If $\ell_1 < \ell_2$ and $\frac{q_{\ell_1+1}^{(2)}}{p_{\ell_1+1}^{(2)}} > \frac{1}{n_2}$, then we deduce the following inequalities for $2 \leq j \leq g$ and $\ell_1 < i \leq \ell_2$:

$$A_3 = \frac{b_{\ell_1+1}^{(2)}}{B_{\ell_1+1}^{(2)}} \overset{(6)}{\leq} \frac{b_i^{(2)}}{B_i^{(2)}} \overset{(14)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}.$$

If $\ell_1 < \ell_2$ and $\frac{q_{\ell_1+1}^{(2)}}{p_{\ell_1+1}^{(2)}} = \frac{1}{n_2}$, then for $2 \leq j \leq g$ and $\ell_1 < i \leq \ell_2$ we get $\frac{q_i^{(2)}}{p_i^{(2)}} = \frac{1}{n_2}$ and

$$A_2 = \frac{b_1^{(2)}}{B_1^{(2)}} \overset{(9)}{<} \frac{1}{e_2} \overset{(9)}{<} \frac{b_{\ell_1+1}^{(3)}}{B_{\ell_1+1}^{(3)}} \overset{(10)}{\leq} \frac{b_i^{(3)}}{B_i^{(3)}} \overset{(17)}{\leq} \frac{b_i^{(j)}}{B_i^{(j)}}.$$

For $k \geq 3$ and $\ell_{k-1} < i \leq \ell_k$ we have that

$$A_2 = \frac{b_1^{(2)}}{B_1^{(2)}} \overset{(9)}{<} \frac{1}{e_2} \overset{k\geq 3}{\leq} \frac{1}{e_{k-1}} \overset{(7)}{<} \frac{b_i^{(k)}}{B_i^{(k)}}.$$

The remaining candidates for the minimum of $\mathcal{B}$ are discarded by (6), (14), and (17). This completes the proof. □

6. In this paper we use a relation between the log canonical threshold and the poles of the motivic zeta function.

Let f be as in Section 1. The *local motivic zeta function* and the *local topological zeta function of* f of Denef and Loeser (see for example [7]) are

$$Z_{mot,f}(T)_0 := \sum_{\emptyset \neq I \subseteq J} (\mathbb{L}-1)^{|I|-1} [E_I^\circ \cap \mu^{-1}(0)] \cdot \prod_{i \in I} \frac{\mathbb{L}^{-(k_i+1)} T^{a_i}}{1 - \mathbb{L}^{-(k_i+1)} T^{a_i}},$$

$$Z_{top,f}(s)_0 := \sum_{\emptyset \neq I \subseteq J} \chi(E_I^\circ \cap \mu^{-1}(0)) \cdot \prod_{i \in I} \frac{1}{a_i s + k_i + 1},$$

where $E_I^\circ = (\bigcap_{i\in I} E_i) - \bigcup_{i \notin I} E_i$, the symbol $[\cdot]$ represents the class of $\cdot$ in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex algebraic varieties, $\mathbb{L}$ is the class $[\mathbb{A}^1]$, and χ is the Euler-Poincaré characteristic. $Z_{mot,f}(T)_0$ and $Z_{top,f}(s)_0$ are independent of the choice of the log resolution μ. The set of poles of $Z_{mot,f}(\mathbb{L}^{-s})_0$ and the set of poles of $Z_{top,f}(s)_0$ are subsets of $\{-(k_i+1)/a_i \mid i \in J\}$.

To compute the log canonical threshold of irreducible quasi-ordinary singularities we will use the following result.

Theorem 6.1 ([13], p. 18; see also [20], 2.7 and 2.8, or [14], 6.3)**.** *The biggest pole of $Z_{mot,f}(\mathbb{L}^{-s})_0$ is equal to $-lct_0(f)$.*

7. We recall some results obtained by the last two authors in [11]. We use the notation of Section 2 and also the definition of the set $\mathcal{B}$ in formula (19). The following result follows from [11, Corollary 3.17].

Theorem 7.1. *If $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ is an irreducible quasi-ordinary polynomial, then the poles of $Z_{mot,f}(\mathbb{L}^{-s})_0$ are contained in the set $\{-\frac{b}{B} \mid \frac{b}{B} \in \mathcal{B}\}$.*

Remark 7.2. Theorem 7.1 is proved by giving a formula for the motivic zeta function in terms of the contact of the jets of arcs with f. The proof uses the change of variable formula for motivic integrals applied to a particular log resolution of f. This log resolution $\mu\colon Y \to U \subset \mathbb{C}^{d+1}$ is built as a composition of toric modifications in [10]. If $b_i^{(j)}/B_i^{(j)} \in \mathcal{B}$, then there exists an exceptional divisor $E_i^{(j)}$ of this log resolution such that $B_i^{(j)}$ (respectively, $b_i^{(j)}$ minus one) is the order of vanishing of the pull-back of f (respectively, of the determinant of the Jacobian of μ) along $E_i^{(j)}$. This is a consequence of Corollary 3.17, Remark 3.19, and Lemma 9.11 of [11].

Remark 7.3. Notice that the pairs $(B_i^{(j)}, b_i^{(j)}) = (0, 1)$ do not contribute to a candidate pole of $Z_{mot,f}(\mathbb{L}^{-s})_0$. The list of candidate poles indicated in Theorem 7.1 arises also in [4] with a different method; see [11] for a comparison.

8. We prove the main results of this paper:

Proof of Theorem 3.1. Since $\mathrm{lct}_0(f)$ is by definition the minimum of $(k_i + 1)/a_i$ for $i \in J$, it follows from Theorem 6.1, Theorem 7.1, and Remark 7.2 that $\mathrm{lct}_0(f) = \min \mathcal{B}$. The result follows then from Proposition 5.4. ☐

Proof of Corollary 3.2. If f is singular and log canonical, then $1 \leq \frac{b_1^{(1)}}{B_1^{(1)}} = \frac{1}{e_0}(1 + \frac{p_1^{(1)}}{q_1^{(1)}})$. Since $\frac{q_1^{(1)}}{p_1^{(1)}} \geq \frac{1}{n_1}$, we deduce that $e_0 - 1 = n_1 \dots n_g - 1 \leq \frac{p_1^{(1)}}{q_1^{(1)}} \leq n_1$. This implies that $g = 1$. If $n_1 = 2$, there are two possible cases: $\lambda_1 = (1, \dots, 1, 1/2, \dots, 1/2, 0, \dots, 0)$ or $\lambda_1 = (1/2, \dots, 1/2, 0, \dots, 0)$. If $n_1 > 2$, we must have $p_1^{(1)} = n_1$ and $q_1^{(1)} = 1$, since $p_1^{(1)}$ divides n_1. By (5) we get that $\lambda_1 = (\frac{1}{n_1}, \dots, \frac{1}{n_1}, 0, \dots, 0)$. ☐

9. We end this paper with some examples.

Example 9.1. Let $\lambda_1 = (1/3, 1/3)$, $\lambda_2 = (7/6, 2/3)$. A polynomial with these characteristic exponents is for example $f = (z^3 - xy)^2 - x^3y^2z^2$. We have $n_1 = 3$ and $n_2 = 2$. By Theorem 3.1, the log canonical threshold comes from $\lambda_{2,1}$ and equals $A_2 = 13/22$. Indeed, $\frac{b_1^{(1)}}{B_1^{(1)}} = 2/3$, $\frac{b_1^{(2)}}{B_1^{(2)}} = 13/22$, $\frac{b_2^{(2)}}{B_2^{(2)}} = 5/8$, and the minimum of these is $13/22$.

Example 9.2. Let us consider a q.o. polynomial with characteristic exponents $\lambda_1 = (1/2, 1/2, 0)$ and $\lambda_2 = (2/3, 2/3, 11/3)$. For instance $f = (y^2 - x_1x_2)^3 - (y^2 - x_1x_2)x_1^6x_2^6x_3^{11}$. We have that $n_1 = 2$ and $n_2 = 3$ and $\mathcal{B} = \{1, 1/2, 10/21, 14/33\}$. We get $\mathrm{lct}_0(f) = 14/33 = A_3$.

Acknowledgement

The first author would like to thank Johns Hopkins University for its hospitality during the writing of this article.

References

[1] S. S. Abhyankar, On the ramification of algebraic functions, Amer. J. Math. 77 (1955), 575–592. MR0071851 (17:193c)

[2] M. Aprodu and D. Naie, Enriques diagrams and log-canonical thresholds of curves on smooth surfaces. Geom. Dedicata 146 (2010), 43–66. MR2644270 (2011f:14030)

[3] E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo, and A. Melle-Hernández, On the log-canonical threshold for germs of plane curves. *Singularities I,* Contemp. Math., 474, Amer. Math. Soc., Providence, RI, 2008, pp. 1–14. MR2454343 (2009m:32050)

[4] ———, *Quasi-ordinary power series and their zeta functions*, Mem. Amer. Math. Soc. 178 (2005), no. 841, vi+85 pp. MR2172403 (2007d:14005)

[5] N. Budur, Singularity invariants related to Milnor fibers: survey. To appear in *Zeta Functions in Algebra and Geometry*, Contemp. Math., Amer. Math. Soc.

[6] T. de Fernex, L. Ein, and M. Mustaţă, Shokurov's ACC conjecture for log canonical thresholds on smooth varieties. Duke Math. J. 152 (2010), no. 1, 93–114. MR2643057 (2011c:14036)

[7] J. Denef and F. Loeser, Geometry on arc spaces of algebraic varieties, *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 327–348. MR1905328 (2004c:14037)

[8] V. Egorin, *Characteristic varieties of algebraic curves.* Ph.D. Thesis, University of Illinois at Chicago, 2004, 80 pp. MR2705805

[9] Y.-N. Gau, Embedded topological classification of quasi-ordinary singularities, Mem. Amer. Math. Soc. 74 (1988), no. 388, 109–129. With an appendix by Joseph Lipman. MR954948 (89m:14002)

[10] P. D. González Pérez, Toric embedded resolutions of quasi-ordinary hypersurface singularities, Ann. Inst. Fourier (Grenoble) 53 (2003), no. 6, 1819–1881. MR2038781 (2005b:32064)

[11] P. D. González Pérez and M. González Villa, Motivic Milnor fibre of a quasi-ordinary hypersurface. arXiv:1105.2480v1.

[12] J.-i. Igusa, On the first terms of certain asymptotic expansions. *Complex analysis and algebraic geometry,* Iwanami Shoten, Tokyo, 1977, pp. 357–368. MR0485881 (58:5680)

[13] L. H. Halle and J. Nicaise, Motivic zeta functions of abelian varieties, and the monodromy conjecture, Adv. Math. 227 (2011), no. 1, 610–653. MR2782205

[14] ———, Motivic zeta functions for degenerations of abelian varieties and Calabi-Yau varieties. arXiv:1012.4969.

[15] J. Kollár, Singularities of pairs. *Algebraic geometry – Santa Cruz 1995,* Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR1492525 (99m:14033)

[16] T. Kuwata, On log canonical thresholds of reducible plane curves. Amer. J. Math. 121 (1999), no. 4, 701–721. MR1704476 (2001g:14047)

[17] R. Lazarsfeld, *Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 49. Springer-Verlag, Berlin, 2004. MR2095472 (2005k:14001b)

[18] J. Lipman, Topological invariants of quasi-ordinary singularities, Mem. Amer. Math. Soc. 74 (1988), no. 388, 1–107. MR954947 (89m:14001)

[19] M. Mustaţă, Singularities of pairs via jet schemes, J. Amer. Math. Soc. 15 (2002), 599–615. MR1896234 (2003b:14005)

[20] W. Veys and W. Zuniga-Galindo, Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra. Trans. Amer. Math. Soc. 360 (2008), no. 4, 2205–2227. MR2366980 (2008i:11140)

Department of Mathematics, University of Notre Dame, 255 Hurley Hall, South Bend, Indiana 46556

E-mail address: nbudur@nd.edu

ICMAT, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040, Madrid, Spain

E-mail address: pgonzalez@mat.ucm.es

ICMAT, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040, Madrid, Spain – and – Mathematics Center Heidelberg (Match), Universität Heidelberg, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany

E-mail address: mgv@mat.ucm.es

E-mail address: villa@mathi.uni-heidelberg.de