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KAZHDAN'S PROPERTY (T) WITH RESPECT TO NON-COMMUTATIVE L_p-SPACES

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Abstract. We show that a group with Kazhdan's property (T) has property (T_B) for B the Haagerup non-commutative $L_p(\mathcal{M})$-space associated with a von Neumann algebra $\mathcal{M}$, $1 < p < \infty$. We deduce that higher rank groups have property $F_{L_p(\mathcal{M})}$.

1. Introduction

Kazhdan's property (T) of a topological group G is an important rigidity property, defined in terms of the unitary representations of G on Hilbert spaces. We recall the precise definition:

Definition 1.1. A pair (G, H) of topological groups, where H is a closed subgroup of G, is said to have relative property (T) if there exist a compact subset Q of G and $\epsilon > 0$ such that whenever a unitary representation π of G on a Hilbert space $\mathcal{H}$ has a (Q, ϵ)-invariant vector, that is, a vector $\xi \in \mathcal{H}$ such that

$$\sup_{g \in Q} ||\pi(g)\xi - \xi|| < \epsilon ||\xi||,$$

then π has a non-zero $\pi(H)$-invariant vector. The pair (Q, ϵ) is called a Kazhdan pair.

A topological group G is said to have property (T) if the pair (G, G) has relative property (T).

For more details on property (T), see the monograph [2].

The following variant of this property for Banach spaces was recently introduced by Bader, Furman, Gelander and Monod in [1]. Let B be a Banach space and $O(B)$ the orthogonal group of B, that is, the group of linear bijective isometries of B. Recall that an orthogonal representation of a topological group G on a Banach space B is a homomorphism $\rho : G \to O(B)$ such that the map $g \mapsto \rho(g)x$ is continuous for every $x \in B$. If $\rho : G \to O(B)$ is an orthogonal representation of a group G, we denote the subspace of $\rho(G)$-invariant vectors by

$$B^{\rho(G)} = \{x \in B \mid \rho(g)x = x \text{ for all } g \in G \}.$$

Observe that $B^{\rho(G)}$ is invariant under G. The representation ρ is said to almost have invariant vectors if it has a (Q, ϵ)-invariant vector for every compact subset Q of G and $\epsilon > 0$.

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Definition 1.2. Let G be a topological group and H be a closed *normal* subgroup of G. The pair (G, H) has relative property (T_B) for a Banach space B if, for any orthogonal representation $\rho : G \to O(B)$, the quotient representation $\rho' : G \to O(B/B^{\rho(H)})$ does not almost have $\rho'(G)$-invariant vectors.

A topological group G has property (T_B) if the pair (G, G) has relative property (T_B).

The authors of [1] studied the case where B is a superreflexive Banach space, and among other things, they showed that a group which has property (T) has property $(T_{L^p(\mu)})$ for μ a σ-finite measure on a standard Borel space $(X, \mathcal{B})$ and $1 < p < \infty$. We will extend this result to the non-commutative setting.

Non-commutative L_p-spaces were introduced by Dixmier [3] and studied by various authors, among them Yeadon [13] and Haagerup [4] (for a survey on these spaces, see Pisier and Xu [6]). Apart from the standard $L^p(\mu)$-spaces, common examples are the p-Schatten ideals

$$S_p = \{x \in \mathcal{B}(\mathcal{H}) \mid \mathrm{tr}(|x|^p) < \infty\},$$

where $\mathcal{H}$ is a separable Hilbert space.

We review below (in Section 2) Haagerup's definition of these non-commutative L_p-spaces. Here is our main result:

Theorem 1.3. *Let G be a topological group and H a closed normal subgroup of G. Assume that the pair (G, H) has relative property (T). For every von Neumann algebra $\mathcal{M}$, the pair (G, H) has relative property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$.*

In particular, if G has property (T), then G has property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$. Property (T_B) has a stronger version which is a fixed point property for affine actions.

Definition 1.4. Let B be a Banach space. A topological group G has property (F_B) if every continuous action of G by affine isometries on B has a G-fixed point.

The authors of [1] showed that higher rank groups and their lattices have property $(F_{L^p(\mu)})$.

Definition 1.5. For $1 \leq i \leq m$, let k_i be local fields and $\mathbb{G}_i(k_i)$ be the k_i-points of connected simple k_i-algebraic groups $\mathbb{G}_i$. Assume that each simple factor $\mathbb{G}_i$ has k_i-rank ≥ 2. The group $G = \prod_{i=1}^{m} \mathbb{G}_i(k_i)$ is called a higher rank group.

Our next result shows that Theorem B in [1] remains true for non-commutative L_p-spaces.

Theorem 1.6. *Let G be a higher rank group and $\mathcal{M}$ a von Neumann algebra. Then G, as well as every lattice in G, has property $F_{L_p(\mathcal{M})}$ for $1 < p < \infty$.*

Theorem 1.6 was proved by Puschnigg in [7] in the case $L_p(\mathcal{M}) = S_p$. The strategy of the proof of Theorem 1.3 (as in [7]) follows the one from [1]. To achieve the result, we will need some results on the Mazur map and the description of the surjective isometries of $L_p(\mathcal{M})$ given by Sherman in [9].

The paper is organized as follows. In Section 2, useful properties of the Mazur map are established. Group representations on $L_p(\mathcal{M})$ are studied in Section 3. The proof of Theorem 1.3 is given in Section 4. In Section 5, we show how Theorem 1.6 can be obtained from a variant of Theorem 1.3.

2. Some properties of the Mazur map

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and equipped with a normal semi-finite weight φ_0. Let $t \mapsto \sigma_t^{\varphi_0}$ be the one-parameter group of modular automorphisms of $\mathcal{M}$ with respect to φ_0. We denote by $\mathcal{N}_{\varphi_0} = \mathcal{M} \rtimes_{\varphi_0} \mathbb{R}$ the crossed product von Neumann algebra, which is a von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H})$ and generated by the operators $\pi_{\varphi_0}(x)$, $x \in \mathcal{M}$, and λ_s, $s \in \mathbb{R}$, defined by

$$\pi_{\varphi_0}(x)(\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t),$$
$$\lambda_s(\xi)(t) = \xi(t-s) \qquad \text{for any } \xi \in L^2(\mathbb{R}, \mathcal{H}) \text{ and } t \in \mathbb{R}.$$

There is a dual action $s \mapsto \theta_s$ of $\mathbb{R}$ on $\mathcal{N}_{\varphi_0}$. Then let τ_{φ_0} be the semi-finite normal trace on $\mathcal{N}_{\varphi_0}$ satisfying

$$\tau_{\varphi_0} \circ \theta_s = \mathrm{e}^{-s}\tau_{\varphi_0} \text{for all } s \in \mathbb{R}.$$

We denote by $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ the *-algebra of τ_{φ_0}-measurable operators affiliated with $\mathcal{N}_{\varphi_0}$. For $1 \leq p \leq \infty$, the Haagerup non-commutative L_p-space associated with $\mathcal{M}$ is defined by

$$L_p(\mathcal{M}) = \{x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \mid \theta_s(x) = \mathrm{e}^{-s/p}x \text{ for all } s \in \mathbb{R}\}.$$

It is known that this space is independent of the weight φ_0 up to isomorphism. The space $L_1(\mathcal{M})$ is isomorphic to $\mathcal{M}_*$. The identification goes as follows: there exists a normal faithful semi-finite operator-valued weight from $\mathcal{N}_{\varphi_0}$ to $\mathcal{M}$ defined by

$$\Phi_{\varphi_0}(x) = \pi_{\varphi_0}^{-1}\Big(\int_{\mathbb{R}} \theta_s(x)\mathrm{d}s\Big) \ , \text{ for } x \in \mathcal{N}_{\varphi_0}.$$

Now, if $\varphi \in \mathcal{M}_*^+$ and $\hat{\varphi}$ denotes the extension of φ to a a normal weight on $\hat{\mathcal{M}}^+$, the extended positive part of $\mathcal{M}$, we then put

$$\tilde{\varphi}^{\varphi_0} = \hat{\varphi} \circ \Phi_{\varphi_0}.$$

We associate to φ the Radon-Nikodým derivative $\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}$ of $\tilde{\varphi}^{\varphi_0}$ with respect to the trace τ_{φ_0}. This isomorphism between $\mathcal{M}_*^+$ and $L_1(\mathcal{M})^+$ extends to the whole spaces by linearity.

If $x \in L_1(\mathcal{M})$, and φ_x is the element of $\mathcal{M}_*^+$ associated to x, we define a linear functional Tr by

$$\mathrm{Tr}(x) = \varphi_x(1)$$

and we have, p' being the conjugate exponent of p,

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) \text{ for } x \in L_p(\mathcal{M}),\ y \in L_{p'}(\mathcal{M}).$$

For $1 \leq p < \infty$, if $x = u|x|$ is the polar decomposition of $x \in L_p(\mathcal{M})$, we define

$$||x||_p = \mathrm{Tr}(|x|^p)^{1/p}.$$

Equipped with $||.||_p$, $L_p(\mathcal{M})$ is a Banach space. For $1 < p < \infty$, the dual space of $L_p(\mathcal{M})$ is $L_{p'}(\mathcal{M})$ and $L_p(\mathcal{M}, \tau)$ is known to be superreflexive.

We now introduce the Mazur map and establish some of its properties.

Definition 2.1. Let $1 \leq p, q < \infty$. For an operator a, let $\alpha|a|$ be its polar decomposition. The map

$$\begin{aligned} M_{p,q} : & L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \to L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}), \\ & x = \alpha|a| \mapsto \alpha|a|^{\frac{p}{q}} \end{aligned}$$

is called the Mazur map.

We will need the following lemma.

Lemma 2.2. *Let $1 \leq p, q, r < \infty$. Then $M_{r,q} \circ M_{p,r} = M_{p,q}$.*

Proof. Let $\alpha|x|$ be the polar decomposition of $x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. Let $\beta > 0$, and set $y = \alpha|x|^\beta$. We claim that the polar decomposition of y is given by α and $|x|^\beta$. To show this, it suffices to prove that $\overline{\mathrm{Im}(|x|^\beta)} = \overline{\mathrm{Im}(|x|)}$.

By taking orthogonals, we have to show that $\mathrm{Ker}(|x|) = \mathrm{Ker}(|x|^\beta)$ for all $\beta > 0$. Recall that the domain $D(|x|^\beta)$ of $|x|^\beta$ is

$$D(|x|^\beta) = \{\xi \mid \int_0^\infty \lambda^{2\beta} d\mu_\xi(\lambda) < \infty\}.$$

If $\xi \in \mathrm{Ker}(|x|)$, we have for all $\eta \in L^2(\mathbb{R}, \mathcal{H})$,

$$\langle |x|\xi, \eta \rangle = \int_0^\infty \lambda d\mu_{\xi,\eta}(\lambda) = 0.$$

In particular, $\mu_\xi(]0, \infty[) = 0$. So $\xi \in D(|x|^\beta)$ and $\xi \in \mathrm{Ker}(|x|^\beta)$ thanks to

$$\langle |x|^\beta \xi, \eta \rangle = \int_0^\infty \lambda^\beta d\mu_{\xi,\eta}(\lambda) = 0.$$

By exchanging the role of $|x|$ and $|x|^\beta$, we get the equality.

Let $1 \leq p, q, r < \infty$, and $\beta = p/r$; then $M_{p,r}(x) = \alpha|x|^\beta$. It follows from what we have just seen that $M_{r,q}(M_{p,r}(x)) = \alpha|x|^{\frac{p}{q}} = M_{p,q}(x)$. □

Proposition 2.3. *Let $1 \leq p, q < \infty$, and $a \in L_p(\mathcal{M})$. Then*

$$||M_{p,q}(a)||_q^q = ||a||_p^p.$$

Proof. We denote again by $\alpha|a|$ the polar decomposition of a. We have already seen that $|M_{p,q}(a)| = |a|^{\frac{p}{q}}$. So we have

$$\mathrm{Tr}(|M_{p,q}(a)|^q) = \mathrm{Tr}(|a|^p).$$ □

Proposition 2.4. *Let $p, q \in]1, \infty[$ be conjugate. The map*

$$\begin{aligned} & L_p(\mathcal{M}) \to L_q(\mathcal{M}), \\ & x \mapsto M_{p,q}(x)^* \end{aligned}$$

is the duality map from $L_p(\mathcal{M})$ to $L_q(\mathcal{M})$.

Proof. We first notice that $M_{p,q}$ sends $L_p(\mathcal{M})$ into $L_q(\mathcal{M})$. Let $x = \alpha|x| \in L_p(\mathcal{M})$ and $s \in \mathbb{R}$. By uniqueness in the polar decomposition, we have $\theta_s(\alpha) = \alpha$ and $\theta_s(|x|) = \mathrm{e}^{-s/p}|x|$, and then

$$\begin{aligned} \theta_s(M_{p,q}(x)) &= \theta_s(\alpha)\theta_s(|x|^{\frac{p}{q}}) \\ &= \alpha(\theta_s(|x|)^{\frac{p}{q}}) \\ &= \mathrm{e}^{-s/q} M_{p,q}(x). \end{aligned}$$

Thanks to the uniqueness of the duality map in superreflexive spaces, we just have to check that $\mathrm{Tr}(M_{p,q}(a)^*a) = 1$ for a in the unit sphere $S(L_p(\mathcal{M}))$ of $L_p(\mathcal{M})$.

Let $a = \alpha|a| \in S(L_p(\mathcal{M}))$; then $M_{p,q}(a) = \alpha|a|^{\frac{p}{q}}$. Since $\alpha^*\alpha|a| = |a|$, it follows that

$$\mathrm{Tr}(|a|^{\frac{p}{q}}\alpha^*\alpha|a|) = \mathrm{Tr}(|a|^{\frac{p}{q}}|a|) = \mathrm{Tr}(|a|^p) = 1. \qquad \square$$

Proposition 2.5. *If $a, b \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and if e, f are two central projections in $\mathcal{N}_{\varphi_0}$ such that $ef = 0$, then $M_{p,q}(ae + bf) = M_{p,q}(ae) + M_{p,q}(bf)$.*

Proof. As is easily checked, we have

$$|ae + bf| = |a|e + |b|f.$$

Let γ be the partial isometry occurring in the polar decomposition of $ae + bf$, and let $a = \alpha|a|$, $b = \beta|b|$ be the polar decompositions of a and b. We claim that $\gamma = \alpha e + \beta f$. Indeed, we have

$$ae + bf = \gamma|ae + bf|,$$
$$\text{and } ae + bf = (\alpha e)(|a|e) + (\beta f)(|b|f) = (\alpha e + \beta f)|ae + bf|.$$

Since αe is zero on $\mathrm{Ker}(|a|e)$ and βf is zero on $\mathrm{Ker}(|b|f)$, $\alpha e + \beta f$ is zero on $\mathrm{Im}(|ae + bf|)^{\perp} = \mathrm{Ker}(|ae + by|) = \mathrm{Ker}(|a|e) \cap \mathrm{Ker}(|b|f)$ $(ef = 0)$.

Using again the fact that $ef = 0$ and that e, f are central elements, we deduce that

$$\begin{aligned} M_{p,q}(ae + bf) &= (\alpha e + \beta f)|ae + bf|^{\frac{p}{q}} \\ &= (\alpha e + \beta f)(e|a|^{\frac{p}{q}} + f|b|^{\frac{p}{q}}) \\ &= M_{p,q}(ae) + M_{p,q}(bf). \end{aligned} \qquad \square$$

Proposition 2.6. *Let J be a Jordan isomorphism of $\mathcal{N}_{\varphi_0}$, and let $1 \leq p, q < \infty$. Then we have*

$$J(x) = M_{p,q} \circ J \circ M_{q,p}(x) \textit{ for all } x \in \mathcal{N}_{\varphi_0}.$$

Proof. By Lemma 3.2 in [10], we have a decomposition $J = J_1 + J_2$ with the following properties: J_1 is a *-homomorphism, J_2 is a *-anti-homomorphism and $J_1(x) = J(x)e$, $J_2(x) = J(x)f$ for all $x \in \mathcal{M}$, with e, f two orthogonal and central projections such that $e + f = I$.

Observe first that, for $a \in \mathcal{N}_{\varphi_0}$ with $a \geq 0$ and a positive real number r, we have

$$J_1(a^r) = J_1(a)^r$$

and the same is true for J_2.

If α is a partial isometry, then $J_1(\alpha)$ and $J_2(\alpha)$ are partial isometries with initial supports $J_1(\alpha^*\alpha)$ and $J_2(\alpha\alpha^*)$, and final supports $J_1(\alpha\alpha^*)$ and $J_2(\alpha^*\alpha)$ respectively.

Let $x = \alpha|x| \in \mathcal{N}_{\varphi_0}$. Since the supports of J_1 and J_2 are orthogonal, it follows from Proposition 2.5 that

$$\begin{aligned} M_{p,q} \circ J \circ M_{q,p}(x) &= M_{p,q}(J_1(M_{q,p}(x)) + J_q(M_{q,p}(x))) \\ &= M_{p,q}(J_1(M_{q,p}(x))) + M_{p,q}(J_2(M_{q,p}(x))). \end{aligned}$$

Moreover, we have

$$\begin{aligned} M_{p,q}(J_1(M_{q,p}(x))) &= M_{p,q}(J_1(\alpha|x|^{\frac{2}{p}})) \\ &= M_{p,q}(J_1(\alpha)J_1(|x|)^{\frac{2}{p}}) \\ &= J_1(x) \end{aligned}$$

and

$$\begin{aligned} M_{p,q}(J_2(M_{q,p}(x))) &= M_{p,q}(J_2(\alpha|x|^{\frac{2}{p}}\alpha^*\alpha)) \\ &= M_{p,q}(J_2(\alpha)J_2(\alpha|x|^{\frac{2}{p}}\alpha^*)) \\ &= M_{p,q}(J_2(\alpha)J_2((\alpha|x|\alpha^*)^{\frac{2}{p}})) \\ &= M_{p,q}(J_2(\alpha)J_2(\alpha|x|\alpha^*)^{\frac{2}{p}}) \\ &= J_2(x). \end{aligned}$$

□

An essential tool for the proof of Theorem 1.3 is the following result about the local uniform continuity of $M_{p,q}$, which is proved in Lemma 3.2 of [8] (for an independent proof in the case $L_p(\mathcal{M},\tau) = S_p$, see [7]).

Proposition 2.7 ([8])**.** *For $1 \leq p, q < \infty$, the Mazur map $M_{p,q}$ is uniformly continuous on the unit sphere $S(L_p(\mathcal{M}))$.*

3. Group representations on $L_p(\mathcal{M})$

Sherman's description of the surjective isometries of $L_p(\mathcal{M})$ in [9] is a crucial tool in the following result (non-surjective isometries in the semi-finite case, and 2-isometries in the general case are described in [14] and [5] respectively). This will allow us to transfer a representation of a group G on $L_p(\mathcal{M})$ to a representation of G on $L_2(\mathcal{M})$.

Proposition 3.1. *For $p > 2$, and $U \in O(L_p(\mathcal{M}))$, the map $V = M_{p,2} \circ U \circ M_{2,p}$ belongs to $O(L_2(\mathcal{M}))$.*

Proof. The fact that $||V(x)||_2 = ||x||_2$ for all $x \in L_2(\mathcal{M})$ follows from Proposition 2.3, and V is bijective by Lemma 2.2. We have to prove that V is linear on $L_2(\mathcal{M})$.

By Theorem 1.2 in [9], there exist a Jordan isomorphism J of $\mathcal{M}$ and a unitary $w \in \mathcal{M}$ such that

$$U(\varphi^{1/p}) = w(\varphi \circ J^{-1})^{1/p} \text{ for all } \varphi \in \mathcal{M}_*^+.$$

It was shown in [12] that J extends to a Jordan *-isomorphism $\widetilde{J}$ between $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0\circ J^{-1}}, \tau_{\varphi_0\circ J^{-1}})$; moreover, $\widetilde{J}$ is an extension of an isomorphism between $\mathcal{N}_{\varphi_0}$ and $\mathcal{N}_{\varphi_0\circ J^{-1}}$ as well as a homeomorphism for the measure topology on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0\circ J^{-1}}, \tau_{\varphi_0\circ J^{-1}})$. The isomorphism $\widetilde{J}$ satisfies the relations

$$\begin{aligned} &\tau_{\varphi_0} \circ \widetilde{J}^{-1} = \tau_{\varphi_0\circ J^{-1}}, \\ &J^{-1} \circ \Phi_{\varphi_0\circ J^{-1}} = \Phi_{\varphi_0} \circ \widetilde{J}^{-1}. \end{aligned}$$

Lemma 3.2. *For $\varphi \in \mathcal{M}_*^+$, we have*

$$\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}} = \widetilde{J}^{-1}\left(\frac{d\widetilde{\varphi \circ J^{-1}}^{\varphi_0\circ J^{-1}}}{d\tau_{\varphi_0\circ J^{-1}}}\right).$$

Proof. For all $\varphi \in \mathcal{M}_*^+$, we have

$$\begin{aligned}
\tau_{\varphi_0}\Big(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}\ .\ \Big) &= \varphi \circ \Phi_{\varphi_0} \\
&= \varphi \circ J^{-1} \circ \Phi_{\varphi_0 \circ J^{-1}} \circ \widetilde{J} \\
&= \tau_{\varphi_0 \circ J^{-1}}\Big(\frac{d\varphi \circ \tilde{J}^{-1\,\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \widetilde{J}(\ .\ )\Big) \\
&= \tau_{\varphi_0} \circ \widetilde{J}^{-1}\Big(\frac{d\varphi \circ \tilde{J}^{-1\,\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \widetilde{J}(\ .\ )\Big) \\
&= \tau_{\varphi_0}\Big(\widetilde{J}^{-1}\Big(\frac{d\varphi \circ \tilde{J}^{-1\,\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}}\Big)\ .\ \Big)\ ,
\end{aligned}$$

where in the last equality we used the fact that $\widetilde{J}$ is a Jordan homomorphism. □

In Lemma 2.1 in [11], it is shown that there exists a topological $*$-isomorphism $\widetilde{\mathcal{K}}$ between $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$ which satisfies the following relation on the Radon-Nikodým derivatives:

$$\widetilde{\mathcal{K}}\Big(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}\Big) = \frac{d\tilde{\varphi}^{\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \text{ for all } \varphi \in \mathcal{M}_*^+.$$

From Lemma 3.2, we obtain

$$\frac{d\varphi \circ \tilde{J}^{-1\,\varphi_0}}{d\tau_{\varphi_0}} = \widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}\Big(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}\Big) \text{ for all } \varphi \in \mathcal{M}_*^+.$$

As a consequence, the linear and bijective isometry U of $L_p(\mathcal{M})$ is given by the following relation on positive elements:

$$U(x) = w\ (\widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(x)) \text{ for all } x \in L_p(\mathcal{M})^+.$$

This relation extends by linearity to the whole $L_p(\mathcal{M})$.

Now notice that $\widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}$ is a Jordan isomorphism on $\mathcal{N}_{\varphi_0}$ and a topological isomorphism (for the measure topology) on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. By Proposition 2.6, for $x \in \mathcal{N}_{\varphi_0}$, we have

$$\begin{aligned}
V(x) &= M_{p,2} \circ U \circ M_{2,p}(x) \\
&= w(M_{p,2} \circ \widetilde{\mathcal{K}}^{-1} \circ \widetilde{J} \circ M_{2,p}(x)) \\
&= w(\widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(x)).
\end{aligned}$$

Recall from [8] that the Mazur map is continuous for the measure topology on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. So by density of $\mathcal{N}_{\varphi_0}$ in $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ for the measure topology, we have

$$V(x) = w(\widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(x)) \text{ for all } x \in L_2(\mathcal{M}),$$

which gives the linearity of V on $L_2(\mathcal{M})$. □

Remark 3.3. The proof of the linearity of the map V in Proposition 3.1 is simpler in the case where $\mathcal{M}$ is a von Neumann algebra equipped with a faithful semi-finite normal trace τ. Indeed, by Theorem 2 in [14], there exist a Jordan isomorphism

J, a positive operator B commuting with $J(\mathcal{M})$, and a partial isometry W in $\mathcal{M}$ with the property that W^*W is the support of B, such that

$$U(x) = WBJ(x) \text{ for all } x \in \mathcal{M} \cap L_p(\mathcal{M}, \tau).$$

Using the fact that B commutes with $J(\mathcal{M})$, and as in the proof of Proposition 2.6, for all $x = \alpha|x| \in \mathcal{M} \cap L_p(\mathcal{M}, \tau)$, we have

$$\begin{aligned} V(x) &= WM_{p,2}(BJ_1(\alpha|x|^{\frac{p}{2}}) + BJ_2(\alpha|x|^{\frac{p}{2}})) \\ &= WM_{p,2}(BJ_1(\alpha|x|^{\frac{p}{2}})) + WM_{p,2}(BJ_2(\alpha|x|^{\frac{p}{2}})) \\ &= WJ_1(\alpha)B^{\frac{p}{2}}J_1(|x|) + WJ_2(\alpha)B^{\frac{p}{2}}J_2(\alpha|x|\alpha^*) \\ &= WB^{\frac{p}{2}}J(x). \end{aligned}$$

The linearity on the whole $L_p(\mathcal{M}, \tau)$ follows from the density of $\mathcal{M} \cap L_p(\mathcal{M}, \tau)$ in $L_p(\mathcal{M}, \tau)$.

Corollary 3.4. *Let G be a topological group, $p \geq 2$, and $U : G \to O(L_p(\mathcal{M}))$ be a representation on $L_p(\mathcal{M})$. For $g \in G$, define $V(g) : L_2(\mathcal{M}) \to L_2(\mathcal{M})$ by*

$$V(g) = M_{p,2} \circ U(g) \circ M_{2,p}.$$

Then V is a representation of G on $L_2(\mathcal{M})$.

Proof. By the previous proposition, $V(g) \in O(L_2(\mathcal{M}))$ for every g in G. Moreover, the map $g \mapsto V(g)x$ is continuous, since $g \mapsto U(g)M_{2,p}(x)$ is continuous and since $M_{p,2} : L_p(\mathcal{M}) \to L_2(\mathcal{M})$ is continuous.

It remains to check that V is a homomorphism. For this, let $g_1, g_2 \in G$. Then, by Lemma 2.2,

$$\begin{aligned} V(g_1)V(g_2) &= M_{p,2} \circ U(g_1) \circ M_{2,p} \circ M_{p,2} \circ U(g_2) \circ M_{2,p} \\ &= M_{p,2} \circ U(g_1) \circ U(g_2) \circ M_{2,p} \\ &= M_{p,2} \circ U(g_1g_2) \circ M_{2,p} \\ &= V(g_1g_2). \end{aligned}$$

□

Let U be a representation of a topological group G on $L_p(\mathcal{M})$ and let

$$L_p(\mathcal{M})^{U(G)} = \{x \in L_p(\mathcal{M}) \mid U(g)x = x \text{ for all } g \in G\}$$

be the space of $U(G)$-invariant vectors in $L_p(\mathcal{M})$. Let p' be the conjugate of p and U^* the contragredient representation of U on the dual space $L_{p'}(\mathcal{M})$ of $L_p(\mathcal{M})$. Since $L_p(\mathcal{M})$ is superreflexive, there exists a complement $L_p(\mathcal{M})^{'}$ for $L_p(\mathcal{M})^{U(G)}$ (see Proposition 2.6 in [1]), and we have

$$L_p(\mathcal{M})^{'} = \{v \in L_p(\mathcal{M}) \mid \mathrm{Tr}(vc) = 0 \text{ for all } c \in L_{p'}(\mathcal{M})^{U^*(G)}\}.$$

Proposition 3.5. *Let $v \in S(L_p(\mathcal{M})^{'})$. Then*

$$d(v, L_p(\mathcal{M})^{U(G)}) \geq \frac{1}{2}.$$

Proof. Assume, by contradiction, that there exists $b \in L_p(\mathcal{M})^{U(G)}$ such that

$$||v - b||_p < \frac{1}{2}.$$

Then $\frac{1}{2} \leq ||b||_p \leq \frac{3}{2}$. Setting $c = \dfrac{b}{||b||_p}$, we have $||b - c||_p \leq \frac{1}{2}$.

Since $c \in L_p(\mathcal{M})^{U(G)}$, it is easily checked that $M_{p,p'}(c)^* \in L_{p'}(\mathcal{M})^{U^*(G)}$; hence

$$\mathrm{Tr}((c-v)M_{p,p'}(c)^*) = \mathrm{Tr}(cM_{p,p'}(c)^*) = ||c||_p^p = 1.$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned} 1 &= \mathrm{Tr}((c-v)M_{p,p'}(c)^*) \\ &\leq ||c-v||_p ||M_{p,p'}(c)^*||_{p'} \\ &= ||c-v||_p ||c||_p^{\frac{p}{p'}} \\ &= ||c-v||_p. \end{aligned}$$

This implies that

$$\begin{aligned} ||v-b||_p &\geq ||v-c||_p - ||c-b||_p \\ &\geq \frac{1}{2}, \end{aligned}$$

and this is a contradiction. □

4. Proof of Theorem 1.3

We follow the strategy of the proof of Theorem A in [1]. Let $p \in]1,\infty[$, and let U be a representation on $L_p(\mathcal{M})$ of a group G. Let H be a closed subgroup of G such that the pair (G,H) has property (T). We claim that the representation U' of G on the complement $L_p(\mathcal{M})^{'}$ of $L_p(\mathcal{M})^{U(H)}$ has no almost $U'(G)$-invariant vectors. This will prove Theorem 1.3.

Let Q be a compact subset in G, and take $\epsilon > 0$. Assume by contradiction that there exist almost $U(G)$-invariant vectors in $L_p(\mathcal{M})^{'}$. Then, we can find, for every n, a unit vector v_n such that

$$\sup_{g\in Q} ||U(g)v_n - v_n||_p < \frac{1}{n}.$$

By Corollary 3.4, $V = M_{p,2} \circ U \circ M_{2,p}$ defines a representation of G on $L_2(\mathcal{M})$. Let w_n be the orthogonal projection of $M_{p,2}(v_n)$ on the orthogonal complement $L_2(\mathcal{M})^{'}$ of $L_2(\mathcal{M})^{V(H)}$. We claim that w_n is (Q,ϵ)-invariant for V for n sufficiently large. This will contradict property (T) for the pair (G,H).

We first show that there exists $\delta^{'} > 0$ such that

$$d(M_{p,2}(v_n), L_2(\mathcal{M})^{V(H)}) \geq \delta^{'} \text{ for all } n.$$

Indeed, otherwise for some n, there exists $a_k \in L_2(\mathcal{M})^{V(H)}$ such that

$$||M_{p,2}(v_n) - a_k||_2 \xrightarrow[k\to\infty]{} 0.$$

By Proposition 2.3, we have

$$||M_{p,2}(v_n)||_2 = ||v_n||_p^{\frac{p}{2}} = 1.$$

Since $||a_k||_2 \xrightarrow[k\to\infty]{} ||M_{p,2}(v_n)||_2 = 1$, we can assume that $||a_k||_2 = 1$. Notice that

$$M_{2,p}(L_2(\mathcal{M})^{V(H)}) = L_p(\mathcal{M})^{U(H)}.$$

Hence, $M_{2,p}(a_k)$ belongs to $L_p(\mathcal{M})^{U(H)}$ for every k. Moreover

$$||v_n - M_{2,p}(a_k)||_p \xrightarrow[k\to\infty]{} 0$$

by the uniform continuity of $M_{2,p}$ on the unit sphere (see Proposition 2.4). This is a contradiction to Proposition 3.5.

In particular, we have

$$||w_n||_2 = d(M_{p,2}(v_n), L_2(\mathcal{M})^{V(H)}) \geq \delta^{'}.$$

For $g \in Q$, we have

$$\begin{aligned}||V(g)w_n - w_n||_2 &\leq ||V(g)M_{p,2}(v_n) - M_{p,2}(v_n)||_2 \\ &= ||M_{p,2}(U(g)v_n) - M_{p,2}(v_n)||_2.\end{aligned}$$

Recall that $||v_n||_p^{\frac{p}{2}} = 1$ and that

$$\sup_{g\in Q} ||U(g)v_n - v_n||_p < \frac{1}{n}.$$

Hence, by the uniform continuity of $M_{p,2}$ on $S(L_2(\mathcal{M}))$, there exists an integer N (depending only on (Q, ϵ)) such that

$$\sup_{g\in Q} ||V(g)w_n - w_n||_2 < \epsilon\delta^{'} \text{ for } n \geq N.$$

Since $||w_n||_2 \geq \delta^{'}$, it follows that

$$\sup_{g\in Q} ||V(g)w_n - w_n||_2 < \epsilon||w_n||_2 \text{ for } n \geq N.$$

This shows that w_n is (Q, ϵ)-invariant for U when $n \geq N$. This finishes the proof of Theorem 1.3.

5. Property $(F_{L_p(\mathcal{M})})$ for higher rank groups

Let H be a closed normal subgroup of G and let L be a closed group of G. Assume that $G = L \ltimes H$. The following strong relative property (T_B) was considered in [1]:

Definition 5.1. A pair $(L \ltimes H, H)$ has property (T_B) if, for any orthogonal representation $\rho : L \ltimes H \to O(B)$, the quotient representation $\rho' : L \to O(B/B^{\rho(H)})$ does not almost have $\rho'(L)$-invariant vectors.

A straightforward modification of our proof of Theorem 1.3 shows that we also have the following result:

Theorem 5.2. *Let $(L \ltimes H, H)$ be a pair with strong relative property (T). Then $(L \ltimes H, H)$ has strong relative property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$.*

Let G be a higher rank group as defined in the introduction. Using an analogue of Howe-Moore's theorem on the vanishing of matrix coefficients, the authors of [1] showed that G has property (F_B) whenever B is a superreflexive Banach space and a certain pair $(L \ltimes H, H)$ of subgroups, which has property (T), has also (T_B). The property $(F_{L_p(\mathcal{M})})$ for higher rank groups in Theorem 1.6 is then a consequence of Theorem 5.2. Moreover, the result for lattices in higher rank groups is obtained by an induction process exactly as in Proposition 8.8 of [1].

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