ON PREHOMOGENEITY OF A RANK VARIETY

MASAYA OUCHI, MICHIO HAMADA, AND TATSUO KIMURA

(Communicated by Lev Borisov)

Abstract. If a linear algebraic group $G$ acts on $M(m,n)$, then it also acts on a rank variety $M^{(r)}(m,n) = \{X \in M(m,n) \mid \text{rank } X = r\}$. In this paper, we give the necessary and sufficient condition that this variety has a Zariski-dense $G$-orbit. We consider everything over the complex number field $\mathbb{C}$.

INTRODUCTION

Let $G$ be a linear algebraic group. Let $\rho : G \to GL(m)$ and $\sigma : G \to GL(n)$ be its rational representations over $\mathbb{C}$. Then $G$ acts on $M(m,n)$ by $\rho \otimes \sigma$, i.e., $X \mapsto \rho(g)X^t \sigma(g) \ (X \in M(m,n), \ g \in G)$. By this action, $G$ also acts on a rank variety $M^{(r)}(m,n) = \{X \in M(m,n) \mid \text{rank } X = r\}$. On the other hand, if a rational representation $\tau : H \to GL(V)$ of a linear algebraic group $H$ on a finite-dimensional vector space $V$ has a Zariski-dense $H$-orbit, we call a triplet $(H, \tau, V)$ a prehomogeneous vector space (abbrev. PV). A point of the Zariski-dense $H$-orbit is called a generic point, and the isotropy subgroup at a generic point is called a generic isotropy subgroup. For the basic facts of PVs, see [K]. In this paper, we shall prove the following theorem.

Theorem 0.1. The following assertions are equivalent.

1. $M^{(r)}(m,n)$ has a Zariski-dense $G$-orbit by the action $\rho \otimes \sigma$.
2. $(G \times GL(r), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*, \ M(m,r) \oplus M(n,r))$ is a PV.

Here the action of (2) is given by $(X,Y) \mapsto (\rho(g)X^tA, \sigma(g)YA^{-1})$ for $(X,Y) \in M(m,r) \oplus M(n,r))$ and $(g,A) \in G \times GL(r)$. Note that we have $m \geq r$ and $n \geq r$ in (2).

1. Proof of theorem

The following lemma is the key for our proof.

Lemma 1.1 (M. Sato). Assume that an algebraic group $G$ acts on both of two irreducible algebraic varieties $W$ and $W'$. Let $\varphi : W \to W'$ be a morphism satisfying

1. $\varphi(gw) = g\varphi(w) \ (g \in G, w \in W),$
2. $\varphi(W) = W'.

Received by the editors May 26, 2011.

2010 Mathematics Subject Classification. Primary 11S90; Secondary 15A03.

Key words and phrases. Rank, Zariski-dense orbit, prehomogeneous vector space.
Then the following assertions are equivalent:

(1) \( W = \overline{G \cdot w} \) for some \( w \in W \); that is, \( W \) is \( G \)-prehomogeneous.
(2) (a) \( W' = \overline{G \cdot w'} \) for some \( w' \in W' \).
   (b) For the above point \( w' \in W' \) in (a), there exists a point \( w \in \varphi^{-1}(w') \) such that \( \varphi^{-1}(w') = \overline{Gw \cdot w} \), where \( Gw = \{ g \in G : gw' = w' \} \) is the isotropy subgroup of \( G \) at \( w' \).

Proof. For the proof, see Lemmas 7.2 and 7.6 in [K]. \( \square \)

Now we shall prove Theorem 0.1. Put \( W = \{(X, Y) \in M(m, r) \oplus M(n, r) | \text{rank } X = \text{rank } Y = r \} \). Then clearly \((G \times GL(r), \rho \otimes \Lambda_1 + \sigma \otimes \Lambda_1^*) \), \( M(m, r) \oplus M(n, r) \) is a PV if and only if \( W \) is \( G \times GL(r) \)-prehomogeneous. Note that \( X'Y = r \) for any \((X, Y) \in W \). Hence we have a \((G \times GL(r))\)-equivariant map \( \varphi : W \to M(\rho), (m, n) \) by \((X, Y) \to X'Y \). Note that \( GL(m) \times GL(n) \) also acts on both \( W \) and \( M(\rho), (m, n) \). Since \( \varphi \) is also \((GL(m) \times GL(n))\)-equivariant and \( M(\rho), (m, n) \) is a single \((GL(m) \times GL(n))\)-orbit, we see that \( \varphi \) is surjective. For \( Z_0 = \varphi(X_0, Y_0) \in M(v), (m, n) \), we shall show that the fiber \( \varphi^{-1}(Z_0) \) is a principal \( GL(r)\)-orbit. Assume that \((X, Y) \in \varphi^{-1}(Z_0) \). If \( X = (x_1, \ldots, x_r) \in M(m, r) \), let \( \langle X \rangle \) be an \( r \)-dimensional subspace of \( \mathbb{C}^m \) generated by column vectors \( x_1, \ldots, x_r \). Since we have \( X'Y = (\sum_{j=1}^{r} y_{ij} x_j : \ldots : \sum_{j=1}^{r} y_{nj} x_j) \) for \( Y = (y_{ij}) \) and rank \( X = \text{rank } X'Y = r \), we have \( \langle X \rangle = \langle X'Y \rangle = \langle Z_0 \rangle = \langle X_0, Y_0 \rangle = \langle X_0 \rangle \), and hence there exists uniquely \( A \in GL(r) \) satisfying \( X = X_0 A^t \). Then we have \( X_0 A^t Y_0 = X_0 A^t Y_0 = X_0 A^t Y_0 \) and since rank \( X_0 = r \), we obtain \( Y_0 = A^t Y_0 \), i.e., \( Y = Y_0 A^{-1} \). This shows that the fiber \( \varphi^{-1}(Z_0) \) is a principal \( GL(r)\)-orbit. Since \( GL(r) \) is contained in the isotropy subgroup of \( Z_0 \), by Lemma 1.4 we obtain Theorem 0.1.

Note that (2) in Theorem 0.1 is equivalent to:
(3) \((G \times GL(r), \rho \otimes \Lambda_1, M(m, r)) \) is a PV with a generic isotropy subgroup \( H \), and \((H, \sigma \otimes \Lambda_1^*, M(n, r)) \) is a PV.

Example 1.2. Let \( \rho : Spin(10) \to GL(16) \) be a half-spin representation of the spin group \( Spin(10) \). Then \( Spin(10) \times GL(14) \) acts on \( M(16, 14) \) and \( M(\rho), (16, 14) \) has the Zariski-dense \((Spin(10) \times GL(14))\)-orbit if and only if \( r = 1, 2, 3, 13, 14 \). By (3), \((Spin(10) \times GL(14)) \times GL(r), \) a half-spin rep. \( \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1^* \), \( M(16, r) \otimes M(14, r)) \) is a PV if and only if \((Spin(10) \times GL(r), \) a half-spin rep. \( \otimes \Lambda_1, M(16, r)) \) is a PV since the latter is a trivial PV (see [K]). So by SK, we have our result. Note that the case \( r = 13 \) (resp. \( r = 14 \)) is a castling transform of the case \( r = 3 \) (resp. \( r = 2 \)).

Remark 1.3. Professors Shigefumi Mori and Yasuo Teranishi proved that (2) in Theorem 0.1 is equivalent to (4) : \( U = \{(X, Y), X'Y \in Grass_r(V(m)) \oplus Grass_s(V(n)) \oplus M(m, n) | (X, Y) \in W \} \) is \( G \)-prehomogeneous, and later Professor Masanobu Taguchi proved that it is equivalent to (5) : \( U' = \{(X, X'Y) \in Grass_r(V(m)) \oplus M(m, n) | (X, Y) \in W \} \) is \( G \)-prehomogeneous (see [KKTI], Remark 1.38).
ON PREHOMOGENEITY OF A RANK VARIETY

REFERENCES


Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: msy2000@math.tsukuba.ac.jp

Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: mhamada@math.tsukuba.ac.jp

Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571, Japan
E-mail address: kimurata@math.tsukuba.ac.jp