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LIMITS OF TANGENTS OF QUASI-ORDINARY HYPERSURFACES

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ABSTRACT. We compute explicitly the limits of tangents of a quasi-ordinary singularity in terms of its special monomials. We show that the set of limits of tangents of Y is essentially a topological invariant of Y.

1. Introduction

The study of the limits of tangents of a complex hypersurface singularity was mainly developed by Lê Dũng Tráng and Bernard Teissier (see [4] and its bibliography). Chunsheng Ban [1] computed the set of limits of tangents Λ of a quasi-ordinary singularity Y when Y has only one very special monomial (see Definition 1.2).

The main achievement of this paper is the explicit computation of the limits of tangents of an arbitrary quasi-ordinary hypersurface singularity (see Theorems 2.9, 2.10 and 2.11). Corollaries 2.12, 2.13 and 2.14 show that the set of limits of tangents of Y comes quite close to being a topological invariant of Y. Corollary 2.13 shows that Λ is a topological invariant of Y when the tangent cone of Y is a hyperplane. Corollary 2.15 shows that the triviality of the set of limits of tangents of Y is a topological invariant of Y.

Let X be a complex analytic manifold. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Let Γ be a germ of a Lagrangian variety of T^*X at a point α. We say that Γ is in *generic position* if $\Gamma \cap \pi^{-1}(\pi(\alpha)) = \mathbb{C}\alpha$. Let Y be a hypersurface singularity of X. Let Γ be the conormal $T^*_Y X$ of Y. The Lagrangian variety Γ is in generic position if and only if Y is the germ of a hypersurface with trivial set of limits of tangents.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. The characteristic variety of $\mathcal{M}$ is a Lagrangian variety of T^*X. The characteristic varieties in generic position have a central role in $\mathcal{D}$-module theory (cf. Corollary 1.6.4 and Theorem 5.11 of [6] and Corollary 3.12 of [5]). It would be quite interesting to have good characterizations of the hypersurface singularities with trivial set of limits of tangents. Corollary 2.15 is a first step in this direction.

After finishing this paper, two questions arose naturally:
Let Y be a hypersurface singularity such that its tangent cone is a hyperplane. Is the set of limits of tangents of Y a topological invariant of Y?

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Is the triviality of the set of limits of tangents of an hypersurface a topological invariant of the hypersurface?

Let $p : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection that takes $(x, y) = (x_1, \ldots, x_n, y)$ into x. Let Y be the germ of a hypersurface of $\mathbb{C}^{n+1}$ defined by $f \in \mathbb{C}\{x_1, \ldots, x_n, y\}$. Let W be the singular locus of Y. The set Z defined by the equations $f = \partial f/\partial y = 0$ is called the *apparent contour* of f relative to the projection p. The set $\Delta = p(Z)$ is called the *discriminant* of f relative to the projection p.

Near $q \in Y \setminus Z$ there is one and only one function $\varphi \in \mathcal{O}_{\mathbb{C}^{n+1},q}$ such that $f(x, \varphi(x)) = 0$. The function f defines implicitly y as a function of x. Moreover,

$$\frac{\partial y}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} = -\frac{\partial f/\partial x_i}{\partial f/\partial y} \text{ on } Y \setminus Z. \tag{1.1}$$

Let $\theta = \xi_1 dx_1 + \cdots + \xi_n dx_n + \eta dy$ be the canonical 1-form of the cotangent bundle $T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{C}_{n+1}$. An element of the projective cotangent bundle $\mathbb{P}^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{P}_n$ is represented by the coordinates

$$(x_1, \ldots, x_n, y; \xi_1 : \cdots : \xi_n : \eta).$$

We will consider in the open set $\{\eta \neq 0\}$ the chart

$$(x_1, \ldots, x_n, y, p_1, \ldots, p_n),$$

where $p_i = -\xi_i/\eta, 1 \leq i \leq n$. Let Γ_0 be the graph of the map from $Y \setminus W$ into $\mathbb{P}_n$ defined by

$$(x, y) \mapsto \left(\frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y} \right).$$

Let Γ be the smallest closed analytic subset of $\mathbb{P}^*\mathbb{C}^{n+1}$ that contains Γ_0. The analytic set Γ is a Legendrian subvariety of the contact manifold $\mathbb{P}^*\mathbb{C}^{n+1}$. The projective algebraic set $\Lambda = \Gamma \cap \pi^{-1}(0)$ is called the *set of limits of tangents* of Y.

Remark 1.1. It follows from (1.1) that

$$\left(\frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y} \right) = \left(-\frac{\partial y}{\partial x_1} : \cdots : -\frac{\partial y}{\partial x_n} : 1 \right) \text{ on } Y \setminus Z.$$

Let $c_1, \ldots, c_n$ be positive integers. We will denote by $\mathbb{C}\{x_1^{1/c_1}, \ldots, x_n^{1/c_n}\}$ the $\mathbb{C}\{x_1, \ldots, x_n\}$ algebra given by the immersion from $\mathbb{C}\{x_1, \ldots, x_n\}$ into $\mathbb{C}\{t_1, \ldots, t_n\}$ that takes x_i into $t_i^{c_i}, 1 \leq i \leq n$. We set $x_i^{1/c_i} = t_i$. Let $a_1, \ldots, a_n$ be positive rationals. Set $a_i = b_i/c_i, 1 \leq i \leq n$, where $(b_i, c_i) = 1$. Given a ramified monomial $M = x_1^{a_1} \cdots x_n^{a_n} = t_1^{b_1} \cdots t_n^{b_n}$ we set $\mathcal{O}(M) = \mathbb{C}\{x_1^{1/c_1}, \ldots, x_n^{1/c_n}\}$.

Let Y be a germ at the origin of a complex hypersurface of $\mathbb{C}^{n+1}$. We say that Y is a quasi-ordinary singularity if Δ is a divisor with normal crossings. We will assume that there is $l \leq m$ such that $\Delta = \{x_1 \cdots x_l = 0\}$.

If Y is an irreducible quasi-ordinary singularity there are ramified monomials $N_0, N_1, \ldots, N_m, g_i \in \mathcal{O}(N_i), 0 \leq i \leq m$, such that $N_0 = 1$, N_{i-1} divides N_i in the ring $\mathcal{O}(N_i)$, g_i is a unit of $\mathcal{O}(N_i), 1 \leq i \leq m$, g_0 vanishes at the origin and the map $x \mapsto (x, \varphi(x))$ is a parametrization of Y near the origin, where

$$\varphi = g_0 + N_1 g_1 + \cdots + N_m g_m. \tag{1.2}$$

Replacing y by $y - g_0$, we can assume that $g_0 = 0$. The monomials $N_i, 1 \leq i \leq m$, are unique and determine the topology of Y (see [3]). They are called the *special monomials* of f. We set $\tilde{\mathcal{O}} = \mathcal{O}(N_m)$.

Definition 1.2. We say that a special monomial N_i, $1 \le i \le m$, is *very special* if $\{N_i = 0\} \ne \{N_{i-1} = 0\}$.

Let $M_1, \dots, M_g$ be the very special monomials of f, where $M_k = N_{n_k}, 1 = n_1 < n_2 < \dots < n_g, 1 \le k \le g$. Set $M_0 = 1, n_{g+1} = n_g + 1$. There are units f_i of $\mathcal{O}(N_{n_{i+1}-1})$, $1 \le i \le g$, such that

$$\varphi = M_1 f_1 + \cdots + M_g f_g. \tag{1.3}$$

2. Limits of tangents

After renaming the variables x_i there are integers $m_k, 1 \le k \le g+1$, and positive rational numbers $a_{kij}, 1 \le k \le g, 1 \le i \le k, 1 \le j \le m_k$ such that

$$M_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} x_{ij}^{a_{kij}}, \qquad 1 \le k \le g. \tag{2.1}$$

The canonical 1-form of $\mathbb{P}^*\mathbb{C}^{n+1}$ becomes

$$\theta = \sum_{i=1}^{g+1} \sum_{j=1}^{m_i} \xi_{ij} dx_{ij}. \tag{2.2}$$

We set $p_{ij} = -\xi_{ij}/\eta, 1 \le i \le g+1, 1 \le j \le m_i$. Observe that

$$\frac{\partial y}{\partial x_{ij}} = a_{iij} \frac{M_i}{x_{ij}} \sigma_{ij}, \tag{2.3}$$

where σ_{ij} is a unit of $\tilde{\mathcal{O}}$.

Theorem 2.1. *If $\sum_{i=1}^{m_1} a_{11i} < 1$, then $\Lambda \subset \{\eta = 0\}$.*

Proof. Set $m = m_1$, $x_i = x_{1i}$, $p_i = p_{1i}$, $\xi_i = \xi_{1i}$ and $a_i = a_{11i}$, $1 \le i \le m$. Given positive integers $c_1, \dots, c_m$, it follows from (2.3) that

$$\prod_{i=1}^{m} p_i^{c_i} = \prod_{i=1}^{m} x_i^{a_i \sum_{j=1}^{m} c_j - c_i} \phi, \tag{2.4}$$

for some unit ϕ of $\tilde{\mathcal{O}}$. By (1.3) and (2.3),

$$\phi(0) = f_1(0)^{\sum_{j=1}^{m} c_j} \prod_{j=1}^{m} a_j^{c_j}. \tag{2.5}$$

Hence

$$\eta^{\sum_{i=1}^{m} c_i} = \psi \prod_{i=1}^{m} \xi_i^{c_i} x_i^{c_i - a_i \sum_{j=1}^{m} c_j}, \tag{2.6}$$

for some unit ψ. If there are integers $c_1, \dots, c_m$ such that the inequalities

$$a_k \textstyle\sum_{j=1}^{m} c_j < c_k, \quad 1 \le k \le m, \tag{2.7}$$

hold, then the result follows from (2.6). Hence it is enough to show that the set Ω of the m-tuples of rational numbers $(c_1, \dots, c_m)$ that satisfy the inequalities (2.7) is nonempty. We will recursively define positive rational numbers l_j, c_j, u_j such that

$$l_j < c_j < u_j, \tag{2.8}$$

$j = 1, \ldots, m$. Let c_1, l_1, u_1 be arbitrary positive rationals satisfying $(2.8)_1$. Let $1 < s \leq m$. If l_i, c_i, u_i are defined for $i \leq s-1$, set

$$l_s = \frac{a_s \sum_{j=1}^{s-1} c_j}{1 - \sum_{j=s}^{m} a_j}, \ u_s = (a_s/a_{s-1}) c_{s-1}. \tag{2.9}$$

Since $\sum_{j \geq s} a_j < 1$ and

$$\begin{aligned} u_s - l_s &= \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{m} a_j)} \left((1 - \sum_{j=s-1}^{m} a_j) c_{s-1} - a_{s-1} \sum_{j<s-1} c_j \right) \\ &= \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{m} a_j)} \left((1 - \sum_{j=s-1}^{m} a_j)(c_{s-1} - l_{s-1}) \right), \end{aligned}$$

it follows from $(2.8)_{s-1}$ that $l_s < u_s$. Let c_s be a rational number such that $l_s < c_s < u_s$. Hence $(2.8)_s$ holds for $s \leq m$.

Let us show that $(c_1, \ldots, c_m) \in \Omega$. Since $c_k < u_k$, then

$$c_k < \frac{a_k}{a_{k-1}} c_{k-1}, \text{ for } k \geq 2.$$

Then, for $j < k$,

$$c_k < \frac{a_k}{a_{k-1}} \frac{a_{k-1}}{a_{k-2}} \cdots \frac{a_{j+1}}{a_j} c_j = \frac{a_k}{a_j} c_j.$$

Hence,

$$a_k c_j < a_j c_k, \text{ for } j > k. \tag{2.10}$$

Since $l_k < c_k$,

$$a_k \sum_{j=1}^{k-1} c_j < c_k - \sum_{j=k}^{m} a_j c_k.$$

Hence, by (2.10),

$$a_k \sum_{j=1}^{k-1} c_j < c_k - \sum_{j=k}^{m} a_k c_j.$$

Therefore $a_k \sum_{j=1}^{m} c_j < c_k$. □

Theorem 2.2. *Let $1 \leq k \leq g$. Let $I \subset \{1, \ldots, m_k\}$. Assume that one of the following three hypotheses is satisfied:*

(1) $\sum_{j \in I} a_{kkj} > 1$;
(2) $k = 1$, $\sum_{j \in I} a_{11j} = 1$ and $\sum_{j=1}^{m_1} a_{11j} > 1$;
(3) $k \geq 2$ *and* $\sum_{j \in I} a_{kkj} = 1$.

Then $\Lambda \subset \{\prod_{j \in I} \xi_{kj} = 0\}$.

Proof.

Case 1. We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_k$. Set $a_i = a_{kki}$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.3) that

$$\prod_{i=1}^{n} \xi_{ki}^{c_i} = \prod_{i=1}^{n} x_{ki}^{a_i \sum_{j=1}^{n} c_j - c_i} \eta^{\sum_{i=1}^{n} c_i} \varepsilon, \tag{2.11}$$

where $\varepsilon \in \widetilde{\mathcal{O}}$. Hence it is enough to show that there are positive rational numbers $c_1, \dots, c_n$ such that

$$a_k(\sum_{j=1}^{n} c_j) - c_k > 0,\ 1 \le k \le n. \tag{2.12}$$

We will recursively define $l_j, c_j, u_j \in]0, +\infty]$ such that $c_j, l_j \in \mathbb{Q}$,

$$l_j < c_j < u_j, \tag{2.13}$$

$j = 1, \dots, n$, and $u_j \in \mathbb{Q}$ if and only if $\sum_{i=j}^{n} a_i < 1$. Choose c_1, l_1, u_1 satisfying (2.13). Let $1 < s \le n-1$. Suppose that l_i, c_i, u_i are defined for $1 \le i \le s-1$. If $\sum_{j=s}^{n} a_j < 1$, set

$$l_s = (a_s/a_{s-1})c_{s-1},\ u_s = \frac{a_s \sum_{j=1}^{s-1} c_j}{1 - \sum_{j=s}^{n} a_j}. \tag{2.14}$$

Since

$$\begin{aligned} u_s - l_s &= \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{n} a_j)} \left(a_{s-1} \sum_{j=1}^{s-2} c_j - c_{s-1}(1 - \sum_{j=s-1}^{n} a_j) \right) \\ &\le \frac{a_s}{a_{s-1}(1 - \sum_{j=s}^{n} a_j)} \left((1 - \sum_{j=s-1}^{n} a_j)(u_{s-1} - c_{s-1}) \right), \end{aligned}$$

it follows from $(2.13)_{s-1}$ that $l_s < u_s$.

If $\sum_{j=s}^{n} a_j \ge 1$, set l_s as above and $u_s = +\infty$. We choose a rational number c_s such that $l_s < c_s < u_s$. Hence $(2.13)_s$ holds for $1 \le s \le n$.

Let us show that $c_1, \dots, c_n$ satisfy (2.12). We will proceed by induction. First we will show that $c_1, \dots, c_n$ satisfy $(2.12)_n$. Suppose that $a_n < 1$. Since $c_n < u_n$, we have that

$$c_n < \frac{a_n \sum_{j=1}^{n-1} c_j}{1 - a_n}.$$

Hence $a_n \sum_{j=1}^{n} c_j > c_n$. If $a_n \ge 1$, then

$$a_n \sum_{j=1}^{n} c_j \ge \sum_{j=1}^{n} c_j > c_n.$$

Hence $(2.12)_n$ is satisfied. Assume that $c_1, \dots, c_n$ satisfy $(2.12)_k$, $2 \le k \le n$. Since $c_k > l_k$,

$$a_k \sum_{j=1}^{n} c_j > c_k > \frac{a_k}{a_{k-1}} c_{k-1}.$$

Hence $a_{k-1} \sum_{j=1}^{n} c_j > c_{k-1}$. Therefore $(c_1, \dots, c_n)$ satisfy $(2.12)_{k-1}$.

Case 2. Set $a_j = a_{11j}$ and $x_j = x_{1j}$. We can assume that $I = \{1, \dots, n\}$, where $1 \le n \le m_1$. Given positive integers $c_1, \dots, c_n$, it follows from (1.2) that

$$\prod_{i=1}^{n} \xi_i^{c_i} = \prod_{i=1}^{n} x_i^{a_i \sum_{j=1}^{n} c_j - c_i} \eta^{\sum_{i=1}^{n} c_i} \varepsilon, \tag{2.15}$$

where $\varepsilon \in \widetilde{\mathcal{O}}$ and $\varepsilon(0) = 0$. Hence it is enough to show that there are positive rational numbers $c_1, \ldots, c_n$, such that

$$a_k \sum_{j=1}^{n} c_j = c_k, \ 1 \leq k \leq n. \tag{2.16}$$

We choose an arbitrary positive integer c_1. Let $1 < s \leq n$. If the c_i are defined for $i < s$, set

$$c_s = \frac{a_s}{a_{s-1}} c_{s-1}. \tag{2.17}$$

Let us show that $c_1, \ldots, c_n$ satisfy (2.16). We will proceed by induction in k. First let us show that $(2.16)_n$ holds.

Let $j < n-1$. By (2.17),

$$c_{n-1} = \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_{j+1}}{a_j} c_j = \frac{a_{n-1}}{a_j} c_j. \tag{2.18}$$

By (2.17), and since $\sum_{j=1}^{n} a_j = 1$,

$$c_n = \frac{a_n}{a_{n-1}} c_{n-1} = \frac{c_{n-1}}{a_{n-1}} (1 - \sum_{j=1}^{n-1} a_j) = \frac{c_{n-1}}{a_{n-1}} - \sum_{j=1}^{n-1} \frac{a_j}{a_{n-1}} c_{n-1}.$$

Hence, by (2.18),

$$c_n = \frac{c_{n-1}}{a_{n-1}} - \sum_{j=1}^{n-1} c_j.$$

Therefore, $\sum_{j=1}^{n} c_j = c_{n-1}/a_{n-1}$. Hence by (2.17),

$$a_n \sum_{j=1}^{n} c_j = a_n \frac{c_{n-1}}{a_{n-1}} = c_n.$$

Therefore $(2.16)_n$ holds.

Assume $(2.16)_k$ holds, for $2 \leq k \leq n$. Then

$$a_k \sum_{j=1}^{n} c_j = c_k = \frac{a_k}{a_{k-1}} c_{k-1}.$$

Hence, $a_{k-1} \sum_{j=1}^{n} c_j = c_{k-1}$.

Case 3. We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_k$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.3) that

$$\prod_{i=1}^{n} \xi_{ki}^{c_i} = \left(\prod_{i=1}^{n} x_{ki}^{a_{kki}(\sum_{j=1}^{n} c_j) - c_i} \right) \eta^{\sum_{i=1}^{n} c_i} \varepsilon,$$

where $\varepsilon \in \tilde{\mathcal{O}}$ and $\varepsilon(0) = 0$. We have reduced the problem to case 2.

□

Theorem 2.3. *If* $\sum_{k=1}^{m_1} a_{11j} = 1$, Λ *is contained in a cone.*

Proof. Set $a_i = a_{11i}, i = 1, \ldots, m_1$. Given positive integers $c_1, \ldots, c_{m_1}$, there is a unit ϕ of $\tilde{\mathcal{O}}$ such that

$$\prod_{i=1}^{m_1} \xi_i^{c_i} = (-1)^{\sum_{j=1}^{m_1} c_j} \phi \prod_{i=1}^{m_1} x_i^{\sum_{j=1}^{m_1} c_j a_i - c_i} \eta^{\sum_{j=1}^{m_1} c_j}. \tag{2.19}$$

By the proof of case 2 of Theorem 2.2, there is one and only one m_1-tuple of integers $c_1, \ldots, c_{m_1}$ such that $(c_1, \ldots, c_{m_1}) = (1)$, $a_i \sum_{j=1}^{m_1} c_j = c_i, 1 \leq i \leq m_1$, and Λ is contained in the cone defined by the equation

$$\prod_{i=1}^{m_1} \xi_i^{c_i} - (-1)^{\sum_{j=1}^{m_1} c_j} \phi(0) \eta^{\sum_{j=1}^{m_1} c_j} = 0, \tag{2.20}$$

where $\phi(0)$ is given by (2.5). ☐

Remark 2.4. Set $D_\varepsilon^* = \{x \in \mathbb{C} : 0 < |x| < \varepsilon\}$, where $0 < \varepsilon \ll 1$. Set $\mu = \sum_{k=1}^{g+1} m_k$. Let $\sigma : \mathbb{C} \to \mathbb{C}^\mu$ be a weighted homogeneous curve parametrized by

$$\sigma(t) = (\varepsilon_{ki} t^{\alpha_{ki}})_{1 \leq k \leq g+1, 1 \leq i \leq m_k}.$$

Notice that the image of σ is contained in $\mathbb{C}^\mu \setminus \Delta$. Set $\theta_0(t) = 1$ and

$$\theta_{ki}(t) = \frac{\partial \varphi}{\partial x_{ki}}(\sigma(t), \varphi(\sigma(t))), \; 1 \leq k \leq g+1, 1 \leq i \leq m_k,$$

for $t \in D_\varepsilon^*$. The curve σ induces a map from D_ε^* into Γ defined by

$$t \mapsto (\sigma(t), \varphi(\sigma(t)); \theta_{11}(t) : \cdots : \theta_{g+1, m_g+1}(t) : \theta_0(t)).$$

Let $\vartheta : D_\varepsilon^* \to \mathbb{P}^\mu$ be the map defined by

$$t \mapsto (\theta_{11}(t) : \cdots : \theta_{g+1, m_g+1}(t) : \theta_0(t)). \tag{2.21}$$

The limit when $t \to 0$ of $\vartheta(t)$ belongs to Λ. The functions θ_{ki} are ramified Laurent series of finite type in the variable t. Let h a be ramified Laurent series of finite type. If $h = 0$, we set $v(h) = \infty$. If $h \neq 0$, we set $v(h) = \alpha$, where α is the only rational number such that $\lim_{t \to 0} t^{-\alpha} h(t) \in \mathbb{C} \setminus \{0\}$. We call α the *valuation* of h. Notice that the limit of ϑ depends only on the functions θ_{ki}, θ_0 of minimal valuation. Moreover, the limit of ϑ depends only on the coefficients of the term of minimal valuation of each θ_{ij}, θ_0. Hence the limit of ϑ depends only on the coefficients of the very special monomials of f. We can assume that $m_{g+1} = 0$ and that there are $\lambda_k \in \mathbb{C} \setminus \{0\}, 1 \leq k \leq g$, such that

$$\varphi = \sum_{k=1}^{g} \lambda_k M_k. \tag{2.22}$$

Remark 2.5. The set of limits of tangents of a plane curve is a finite set, the union of the tangent cones of its irreducible components. If the curve is irreducible, its limit of tangents is always trivial. The set of limits of tangents of a surface Y is the union of the dual variety of the tangent cone of Y with a finite family of projective lines. When we consider a hypersurface Y of $\mathbb{C}^n$ with $n \geq 4$, the set of limits of tangents Λ of Y has a more complex structure. In order to describe Λ, Lê and Teissier introduced a finite family of subsets of the tangent cone of Y, the *halo* of Y. The halo of Y is called “la auréole” of Y in [4].

Remark 2.6. Let L be a finite set. Set $\mathbb{C}^L = \{(x_a)_{a\in L} : x_a \in \mathbb{C}\}$. Let $\sum_{a\in L} \xi_a dx_a$ be the canonical 1-form of $T^*\mathbb{C}^L$. Let Λ be the subset of $\mathbb{P}_L$ defined by the equations

$$\prod_{a\in I} \xi_a = 0,\ I \in \mathcal{I}, \tag{2.23}$$

where $\mathcal{I} \subset \mathcal{P}(L)$. Set $\mathcal{I}' = \{J \subset L : J \cap I \neq \emptyset \text{ for all } I \in \mathcal{I}\}$, $\mathcal{I}^* = \{J \in \mathcal{I}'$ such that there is no $K \in \mathcal{I}' : K \subset J, K \neq J\}$. The irreducible components of Λ are the linear projective sets $\Lambda_J, J \in \mathcal{I}^*$, where Λ_J is defined by the equations

$$\xi_a = 0, \qquad a \in J.$$

Let Y be a germ of hypersurface of $(\mathbb{C}^L, 0)$. Let Λ be the set of limits of tangents of Y. For each irreducible component Λ_J of Λ there is a cone V_J contained in the tangent cone of Y such that Λ_J is the dual of the projectivization of V_J. The halo of Y is the union of the cones V_J, $J \in \mathcal{I}^*$.

Remark 2.7. If Λ is defined by the equations (2.23), the halo of Y equals the union of the linear subsets $V_J, J \in \mathcal{I}^*$ of $\mathbb{C}^{\ L}$, where V_J is defined by the equations

$$x_a = 0, \qquad a \in L \setminus J.$$

Lemma 2.8. *The determinant of the $n \times n$ matrix $(\lambda_i - \delta_{ij})$ equals*

$$(-1)^n(1 - \sum_{i=1}^n \lambda_i).$$

Proof. Notice that $\det(\lambda_i - \delta_{ij}) =$

$$\left|\begin{array}{ccc|c} & & & 1 \\ & -I_{n-1} & & \vdots \\ & & & 1 \\ \hline \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n - 1 \end{array}\right| = \left|\begin{array}{ccc|c} & & & 1 \\ & -I_{n-1} & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & \sum_{i=1}^n \lambda_i - 1 \end{array}\right|.$$

□

Theorem 2.9. *Assume that $\sum_{i=1}^{m_1} a_{11i} < 1$. Set*

$$L = \bigcup_{k=2}^g \{k\} \times \{1, \ldots, m_k\},\ \mathcal{I} = \bigcup_{k=2}^g \{\{k\} \times I : \sum_{j\in I} a_{kkj} \geq 1\}.$$

The set Λ is the union of the irreducible linear projective sets $\Lambda_J, J \in \mathcal{I}^*$, defined by the equations $\eta = 0$ and

$$\xi_{kj} = 0,\ (k, j) \in J. \tag{2.24}$$

The tangent cone of Y equals $\{x_{11} \cdots x_{1m_1} = 0\}$. The halo of Y is the union of the cones V_J, $J \in \mathcal{I}^*$, where V_J is defined by the equations $x_{1j} = 0$, $1 \leq j \leq m_1$, and

$$x_{kj} = 0, (k, j) \in L \setminus J. \tag{2.25}$$

Proof. Let us show that $\Lambda_J \subset \Lambda$. We can assume that there are integers $n_1, \ldots, n_g$, $1 \leq n_k \leq m_k$, $1 \leq k \leq g$, such that $J = \bigcup_{k=1}^g \{k\} \times \{n_k + 1, \ldots, m_k\}$. We will use the notation of Remark 2.4.

Set $m = \sum_{k=1}^g m_k, n = m - \#J$. Assume that there are positive rational numbers $\alpha_k, \beta_k, 1 \leq k \leq g$, such that $\alpha_{ki} = \alpha_k$ if $1 \leq i \leq n_k$, $\alpha_{ki} = \beta_k$ if $n_k + 1 \leq i \leq m_k$, and $\alpha_k > \beta_k$, $1 \leq k \leq g$. Since $v(\theta_{ki}) = v(M_k) - v(x_{ki}) = v(M_k) - \alpha_{ki}$,

$$\lim_{t\to 0} \vartheta(t) \in \Lambda_J.$$

Let $\psi : (\mathbb{C} \setminus \{0\})^n \to \Lambda_J$ be the map defined by

$$\psi(\varepsilon_{ij}) = \lim_{t \to 0} \vartheta(t). \tag{2.26}$$

The map ψ has components ψ_{ki}, $1 \le i \le n_k, 1 \le k \le g$. In order to prove the theorem it is enough to show that we can choose the rational numbers α_k, β_k in such a way that the Jacobian of ψ does not vanish identically. We will proceed by induction in k. Let $k = 1$. Since $\sum_{i=1}^{m_1} a_{11i} < 1$, $n_1 = m_1$. Choose positive rationals α_1, β_1, $\alpha_1 > \beta_1$. There is a rational number $v_0 < 0$ such that $v(\theta_{1i}) = v_0$, for all $1 \le i \le n_1$.

Assume that there are α_k, β_k such that $v(\theta_{ki}) = v_0$ for $1 \le i \le n_k$ and $v(\theta_{ki}) > v_0$ for $n_k + 1 \le i \le m_k$, $k = 1, \ldots, u$. Set

$$\underline{\alpha}_{u+1} = \frac{\alpha_u + \sum_{k=1}^{u} \sum_{i=1}^{m_k} (a_{u+1,k,i} - a_{uki})\alpha_{ki}}{1 - \sum_{i=1}^{n_{u+1}} a_{u+1,u+1,i}}.$$

Since the special monomials are ordered by valuation and, by construction of Λ_J, $\sum_{i=1}^{n_k} a_{kki} < 1$ for all $1 \le k \le g$, $\underline{\alpha}_{u+1}$ is a positive rational number. Choose a rational number β_{u+1} such that $0 < \beta_{u+1} < \underline{\alpha}_{u+1}$. Set

$$\alpha_{u+1} = \underline{\alpha}_{u+1} + \frac{\sum_{i=n_{u+1}+1}^{m_{u+1}} a_{u+1,u+1,i}\beta_{u+1}}{1 - \sum_{i=1}^{n_{u+1}} a_{u+1,u+1,i}}.$$

Then, $v(\theta_{u+1,i}) = v(M_{u+1}) - \alpha_{u+1} = v(M_u) - \alpha_u = v_0$ for $1 \le i \le n_{u+1}$.

Set $\widehat{M}_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} \varepsilon_{ij}^{a_{kij}}, 1 \le i \le n_k, 1 \le k \le g$. With these choices of α_{ki}, we have that

$$\psi_{ki} = \frac{1}{\varepsilon_{ki}} \sum_{l=k}^{g} a_{kli} \widehat{M}_l,\ 1 \le i \le m_k, 1 \le k \le g.$$

Let D be the Jacobian matrix of ψ. The matrix D has $n_r \times\ n_s$ blocks D_{rs}, $1 \le r, s \le g$. If $r < s$, the entries of D_{rs} are

$$\frac{1}{\varepsilon_{ri}\varepsilon_{sj}} \sum_{l=s}^{g} a_{rli} a_{slj} \widehat{M}_l.$$

Moreover, D_{rr} has entries

$$\frac{\widehat{M}_r}{\varepsilon_{ri}\varepsilon_{rj}} \left(a_{rri}(a_{rrj} - \delta_{ij}) + \sum_{l=r+1}^{g} a_{rri} a_{rrj} \widehat{M}_l \right).$$

Let m be the maximal ideal of the ring $\mathcal{O}(\widehat{M}_g)$. If $r \le s$ the entry (i, j) of D_{rs} belongs to the ideal generated by $\widehat{M}_s/(\varepsilon_{ri}\varepsilon_{rj})$. Hence $\det(D_{rr})$ belongs to the ideal I_r generated by

$$\left(\widehat{M}_r^{m_r} \Big/ \prod_{i=1}^{m_r} \varepsilon_{ri} \right)^2,\ 1 \le r \le g. \tag{2.27}$$

Moreover, $\det(D)$ belongs to the ideal I generated by

$$\left(\prod_{l=1}^{g} \widehat{M}_l^{m_l} \Big/ \prod_{l=1}^{g} \prod_{i=1}^{m_l} \varepsilon_{li} \right)^2. \tag{2.28}$$

Let σ be a permutation of $\{(1, 1), \ldots, (1, m_1), \ldots, (g, 1), \ldots, (g, m_g)\}$. If there are $(r, i), (s, j)$ such that $\sigma(r, i) = (s, j)$ and $r \ne s$, the product of the entries $(r, i), \sigma(r, i)$ of D belongs to the ideal Im. Therefore $\det(D)$ is congruent modulo Im to the product of the determinants of the diagonal blocks $D_{rr}, 1 \leq r \leq g$. Moreover, $\det(D_{rr})$ is congruent modulo $I_r m$ to the determinant of the matrix D_r with entries

$$\frac{\widehat{M}_r}{\varepsilon_{ri}\varepsilon_{rj}} a_{rri}(a_{rri} - \delta_{ij}).$$

By Lemma 2.8, $\det(D_r)$ equals the product of (2.27) by a nonvanishing complex number. Therefore there are $\lambda \in \mathbb{C} \setminus \{0\}$ and $\varepsilon \in m$ such that $\det(D)$ equals the product of (2.28) by a unit of $\mathcal{O}(\widehat{M}_g)$. Hence $\det(D)$ does not vanish identically and Λ contains an open set of Λ_J. Since Λ is a projective variety and Λ_J is irreducible, Λ contains Λ_J. □

Theorem 2.10. *Assume that* $\sum_{i=1}^{m_1} a_{11i} > 1$. *Set*

$$L = \bigcup_{k=1}^{g} \{k\} \times \{1, \ldots, m_k\}, \ \mathcal{I} = \bigcup_{k=1}^{g} \{\{k\} \times I : \sum_{j \in I} a_{kkj} \geq 1\}.$$

The set Λ *is the union of the irreducible linear projective sets* $\Lambda_J, J \in \mathcal{I}^*$, *defined by the equations* (2.24). *The tangent cone of* Y *equals* $\{y = 0\}$. *The halo of* Y *is the union of the cones* V_J, $J \in \mathcal{I}^*$, *where* V_J *is defined by the equations* $y = 0$ *and* (2.25).

Proof. The proof is analogous to the proof of Theorem 2.9. On the first induction step we choose

$$\beta_1 = \left(\frac{1 - \sum_{i=1}^{n_1} a_{11i}}{\sum_{i=n_1+1}^{m_1} a_{11i}} \right) \alpha_1.$$

Hence $\beta_1 < \alpha_1$, $v(\theta_{1i}) = v(\eta) = 0$ for $1 \leq i \leq n_1$ and $v(\theta_{1i}) > 0$ for $n_1 + 1 \leq i \leq m_1$. The rest of the proof proceeds as in the previous case. □

Theorem 2.11. *Assume that* $\sum_{i=1}^{m_1} a_{11i} = 1$. *Set*

$$L = \bigcup_{k=2}^{g} \{k\} \times \{1, \ldots, m_k\}, \ \mathcal{I} = \bigcup_{k=2}^{g} \{\{k\} \times I : \sum_{j \in I} a_{kkj} \geq 1\}.$$

The set Λ *is the union of the irreducible projective algebraic sets* $\Lambda_J, J \in \mathcal{I}^*$, *where* Λ_J *is defined by the equations* (2.20) *and* (2.24).

There are integers c, d_i *such that* $a_{11i} = d_i/c$, $1 \leq i \leq m_1$, *and* c *is the l.c.d. of* $d_1, \ldots, d_{m_1}$. *The tangent cone of* Y *equals*

$$y^c - f(0)^c \prod_{i=1}^{m_1} x_{1i}^{d_i} = 0. \tag{2.29}$$

The halo of Y *is the union of the cones* V_J, $J \in \mathcal{I}^*$, *where* V_J *is defined by the equations* (2.25) *and* (2.29).

Proof. Following the arguments of Theorem 2.9, it is enough to show that $\Lambda_J \subset \Lambda$ for each $J \in \mathcal{I}^*$. Choose $J \in \mathcal{I}^*$. Let $\tilde{\Lambda}_J$ be the linear projective variety defined by the equations (2.24). We follow an argument analogous to the one used in Theorem 2.9. We have $n_1 = m_1$. We choose positive rational numbers α_1, β_1 such that $\beta_1 < \alpha_1$. Then $v(\theta_{1i}) = 0$ for all $i = 1, \ldots, m_1$. The remaining steps of the proof proceed as before. Hence

$$\lim_{t \to 0} \vartheta(t) \in \tilde{\Lambda}_J.$$

Let $\psi : (\mathbb{C}\setminus\{0\})^n \to \tilde{\Lambda}_J$ be the map defined by (2.26). By Theorem 2.3 the image of ψ is contained in Λ_J. By Lemma 2.8, $\det(D_1) = 0$. Let D_1' be the matrix obtained from D_1 by eliminating the m_1-th line and column. The argument of the proof of Theorem 2.9 works when we replace D_1 by D_1'. Hence, $\Lambda_J \subset \Lambda$. □

Let Y be a quasi-ordinary hypersurface singularity.

Corollary 2.12. *The set of limits of tangents of Y depends only on the tangent cone of Y and the topology of Y.*

Corollary 2.13. *If the tangent cone of Y is a hyperplane, the set of limits of tangents of Y depends only on the topology of Y.*

Corollary 2.14. *Let $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ be the first special monomial of Y. If $\alpha_1 + \cdots + \alpha_k \neq 1$, the set of limits of tangents of Y depends only on the topology of Y.*

Corollary 2.15. *The triviality of the set of limits of tangents of Y is a topological invariant of Y.*

Proof. The set of limits of tangents of Y is trivial if and only if all the exponents of all the special monomials of Y are greater than or equal to 1. □

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