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TOPOLOGICAL METHOD FOR DETECTING FIXED POINTS OF MAPS HOMOTOPIC TO SELFMAPS OF COMPACT ENRS

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Abstract. Assume that X is a metric space and $B \subset X$ is compact. Let $f : B \to X$ be a continuous map homotopic to g, the selfmap of B. The aim of this paper is to present a method for detecting fixed points of f. It is based on the notion of the Ważewski set for the homotopy F between g and f.

1. Introduction

The notion of a Ważewski set is a very useful tool for detecting invariant sets in dynamical systems generated by differential equations (see [Sr1, Sr2, SWZ]). In particular it is frequently used for detecting fixed points of a time-one map of a semi-flow on the space X. We recall very briefly the main idea of the Ważewski method. A continuous map $\phi : [0, +\infty) \times X \to X$ is a semi-flow if

$$\phi(0, x) = x, \quad \phi(s, \phi(t, x)) = \phi(s + t, x), \quad s, t \in [0, +\infty),\ x \in X.$$

Let $B \subset X$. One can define the exit set of B by

$$B^- = \{x \in B : \phi([0, t] \times \{x\}) \not\subset B \text{ for all } t > 0\}.$$

We say that B is a Ważewski set if B and B^- are compact. Then one can show that the function

$$\sigma : B \ni x \to \sup\{t \geq 0 : \phi([0, t], x) \subset B\} \in [0, +\infty]$$

is continuous. It was proved by Srzednicki that if B and B^- are additionally ENRs and their Euler characteristics are not equal (i.e. $\chi(B) \neq \chi(B^-)$), then B contains a stationary point of ϕ. The continuity of σ and the Lefschetz fixed point theorem are essential to show that for $t > 0$ the map $\phi(t, \cdot) : B \to X$ has a fixed point in B.

The aim of this paper is to present a generalization of the method above for detecting a fixed point in the following context: assume that X is a metric space and $B \subset X$ is compact. Let $f : B \to X$ be a continuous map homotopic to $g : B \to X$ such that $g(B) \subset B$; i.e., there exists a continuous map $F : [0, 1] \times B \to X$ such that

- $F_1 := F(1, \cdot) = f$,
- $F_0 := F(0, \cdot) = g$.

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In order to find fixed points of f we define the Ważewski set for the homotopy F.

2. Ważewski sets for a homotopy

By $\pi_1 : [0,1] \times X \to [0,1]$ and $\pi_2 : [0,1] \times X \to X$ we denote projections. For $Z \subset [0,1] \times X$ and $t \in [0,1]$ we put $Z_t = \{x \in X : (t,x) \in Z\}$.

For a homotopy $F : [0,1] \times B \to X$ we denote

$$F_t : B \ni x \to F(t,x) \in X, \quad t \in [0,1].$$

Let $W \subset [0,1] \times X$ be a compact subset such that $W_0 = B$. We define an exit-time function for W:

$$\tau_W : B \ni x \to \sup\{t \in [0,1] : \ F(s,x) \in W_s \ \forall s \in [0,t]\} \in [0,1].$$

We say that W is a Ważewski set for F if its exit-time function is continuous. The set $W^- = \{(\tau_W(x), F(\tau_W(x), x)) : \ x \in B\} \subset W$ is called the exit set of W. It follows that W^- is compact.

Definition 2.1. The Ważewski set W for F is called a *segment* if the following conditions hold:

- there exists a compact subset W^{--} (called the essential exit set) of W^- such that
$$W^- \cap ([0,1) \times X) \subset W^{--}, \ W^{--} = \mathrm{cl}(W^{--} \cap ([0,1) \times X)),$$
- there exists a homeomorphism $h : [0,1] \times B \to W$ such that $\pi_1 \circ h = \pi_1$ and
$$W^{--} \subset h([0,1] \times W_0^{--}), \quad \pi_2(h(1, W_0^{--})) \subset W_0^{--},$$
- $W_1 = W_0$, $W_1^{--} \subset W_0^{--}$ and $g(W_0^{--}) \subset W_0^{--}$.

Definition 2.2. Let W be a segment and h be a homeomorphism from Definition 2.1. Then $m : W_0 \to W_0$ defined by

$$m(x) := \pi_2 h(1, \pi_2 h^{-1}(0,x)), \quad x \in W_0,$$

is the *monodromy homeomorphism* of the segment W.

Let us note that $m(W_0^{--}) \subset W_0^{--}$. Moreover, two morphisms induced in homologies: $H(m) : H(W_0) \to H(W_0)$ and $H(m \circ g) : H(W_0) \to H(W_0)$ do not depend on the choice of h.

Indeed, let $h : [0,1] \times B \to W$, $k : [0,1] \times B \to W$ be homeomorphisms satisfying conditions from the definition of a segment and let m_h and m_k be their corresponding monodromy homeomorphisms. We will prove that $m_h \circ g$ and $m_k \circ g$ are homotopic (for $g = id$, m_h and m_k are homotopic); thus they induce the same morphisms in homologies. Let

$$h_t : W_0 \ni x \to \pi_2 h(t, \pi_2 h^{-1}(0,x)) \in W_t$$

and

$$k^t : W_t \ni x \to \pi_2 k(1, \pi_2 k^{-1}(t,x)) \in W_0.$$

Then

$$H : [0,1] \times B \ni (s,x) \to k^s \circ h_s \circ g(x) \in B$$

is a homotopy such that $H(0,\cdot) = k^0 \circ h_0 \circ g = m_k \circ g$ and $H(1,\cdot) = k^1 \circ h_1 \circ g = m_h \circ g$.

Example 2.3. We describe this construction in the case of a time-one map of a semi-flow ϕ on X. For a subset $B \subset X$ one can consider a homotopy $F : [0,1] \times B \to X$ given by

$$F(t,x) = \phi(t,x).$$

Then $f = F_1 = \phi(1,\cdot)|_B$ is a restriction of the time-one map to the set B and $g = F_0 = \phi(0,\cdot)|_B = i$ is the inclusion map. In the extended phase space $[0,+\infty) \times X$ we consider a set $W = [0,1] \times B$. Assume that the exit-time function

$$\tau_W : B \ni x \to \sup\{t \in [0,1] : \ (s, \phi(s,x)) \in W \ \forall s \in [0,t]\} \in [0,1]$$

is continuous. The exit set is given by

$$W^- = \{(\tau_W(x), \phi(\tau_W(x), x)) : x \in B\}.$$

If B is a Ważewski set for the semi-flow ϕ (i.e., B, B^- are compact), then W is a segment for F. Indeed,

$$\tau_W(x) = \begin{cases} \sigma(x), & \text{if } \sigma(x) \le 1; \\ 1, & \text{if } \sigma(x) \ge 1. \end{cases}$$

Then W^- is given by

$$W^- = \{(\sigma(x), \phi(\sigma(x), x)) : x \in \sigma^{-1}([0,1])\} \cup \{(1, \phi(1,x)) : x \in \sigma^{-1}([1,+\infty))\},$$

and the essential exit set of W is equal to

$$W^{--} = \{(\sigma(x), \phi(\sigma(x), x)) : x \in \sigma^{-1}([0,1])\}.$$

In particular,

$$W_0 = B, \quad W_0^{--} = B^-.$$

Example 2.4. Let us mention that if X is contractible, then each map $f : B \to X$ is homotopic to the inclusion. In particular, f is not necessarily a time-one map for any semi-flow even if g is the inclusion map.

Example 2.5. We illustrate the notion of a segment for a homotopy with the following simple example. Let $B = \{z \in \mathbb{C} : |z| \le 1\}$ and $f : B \to \mathbb{C}$ be a continuous map such that

$$\{(1-t)z + tf(z) : t \in [0,1]\} \cap B = \{z\}, \quad z \in \partial B.$$

Consider a homotopy

$$F : [0,1] \times B \ni (t,z) \to (1-t)z + tf(z) \in \mathbb{C}.$$

We define $W = [0,1] \times B$. In order to show that W is a Ważewski set for F it is sufficient to show that the exit-time function $\tau_W : B \to [0,1]$ is continuous. But $\tau_W(z)$ is determined by the intersection point of the segment joining points z and $f(z)$ with the set

$$Y = [0,1] \times \partial B \cup \{1\} \times B,$$

so it is continuous. We will argue that Y is equal to the exit set W^-. By definition, $W^- = w(B)$, where

$$w : B \ni z \to (\tau_W(z), F(\tau_W(z), z)) \in Y.$$

Since $w(z) = z$ for $z \in \partial B$ and $w|_{\partial B}$ has an extension on B, so $w|_{\partial B} : \partial B \to W^- = w(B)$ is homotopic to a constant map in W^-; hence $W^- = Y$. It follows that W

is a segment for F with the identity on W as a homeomorphism h. Observe that

$$W_0 = B, W_0^{--} = \partial B,$$

and m is homotopic to the identity on W_0.

3. Detection of fixed points

We use standard notation concerning fixed point theory ([JM]). In particular, $\mathrm{Fix}\,(f) = \{x \in U : f(x) = x\}$ denotes the set of fixed points of the map $f : U \to X$, where $U \subset X$. Recall (see [JM]) that if X is an ENR, $U \subset X$ is open and $\mathrm{Fix}\,(f)$ is compact, then the fixed point index $\mathrm{ind}(f, \mathrm{Fix}\,(f)) \in \mathbb{Z}$ is well defined. Moreover, if $f : X \to X$ is a selfmap of a compact ENR, then its Lefschetz number is well defined by

$$L(f) := L(H(f)) = \sum_{n\geq 0}(-1)^n \mathrm{trace}(H_n(f)),$$

where $H(f) : H(X) \to H(X)$ is a map induced by f in singular homologies with rational coefficients.

Theorem 3.1. *Assume that W is a segment for F, $g = F_0 : B \to X$, $g(B) \subset B$ and $f = F_1 : B \to X$. The set*

$$U = \{x \in W_0 : F(t, x) \in W_t \setminus W_t^{--}, \text{ for all } t \in [0, 1]\}$$

is open in $B = W_0$ and the set of fixed points of $f|_U : U \to B$ is compact. Moreover, if W and W^{--} are ENRs, then

$$\mathrm{ind}(f|_U, \mathrm{Fix}\,(f|_U)) = L(m \circ g) - L(m \circ g|_{W_0^{--}}).$$

In particular, if $L(m \circ g) - L(m \circ g|_{W_0^{--}}) \neq 0$, then $f : B \to X$ has a fixed point in B.

Proof: It is easy to check that

$$U = W_0 \setminus \bigcup_{t\in[0,1]} F_t^{-1}(W_t^{--}).$$

If for each n, $x_n \in \bigcup_{t\in[0,1]} F_t^{-1}(W_t^{--})$ and $\lim_{n\to\infty} x_n = x$, then

$$\lim_{n\to\infty} F(\tau_W(x_n), x_n) = F(\tau_W(x), x).$$

Moreover, for each n, $F(\tau_W(x_n), x_n) \in W^{--}$; hence $F(\tau_W(x), x) \in W^{--}$ and $x \in \bigcup_{t\in[0,1]} F_t^{-1}(W_t^{--})$. Thus $\bigcup_{t\in[0,1]} F_t^{-1}(W_t^{--})$ is closed and U is open in W_0.

We define maps

$$m_s : W_s \ni x \to \pi_2 h(1, \pi_2 h^{-1}(s, x)) \in W_0 = W_1, \quad s \in [0, 1].$$

In particular, $m_0 = m : W_0 \to W_0$ and $m_1 = \mathrm{id}_{W_0}$. Consider a homotopy $H : [0, 1] \times W_0 \to W_0$ given by

$$H_t(x) = \begin{cases} m_{\tau_W(x)}(F(\tau_W(x), x)), & \text{if } \tau_W(x) \leq 1 - t; \\ m_{1-t}(F(1 - t, x)), & \text{if } \tau_W(x) \geq 1 - t. \end{cases}$$

In particular, $H_1 = m \circ g$. By the homotopy property of the Lefschetz number we obtain

$$L(H_0) = L(H_1) = L(m \circ g).$$

By definition we have that

$$H_0(x) = m_{\tau_W(x)}(F(\tau_W(x), x)), \quad x \in W_0.$$

It is easy to check that

$$U = \{x \in W_0 : \tau_W(x) = 1,\ F(1,x) \in W_0 \setminus W_1^{--}\}$$

and $(H_0)^{-1}(W_0 \setminus W_0^{--}) \subset U \subset (H_0)^{-1}(W_0 \setminus W_1^{--})$. $H_0(x) = F(1,x) = f(x)$ if $\tau_W(x) = 1$; hence

$$H_0|_U = f|_U.$$

If $x \in W_0 \setminus W_0^{--}$ and $H_0(x) = x$, then it follows that $\tau_W(x) = 1$ (since in the other case $H_0(x) \in W_0^{--}$); hence $x \in U$. Moreover,

$$\mathrm{Fix}\,(f|_U) = \mathrm{Fix}\,(H_0) \cap \tau_W^{-1}(1);$$

hence $\mathrm{Fix}\,(f|_U)$ is compact. Put $V = \tau_W^{-1}([0,1))$. It follows that V is open in W_0, $W_0^{--} \subset V$ and $H_0(V) \subset W_0^{--}$. By the compactness of W_0^{--} it follows that there is an open neighborhood D of W_0^{--} such that $g(D) \subset V$; hence $H_0(D) \subset W_0^{--}$. It follows that

$$\mathrm{Fix}\,(H_0) = \mathrm{Fix}\,(H_0|_U) \cup \mathrm{Fix}\,(H_0|_D) = \mathrm{Fix}\,(f|_U) \cup \mathrm{Fix}\,(m \circ g|_{W_0^{--}}).$$

Since the sets $\mathrm{Fix}\,(f|_U)$ and $\mathrm{Fix}\,(m \circ g|_{W_0^{--}})$ are compact and disjoint, hence by the Lefschetz fixed point theorem and the additivity property of the fixed point index we get

$$L(m \circ g) = L(H_0) = \mathrm{ind}\,(H_0|_U, \mathrm{Fix}\,(f|_U)) + \mathrm{ind}\,(H_0|_V, \mathrm{Fix}\,(m \circ g|_{W_0^{--}})).$$

By the commutativity property of the fixed point index and the Lefschetz fixed point theorem we have

$$\mathrm{ind}\,(H_0|_V, \mathrm{Fix}\,(m \circ g|_{W_0^{--}})) = \mathrm{ind}\,(H_0|_{W_0^{--}}, \mathrm{Fix}\,(m \circ g|_{W_0^{--}})) = L(m \circ g|_{W_0^{--}}),$$

and consequently

$$\mathrm{ind}\,(f|_U, \mathrm{Fix}\,(f|_U)) = \mathrm{ind}\,(H_0|_U, \mathrm{Fix}\,(f|_U)) = L(m \circ g) - L(m \circ g|_{W_0^{--}}).$$

□

Example 3.2. Assume that $W = [0,1] \times B$ is a segment for the homotopy $F : [0,1] \times B \to X$ with the homeomorphism $h : W \ni x \to x \in W$ and g is the inclusion map. Then $L(m \circ g) - L(m \circ g|_{W_0^{--}}) = \chi(B) - \chi(W_0^{--})$, so if $\chi(B) \neq \chi(W_0^{--})$, then $f = F_1$ has a fixed point in B.

4. Relation to the Nielsen number

The classical Nielsen number theory is concerned with the determination of the minimal number of fixed points for all maps in the homotopy class of a given map $f : X \to X$ (see [JM, J]). The Nielsen number $N(f)$ provides a homotopy invariant lower bound for the number of fixed points of f. Assume that X is a compact ENR and $f : X \to X$ is a continuous map. In the set of fixed points $\mathrm{Fix}\,(f)$ we define the equivalence relation in the following way: two fixed points are in the Nielsen relation if they can be joined by a path which is homotopic relative to the end points to its own f-image. Then $\mathrm{Fix}\,(f)$ splits into a disjoint union of Nielsen classes. Each Nielsen class F is compact, so its fixed point index $\mathrm{ind}\,(f,F)$ is well defined. The Nielsen number of f ($N(f)$) is defined as the number of essential fixed point classes, i.e., such that $\mathrm{ind}\,(f,F) \neq 0$. In this paper we will use the relative Nielsen theory for selfmaps $f : (X,A) \to (X,A)$ of pairs of compact ENRs introduced in [S2] (see also [Z]). Let F be a Nielsen class of $f : X \to X$. We say

that F assumes its index in A if $\operatorname{ind}(f,F) = \operatorname{ind}(f|_A, F\cap A)$. The Nielsen number of the closure $N(f, \operatorname{cl}(X\setminus A))$ is defined as the number of the Nielsen classes of f which do not assume its index in A. It is a relative homotopy invariant lower bound for the number of fixed points of f on $\operatorname{cl}(X\setminus A)$.

Theorem 4.1. *Assume that W is a segment for the homotopy F. Then*

$$\operatorname{card}\operatorname{Fix}(f|_U) \geq N(m\circ g, \operatorname{cl}(W_0\setminus W_0^{--})),$$

where $U = \{x\in W_0 : \tau_W(x) = 1,\ F(1,x)\in W_0\setminus W_0^{--}\}$.

Proof: Let $H : [0,1]\times W_0 \to W_0$ be a homotopy used in the proof of Theorem 3.1. Let us observe that for $x\in W_0^{--}$ we have $\tau_W(x) = 0$, so then

$$H_t(x) = m\circ g(x), \quad t\in[0,1];$$

hence $H_t(W_0^{--}) \subset W_0^{--}$ if $t\in[0,1]$, and thus H is a homotopy of the pair (W_0, W_0^{--}). Since $H_1 = m\circ g : (W_0, W_0^{--}) \to (W_0, W_0^{--})$, then by the homotopy property of the relative Nielsen number we must show that

$$\operatorname{card}\operatorname{Fix}(f|_U) \geq N(H_0, \operatorname{cl}(W_0\setminus W_0^{--})).$$

It is sufficient to prove that if $K\subset W_0$ is a Nielsen class of H_0 and K does not assume its index in W_0^{--}, then $(\operatorname{Fix}(f)\cap U)\cap K$ is nonempty. Suppose that

$$\operatorname{Fix}(f|_U)\cap K = \emptyset.$$

By the same arguments as in the proof of Theorem 3.1 we have that

$$\operatorname{Fix}(H_0) = \operatorname{Fix}(f|_U)\cup \operatorname{Fix}(m\circ g|_{W_0^{--}}),$$

so $K\subset W_0^{--}$. It follows by the arguments in the proof of Theorem 3.1 that there is an open neighborhood D of W_0^{--} such that $H_0(D)\subset W_0^{--}$; hence by the commutativity property of the fixed point index we obtain that

$$\operatorname{ind}(H_0, K) = \operatorname{ind}(H_0|_{W_0^{--}}, K) = \operatorname{ind}(m\circ g|_{W_0^{--}}, K),$$

a contradiction. □

Remark 4.2. In the remarks below we assume that $F_0 = g : B \hookrightarrow X$ is the inclusion.

(1) It follows that $\operatorname{cl}(W_0\setminus W_0^{--}) = W_0$ but in general $N(m, \operatorname{cl}(W_0\setminus W_0^{--}))$ is not equal to the classical Nielsen number $N(m)$ of $m : W_0\to W_0$.

(2) The relative Nielsen $N(m, \operatorname{cl}(W_0\setminus W_0^{--}))$ in Theorem 4.1 cannot be replaced by the Nielsen number $N(m)$. For example we consider $B = [1,2]\subset \mathbb{R}$ and the homotopy $F : [0,1]\times B\to\mathbb{R}$ given by

$$F(t,x) := t + x.$$

Let $W = [0,1]\times B$ with a homeomorphism $h : W\ni x\to x\in W$. Let us observe that $\tau_W(x) = 2 - x$ for $x\in[1,2]$; hence W is a segment for F with the exit set

$$W^- = W^{--} = [0,1]\times\{2\}.$$

Since the monodromy map $m : \{2\} \to \{2\}$ is the identity map and $\{2\}$ is contractible, hence $N(m) = 1$. On the other hand the map $f : [1,2] \ni x \to F(1,x) = 1+x \in \mathbb{R}$ is fixed point free. Observe that $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--})) = 0$.

(3) If $N(m|_{W_0^{--}}) = 0$, then $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--})) = N(m)$. Indeed, let $K \subset W_0$ be a Nielsen class of m. It follows by results in [S1] then that $K \cap W_0^{--}$ is either empty or the union of Nielsen classes of $m|_{W_0^{--}}$. Hence $\mathrm{ind}\,(m|_{W_0^{--}}) = 0$, because $N(m|_{W_0^{--}}) = 0$. It follows that K does not assume its index in W_0^{--} if and only if K is an essential Nielsen class for m.

(4) If $B = W_0$ is contractible, then $L(m) \neq 0$ iff $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--})) = 1$ and $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--})) = 0$ if $L(m) - L(m|_{W_0^{--}}) = 0$, so in that case $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--}))$ do not carry more information concerning the structure of $\mathrm{Fix}\,(f)$ than the Lefschetz number.

(5) For a map of pairs of compact ENRs one can define another relative Nielsen number denoted by $N(f, X, A)$ ([S3, Z]). It is defined in the following way. A Nielsen class of f is said to be a common Nielsen class if it contains an essential fixed point class of $f|_A$. By $N(f, f|_A)$ we denote the number of common and essential Nielsen classes of f. We define

$$N(f, X, A) = N(f) + N(f|_A) - N(f, f|_A).$$

The relative Nielsen number $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--}))$ cannot be replaced in Theorem 4.1 by $N(m, W_0, W_0^{--})$. To see this let us consider the homotopy given by the flow generated by the linear planar differential equation $z' = \overline{z}$ ($z = (x,y) \in \mathbb{C}$), namely

$$F(t, (x,y)) = (xe^t, ye^{-t}).$$

Let $B = [-1,1] \times [-1,1]$. One can check that $W = [0,1] \times B$ is a segment for $F : [0,1] \times B \to \mathbb{C}$ with $W_0^{--} = \{\pm 1\} \times [-1,1]$ and the monodromy map m being the identity on the pair (W_0, W_0^{--}). One can check that $N(m, W_0, W_0^{--}) = 2$ and $N(m, \mathrm{cl}\,(W_0 \setminus W_0^{--})) = 1$. Obviously the map $f(x,y) = F(1,(x,y)) = (ex, e^{-1}y)$ has exactly one fixed point.

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