TOPOLOGICAL METHOD FOR DETECTING FIXED POINTS OF MAPS HOMOTOPIC TO SELFMAPS OF COMPACT ENRS

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(Communicated by Yingfei Yi)

Abstract. Assume that $X$ is a metric space and $B \subset X$ is compact. Let $f : B \rightarrow X$ be a continuous map homotopic to $g$, the selfmap of $B$. The aim of this paper is to present a method for detecting fixed points of $f$. It is based on the notion of the Ważewski set for the homotopy $F$ between $g$ and $f$.

1. Introduction

The notion of a Ważewski set is a very useful tool for detecting invariant sets in dynamical systems generated by differential equations (see [Sr1, Sr2, SWZ]). In particular it is frequently used for detecting fixed points of a time-one map of a semi-flow on the space $X$. We recall very briefly the main idea of the Ważewski method. A continuous map $\phi : [0, +\infty) \times X \rightarrow X$ is a semi-flow if

$$\phi(0, x) = x, \quad \phi(s, \phi(t, x)) = \phi(s + t, x), \quad s, t \in [0, +\infty), \quad x \in X.$$ 

Let $B \subset X$. One can define the exit set of $B$ by

$$B^- = \{ x \in B : \phi([0, t] \times \{ x \}) \not\subset B \text{ for all } t > 0 \}.$$ 

We say that $B$ is a Ważewski set if $B$ and $B^-$ are compact. Then one can show that the function

$$\sigma : B \ni x \rightarrow \sup \{ t \geq 0 : \phi([0, t], x) \subset B \} \in [0, +\infty]$$

is continuous. It was proved by Srzednicki that if $B$ and $B^-$ are additionally ENRs and their Euler characteristics are not equal (i.e. $\chi(B) \neq \chi(B^-)$), then $B$ contains a stationary point of $\phi$. The continuity of $\sigma$ and the Lefschetz fixed point theorem are essential to show that for $t > 0$ the map $\phi(t, \cdot) : B \rightarrow X$ has a fixed point in $B$.

The aim of this paper is to present a generalization of the method above for detecting a fixed point in the following context: assume that $X$ is a metric space and $B \subset X$ is compact. Let $f : B \rightarrow X$ be a continuous map homotopic to $g : B \rightarrow X$ such that $g(B) \subset B$; i.e., there exists a continuous map $F : [0, 1] \times B \rightarrow X$ such that

- $F_1 := F(1, \cdot) = f$,
- $F_0 := F(0, \cdot) = g$.

Received by the editors November 1, 2010 and, in revised form, June 3, 2011 and June 9, 2011. 2000 Mathematics Subject Classification. Primary 37C25, 58J20; Secondary 32S50. Key words and phrases. Lefschetz number, relative Nielsen number, fixed point index.

This work was co-financed with budget funds allocated to education in 2010-2011 as a research project by the grant NN201 411439.
In order to find fixed points of $f$ we define the Ważewski set for the homotopy $F$.

2. Ważewski sets for a homotopy

By $\pi_1 : [0,1] \times X \to [0,1]$ and $\pi_2 : [0,1] \times X \to X$ we denote projections. For $Z \subset [0,1] \times X$ and $t \in [0,1]$ we put $Z_t = \{ x \in X : (t,x) \in Z \}$.

For a homotopy $F : [0,1] \times B \to X$ we denote

$$F_t : B \ni x \to F(t,x) \in X,$$ $t \in [0,1]$. 

Let $W \subset [0,1] \times X$ be a compact subset such that $W_0 = B$. We define an exit-time function for $W$:

$$\tau_W : B \ni x \to \sup\{ t \in [0,1] : F(s,x) \in W_s \forall s \in [0,t] \} \in [0,1].$$

We say that $W$ is a Ważewski set for $F$ if its exit-time function is continuous. The set $W^- = \{ (\tau_W(x), F(\tau_W(x),x)) : x \in B \} \subset W$ is called the exit set of $W$. It follows that $W^-$ is compact.

**Definition 2.1.** The Ważewski set $W$ for $F$ is called a *segment* if the following conditions hold:

- there exists a compact subset $W^{---}$ (called the essential exit set) of $W^-$ such that

  $$W^- \cap ([0,1] \times X) \subset W^{--}, \quad W^{--} = \text{cl}(W^{---} \cap ([0,1] \times X)),$$

- there exists a homeomorphism $h : [0,1] \times B \to W$ such that $\pi_1 \circ h = \pi_1$ and

  $$W^{---} \subset h([0,1] \times W_0^{---}), \quad \pi_2(h(1,W_0^{---})) \subset W_0^{--},$$

- $W_1 = W_0, W_1^{--} \subset W_0^{--}$ and $g(W_0^{--}) \subset W_0^{--}$.

**Definition 2.2.** Let $W$ be a segment and $h$ be a homeomorphism from Definition 2.1. Then $m : W_0 \to W_0$ defined by

$$m(x) := \pi_2 h(1, \pi_2 h^{-1}(0,x)), \quad x \in W_0,$$

is the *monodromy homeomorphism* of the segment $W$.

Let us note that $m(W_0^{---}) \subset W_0^{--}$. Moreover, two morphisms induced in homologies: $H(m) : H(W_0) \to H(W_0)$ and $H(m \circ g) : H(W_0) \to H(W_0)$ do not depend on the choice of $h$.

Indeed, let $h : [0,1] \times B \to W$, $k : [0,1] \times B \to W$ be homeomorphisms satisfying conditions from the definition of a segment and let $m_k$ and $m_k$ be their corresponding monodromy homeomorphisms. We will prove that $m_k \circ g$ and $m_k \circ g$ are homotopic (for $g = \text{id}$, $m_k$ and $m_k$ are homotopic); thus they induce the same morphisms in homologies. Let

$$h_t : W_0 \ni x \to \pi_2 h(t, \pi_2 h^{-1}(0,x)) \in W_t$$

and

$$k^t : W_t \ni x \to \pi_2 k(1, \pi_2 k^{-1}(t,x)) \in W_0.$$

Then

$$H : [0,1] \times B \ni (s,x) \to k^s \circ h_s \circ g(x) \in B$$

is a homotopy such that $H(0,\cdot) = k^0 \circ h_0 \circ g = m_k \circ g$ and $H(1,\cdot) = k^1 \circ h_1 \circ g = m_k \circ g$. 


Example 2.3. We describe this construction in the case of a time-one map of a semi-flow \( \phi \) on \( X \). For a subset \( B \subset X \) one can consider a homotopy \( F : [0, 1] \times B \to X \) given by
\[
F(t, x) = \phi(t, x).
\]
Then \( f = F_1 = \phi(1, \cdot)|_B \) is a restriction of the time-one map to the set \( B \) and \( g = F_0 = \phi(0, \cdot)|_B = i \) is the inclusion map. In the extended phase space \([0, +\infty) \times X\) we consider a set \( W = [0, 1] \times B \). Assume that the exit-time function
\[
\tau_W : B \ni x \to \sup \{ t \in [0, 1] : (s, \phi(s, x)) \in W \forall s \in [0, t] \} \in [0, 1]\]
is continuous. The exit set is given by
\[
W^- = \{(\tau_W(x), \phi(\tau_W(x), x)) : x \in B\}.
\]
If \( B \) is a Ważewski set for the semi-flow \( \phi \) (i.e., \( B, B^- \) are compact), then \( W \) is a segment for \( F \). Indeed,
\[
\tau_W(x) = \begin{cases} 
\sigma(x), & \text{if } \sigma(x) \leq 1; \\
1, & \text{if } \sigma(x) \geq 1.
\end{cases}
\]
Then \( W^- \) is given by
\[
W^- = \{(\sigma(x), \phi(\sigma(x), x)) : x \in \sigma^{-1}([0, 1]) \}\cup\{(1, \phi(1, x)) : x \in \sigma^{-1}([1, +\infty))\},
\]
and the essential exit set of \( W \) is equal to
\[
W^- = \{(\sigma(x), \phi(\sigma(x), x)) : x \in \sigma^{-1}([0, 1])\}.
\]
In particular,
\[
W_0 = B, \quad W_0^- = B^-.
\]
Example 2.4. Let us mention that if \( X \) is contractable, then each map \( f : B \to X \) is homotopic to the inclusion. In particular, \( f \) is not necessarily a time-one map for any semi-flow even if \( g \) is the inclusion map.

Example 2.5. We illustrate the notion of a segment for a homotopy with the following simple example. Let \( B = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and \( f : B \to \mathbb{C} \) be a continuous map such that
\[
\{(1 - t)z + tf(z) : t \in [0, 1] \} \cap B = \{ z \}, \quad z \in \partial B.
\]
Consider a homotopy
\[
F : [0, 1] \times B \ni (t, z) \to (1 - t)z + tf(z) \in \mathbb{C}.
\]
We define \( W = [0, 1] \times B \). In order to show that \( W \) is a Ważewski set for \( F \) it is sufficient to show that the exit-time function \( \tau_W : B \to [0, 1] \) is continuous. But \( \tau_W(z) \) is determined by the intersection point of the segment joining points \( z \) and \( f(z) \) with the set
\[
Y = [0, 1] \times \partial B \cup \{1\} \times B,
\]
so it is continuous. We will argue that \( Y \) is equal to the exit set \( W^- \). By definition, \( W^- = w(B) \), where
\[
w : B \ni z \to (\tau_W(z), F(\tau_W(z), z)) \in Y.
\]
Since \( w(z) = z \) for \( z \in \partial B \) and \( w|_{\partial B} \) has an extension on \( B \), so \( w|_{\partial B} : \partial B \to W^- = w(B) \) is homotopic to a constant map in \( W^- \); hence \( W^- = Y \). It follows that \( W \)
is a segment for $F$ with the identity on $W$ as a homeomorphism $h$. Observe that
\[ W_0 = B, W_0^{-} = \partial B, \]
and $m$ is homotopic to the identity on $W_0$.

3. Detection of fixed points

We use standard notation concerning fixed point theory ([JM]). In particular, Fix$(f) = \{x \in U : f(x) = x\}$ denotes the set of fixed points of the map $f : U \to X$, where $U \subset X$. Recall (see [JM]) that if $X$ is an ENR, $U \subset X$ is open and Fix$(f)$ is compact, then the fixed point index ind$(f, \text{Fix}(f)) \in \mathbb{Z}$ is well defined. Moreover, if $f : X \to X$ is a selfmap of a compact ENR, then its Lefschetz number is well defined by
\[ L(f) := L(H(f)) = \sum_{n \geq 0} (-1)^n \text{trace}(H_n(f)), \]
where $H(f) : H(X) \to H(X)$ is a map induced by $f$ in singular homologies with rational coefficients.

**Theorem 3.1.** Assume that $W$ is a segment for $F$, $g = F_0 : B \to X$, $g(B) \subset B$ and $f = F_1 : B \to X$. The set
\[ U = \{ x \in W_0 : F(t, x) \in W_t \setminus W_t^{-}, \text{ for all } t \in [0, 1] \} \]
is open in $B = W_0$ and the set of fixed points of $f|_U : U \to B$ is compact. Moreover, if $W$ and $W^{-}$ are ENRs, then
\[ \text{ind}(f|_U, \text{Fix}(f|_U)) = L(m \circ g) - L(m \circ g|_{W_0^{-}}). \]
In particular, if $L(m \circ g) - L(m \circ g|_{W_0^{-}}) \neq 0$, then $f : B \to X$ has a fixed point in $B$.

**Proof:** It is easy to check that
\[ U = W_0 \setminus \bigcup_{t \in [0, 1]} F_t^{-1}(W_t^{-}). \]

If for each $n$, $x_n \in \bigcup_{t \in [0, 1]} F_t^{-1}(W_t^{-})$ and $\lim_{n \to \infty} x_n = x$, then
\[ \lim_{n \to \infty} F(\tau_W(x_n), x_n) = F(\tau_W(x), x). \]

Moreover, for each $n$, $F(\tau_W(x_n), x_n) \in W^{-}$; hence $F(\tau_W(x), x) \in W^{-}$ and $x \in \bigcup_{t \in [0, 1]} F_t^{-1}(W_t^{-})$. Thus $\bigcup_{t \in [0, 1]} F_t^{-1}(W_t^{-})$ is closed and $U$ is open in $W_0$.

We define maps
\[ m_s : W_s \ni x \to \pi_2 h(1, \pi_2 h^{-1}(s, x)) \in W_0 = W_1, \quad s \in [0, 1]. \]
In particular, $m_0 = m : W_0 \to W_0$ and $m_1 = \text{id}_{W_0}$. Consider a homotopy $H : [0, 1] \times W_0 \to W_0$ given by
\[ H_t(x) = \begin{cases} m_{\tau_W(x)}(F(\tau_W(x), x)), & \text{if } \tau_W(x) \leq 1-t; \\ m_{1-t}(F(1-t, x)), & \text{if } \tau_W(x) \geq 1-t. \end{cases} \]
In particular, $H_1 = m \circ g$. By the homotopy property of the Lefschetz number we obtain
\[ L(H_0) = L(H_1) = L(m \circ g). \]

By definition we have that
\[ H_0(x) = m_{\tau_W(x)}(F(\tau_W(x), x)), \quad x \in W_0. \]
It is easy to check that

\[ U = \{ x \in W_0 : \tau_W(x) = 1, \ F(1, x) \in W_0 \setminus W_1^{-} \} \]

and \((H_0)^{-1}(W_0 \setminus W_0^{-}) \subset U \subset (H_0)^{-1}(W_0 \setminus W_1^{-})\). \(H_0(x) = F(1, x) = f(x)\) if \(\tau_W(x) = 1\); hence

\[ H_0|_U = f|_U. \]

If \(x \in W_0 \setminus W_0^{-}\) and \(H_0(x) = x\), then it follows that \(\tau_W(x) = 1\) (since in the other case \(H_0(x) \in W_0^{-}\)); hence \(x \in U\). Moreover,

\[ \text{Fix}(f|_U) = \text{Fix}(H_0) \cap \tau_W^{-1}(1); \]

hence \(\text{Fix}(f|_U)\) is compact. Put \(V = \tau_W^{-1}([0, 1])\). It follows that \(V\) is open in \(W_0\), \(W_0^{-} \subset V\) and \(H_0(V) \subset W_0^{-}\). By the compactness of \(W_0^{-}\) it follows that there is an open neighborhood \(D\) of \(W_0^{-}\) such that \(g(D) \subset V\); hence \(H_0(D) \subset W_0^{-}\). It follows that

\[ \text{Fix}(H_0) = \text{Fix}(H_0|_U) \cup \text{Fix}(H_0|_D) = \text{Fix}(f|_U) \cup \text{Fix}(m \circ g|_{W_0^{-}}). \]

Since the sets \(\text{Fix}(f|_U)\) and \(\text{Fix}(m \circ g|_{W_0^{-}})\) are compact and disjoint, hence by the Lefschetz fixed point theorem and the additivity property of the fixed point index we get

\[ L(m \circ g) = L(H_0) = \text{ind}(H_0|_U, \text{Fix}(f|_U)) + \text{ind}(H_0|_V, \text{Fix}(m \circ g|_{W_0^{-}})). \]

By the commutativity property of the fixed point index and the Lefschetz fixed point theorem we have

\[ \text{ind}(H_0|_V, \text{Fix}(m \circ g|_{W_0^{-}})) = \text{ind}(H_0|_{W_0^{-}}, \text{Fix}(m \circ g|_{W_0^{-}})) = L(m \circ g|_{W_0^{-}}), \]

and consequently

\[ \text{ind}(f|_U, \text{Fix}(f|_U)) = \text{ind}(H_0|_U, \text{Fix}(f|_U)) = L(m \circ g) - L(m \circ g|_{W_0^{-}}). \]

\[ \square \]

**Example 3.2.** Assume that \(W = [0, 1] \times B\) is a segment for the homotopy \(F : [0, 1] \times B \to X\) with the homeomorphism \(h : W \ni x \mapsto x \in W\) and \(g\) is the inclusion map. Then \(L(m \circ g) - L(m \circ g|_{W_0^{-}}) = \chi(B) - \chi(W_0^{-})\), so if \(\chi(B) \neq \chi(W_0^{-})\), then \(f = F_1\) has a fixed point in \(B\).

### 4. Relation to the Nielsen Number

The classical Nielsen number theory is concerned with the determination of the minimal number of fixed points for all maps in the homotopy class of a given map \(f : X \to X\) (see \([\text{JM} \text{, J}]\)). The Nielsen number \(N(f)\) provides a homotopy invariant lower bound for the number of fixed points of \(f\). Assume that \(X\) is a compact \(ENR\) and \(f : X \to X\) is a continuous map. In the set of fixed points \(\text{Fix}(f)\) we define the equivalence relation in the following way: two fixed points are in the Nielsen relation if they can be joined by a path which is homotopic relative to the end points to its own \(f\)-image. Then \(\text{Fix}(f)\) splits into a disjoint union of Nielsen classes. Each Nielsen class \(F\) is compact, so its fixed point index \(\text{ind}(f, F)\) is well defined. The Nielsen number of \(f\) \((N(f))\) is defined as the number of essential fixed point classes, i.e., such that \(\text{ind}(f, F) \neq 0\). In this paper we will use the relative Nielsen theory for selfmaps \(f : (X, A) \to (X, A)\) of pairs of compact \(ENRs\) introduced in \([\text{S2}]\) (see also \([\text{Z}]\)). Let \(F\) be a Nielsen class of \(f : X \to X\). We say
that $F$ assumes its index in $A$ if $\text{ind}(f, F) = \text{ind}(f|_A, F \cap A)$. The Nielsen number of the closure $N(f, \text{cl}(X \setminus A))$ is defined as the number of the Nielsen classes of $f$ which do not assume its index in $A$. It is a relative homotopy invariant lower bound for the number of fixed points of $f$ on $\text{cl}(X \setminus A)$.

**Theorem 4.1.** Assume that $W$ is a segment for the homotopy $F$. Then

$$\text{card Fix}(f|_U) \geq N(m \circ g, \text{cl}(W_0 \setminus W_0^-)),$$

where $U = \{x \in W_0 : \tau_W(x) = 1, F(1, x) \in W_0 \setminus W_0^-\}$.

**Proof:** Let $H : [0, 1] \times W_0 \to W_0$ be a homotopy used in the proof of Theorem 3.1. Let us observe that for $x \in W_0^-$ we have $\tau_W(x) = 0$, so then

$$H_t(x) = m \circ g(x), \quad t \in [0, 1];$$

hence $H_t(W_0^-) \subset W_0^-$ if $t \in [0, 1]$, and thus $H$ is a homotopy of the pair $(W_0, W_0^-)$. Since $H_1 = m \circ g : (W_0, W_0^-) \to (W_0, W_0^-)$, then by the homotopy property of the relative Nielsen number we must show that

$$\text{card Fix}(f|_U) \geq N(h_0, \text{cl}(W_0 \setminus W_0^-)).$$

It is sufficient to prove that if $K \subset W_0$ is a Nielsen class of $H_0$ and $K$ does not assume its index in $W_0^-$, then $(\text{Fix}(f) \cap U) \cap K$ is nonempty. Suppose that

$$\text{Fix}(f|_U) \cap K = \emptyset.$$

By the same arguments as in the proof of Theorem 3.1 we have that

$$\text{Fix}(H_0) = \text{Fix}(f|_U) \cup \text{Fix}(m \circ g|_{W_0^-}),$$

so $K \subset W_0^-$. It follows by the arguments in the proof of Theorem 3.1 that there is an open neighborhood $D$ of $W_0^-$ such that $H_0(D) \subset W_0^-$. Hence by the commutativity property of the fixed point index we obtain that

$$\text{ind}(H_0, K) = \text{ind}(H_0|_{W_0^-}, K) = \text{ind}(m \circ g|_{W_0^-}, K),$$

a contradiction. $\square$

**Remark 4.2.** In the remarks below we assume that $F_0 = g : B \to X$ is the inclusion.

1. It follows that $\text{cl}(W_0 \setminus W_0^-) = W_0$ but in general $N(m, \text{cl}(W_0 \setminus W_0^-))$ is not equal to the classical Nielsen number $N(m)$ of $m : W_0 \to W_0$.

2. The relative Nielsen $N(m, \text{cl}(W_0 \setminus W_0^-))$ in Theorem 4.1 cannot be replaced by the Nielsen number $N(m)$. For example we consider $B = [1, 2] \subset \mathbb{R}$ and the homotopy $F : [0, 1] \times B \to \mathbb{R}$ given by

$$F(t, x) := t + x.$$

Let $W = [0, 1] \times B$ with a homeomorphism $h : W \ni x \to x \in W$. Let us observe that $\tau_W(x) = 2 - x$ for $x \in [1, 2]$; hence $W$ is a segment for $F$ with the exit set

$$W^- = W^- = [0, 1] \times \{2\}.$$
Since the monodromy map $m : \{2\} \to \{2\}$ is the identity map and $\{2\}$ is contractible, hence $N(m) = 1$. On the other hand the map $f : [1, 2] \ni x \to F(1, x) = 1 + x \in \mathbb{R}$ is fixed point free. Observe that $N(m, \text{cl}(W_0 \setminus W_0^-)) = 0$.

(3) If $N(m|_{W_0^-}) = 0$, then $N(m, \text{cl}(W_0 \setminus W_0^-)) = N(m)$. Indeed, let $K \subset W_0$ be a Nielsen class of $m$. It follows by results in [S1] then that $K \cap W_0^-$ is either empty or the union of Nielsen classes of $m|_{W_0^-}$. Hence $\text{ind}(m|_{W_0^-}) = 0$, because $N(m|_{W_0^-}) = 0$. It follows that $K$ does not assume its index in $W_0^-$ if and only if $K$ is an essential Nielsen class for $m$.

(4) If $B = W_0$ is contractible, then $L(m) \neq 0$ iff $N(m, \text{cl}(W_0 \setminus W_0^-)) = 1$ and $N(m, \text{cl}(W_0 \setminus W_0^-)) = 0$ if $L(m) = L(m|_{W_0^-}) = 0$, so in that case $N(m, \text{cl}(W_0 \setminus W_0^-))$ do not carry more information concerning the structure of $\text{Fix}(f)$ than the Lefschetz number.

(5) For a map of pairs of compact ENRs one can define another relative Nielsen number denoted by $N(f, X, A)$ (S3 [Z]). It is defined in the following way. A Nielsen class of $f$ is said to be a common Nielsen class if it contains an essential fixed point class of $f|_A$. By $N(f, f|_A)$ we denote the number of common and essential Nielsen classes of $f$. We define

$$N(f, X, A) = N(f) + N(f|_A) - N(f, f|_A).$$

The relative Nielsen number $N(m, \text{cl}(W_0 \setminus W_0^-))$ cannot be replaced in Theorem 4.1 by $N(m, W_0, W_0^-)$. To see this let us consider the homotopy given by the flow generated by the linear planar differential equation $z' = z$ ($z = (x, y) \in \mathbb{C}$), namely

$$F(t, (x, y)) = (xe^t, ye^{-t}).$$

Let $B = [-1, 1] \times [-1, 1]$. One can check that $W = [0, 1] \times B$ is a segment for $F : [0, 1] \times B \to \mathbb{C}$ with $W_0^- = \{\pm 1\} \times [-1, 1]$ and the monodromy map $m$ being the identity on the pair $(W_0, W_0^-)$. One can check that $N(m, W_0, W_0^-) = 2$ and $N(m, \text{cl}(W_0 \setminus W_0^-)) = 1$. Obviously the map $f(x, y) = F(1, (x, y)) = (ex, e^{-1}y)$ has exactly one fixed point.

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