

A NEW SHORT PROOF FOR THE UNIQUENESS OF THE UNIVERSAL MINIMAL SPACE

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ABSTRACT. We give a new short proof for the uniqueness of the universal minimal space. The proof holds for the uniqueness of the universal object in every collection of topological dynamical systems closed under taking projective limits and possessing universal objects.

1. INTRODUCTION

In topological dynamics one studies jointly continuous actions $G \times X \rightarrow X$ of (Hausdorff) topological groups G on nonempty (Hausdorff) compact spaces X . Such an action is called a *topological dynamical system*, and we call X a G -space. A G -space X is said to be *minimal* if X and \emptyset are the only G -invariant closed subsets of X . By Zorn's lemma each G -space contains a minimal G -subspace. These minimal objects are in some sense the most basic ones in the category of G -spaces. For various topological groups G they have been the object of extensive study. Given a topological group G , one is naturally interested in describing all of the minimal G -spaces up to isomorphism. Such a description is given by the following construction: one can show that there exists a minimal G -space U_G with the universal property that every minimal G -space X is a *factor* of U_G ; i.e., there is a continuous G -equivariant map from U_G onto X . Any such G -space is called a *universal minimal G -space*; however it can be shown to be unique up to isomorphism. The existence of a universal minimal G -space is easy to demonstrate by choosing a minimal G -subspace of the product over all minimal G -spaces (one representative from each isomorphism class; the collection of isomorphism classes of minimal G -spaces is a set). The uniqueness turns out to be harder to show, since for two universal minimal G -spaces X and Y , there could be more than one epimorphism from X to Y , where an *epimorphism* is a surjective G -equivariant continuous map. An easy observation is that it suffices to show that a universal minimal G -space X is *coalescent*; i.e. every epimorphism $\phi : X \rightarrow X$ is an isomorphism. If M_1 and M_2 are universal minimal G -spaces, then by universality we have epimorphisms $\phi_1 : M_1 \rightarrow M_2$ and $\phi_2 : M_2 \rightarrow M_1$. If in addition M_1 is coalescent, then $\phi_2 \circ \phi_1$ must be an isomorphism, and hence ϕ_1 and ϕ_2 are isomorphisms.

In the literature two different approaches for the proof of uniqueness have been offered. The algebraic approach using Ellis semigroups was laid out by Ellis in [E69] (he only treated the case of discrete G , but the proof works for any G). The details can be found in [dV93, IV(4.30)] and [U00, Appendix 3]. The main tool is

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the theorem that any minimal subsystem of an enveloping semigroup (see [dV93, IV(3.5(1))]) is coalescent. This theorem is proven by using the Ellis-Numakura Lemma ([E58], [N52]): every nonempty compact Hausdorff right semitopological semigroup contains an idempotent. Another proof was given by Auslander in [A88]. The main idea is to show that given a minimal system (G, X) there is a cardinal κ such that (G, X^κ) contains a minimal coalescent subsystem. The main tool is the so-called *almost periodic sets*.

We will give a proof for the uniqueness of the universal minimal space that holds in a more general setting. Given a collection \mathcal{K} of G -spaces, we call $M \in \mathcal{K}$, \mathcal{K} -*universal* if any $N \in \mathcal{K}$ is a G -factor of M . We show that if \mathcal{K} is closed under taking projective limits and possesses universal spaces, then it has a unique universal space (up to isomorphism). For example, the reader will be able to verify easily that our result also applies to the universal equicontinuous minimal space and the universal distal minimal space.

2. UNIQUENESS

Theorem 1. *Let G be a topological group and \mathcal{K} a collection of G -spaces. Assume that \mathcal{K} is closed under taking projective limits. Let M_1, M_2 be \mathcal{K} -universal spaces. Then M_1 and M_2 are isomorphic.*

Proof. As noted in the introduction it suffices to prove that every \mathcal{K} -universal space M is coalescent. Let $\varphi : M \rightarrow M$ be an epimorphism. We will show that φ is injective. Assume not. Let β be an ordinal with $|\beta| > |M^2|$. We will define for any ordinal $\alpha \leq \beta$ a G -space $X_\alpha \in \mathcal{K}$ and epimorphisms $\phi_{\gamma, \alpha} : X_\alpha \rightarrow X_\gamma$ for any ordinal $\gamma \leq \alpha$ with the following properties:

- (1) (Compatibility) $\phi_{\delta, \alpha} = \phi_{\delta, \gamma} \circ \phi_{\gamma, \alpha}$ for $\alpha \geq \gamma \geq \delta$.
- (2) For each ordinal $\gamma < \alpha$, there exist distinct $x_\gamma, y_\gamma \in X_{\gamma+1}$ with $\phi_{\gamma, \gamma+1}(x_\gamma) = \phi_{\gamma, \gamma+1}(y_\gamma)$. We denote this common image by z_γ .

By the universality of (G, M) one has an epimorphism $M \rightarrow X_\beta$. This implies that $|X_\beta| \leq |M|$. For each $\gamma < \beta$, since $\phi_{\gamma+1, \beta}$ is surjective, we can find $\tilde{x}_\gamma, \tilde{y}_\gamma \in X_\beta$ with $\phi_{\gamma+1, \beta}(\tilde{x}_\gamma) = x_\gamma$ and $\phi_{\gamma+1, \beta}(\tilde{y}_\gamma) = y_\gamma$. For any $\gamma < \alpha < \beta$, one has $\phi_{\gamma+1, \beta}(\tilde{x}_\alpha) = \phi_{\gamma+1, \alpha}(z_\alpha) = \phi_{\gamma+1, \beta}(\tilde{y}_\alpha)$, and $\phi_{\gamma+1, \beta}(\tilde{x}_\gamma) = x_\gamma \neq y_\gamma = \phi_{\gamma+1, \beta}(\tilde{y}_\gamma)$, and hence $(\tilde{x}_\alpha, \tilde{y}_\alpha) \neq (\tilde{x}_\gamma, \tilde{y}_\gamma)$. This implies that the map $\{\gamma \mid 0 \leq \gamma < \beta\} \rightarrow X_\beta \times X_\beta$ given by $\gamma \mapsto (\tilde{x}_\gamma, \tilde{y}_\gamma)$ is injective, which in turn implies that $|\beta| \leq |X_\beta^2|$. Putting all the inequalities together, including our initial choice $|\beta| > |M^2|$, we get $|M^2| < |\beta| \leq |X_\beta^2| \leq |M^2|$, which is impossible.

The construction is carried out through transfinite induction. Let $X_0 = M$. If α is a successor ordinal, take an epimorphism $f_\alpha : M \rightarrow X_{\alpha-1}$ using the universality of M and define $(G, X_\alpha) = (G, M)$, $\phi_{\alpha-1, \alpha} = f_\alpha \circ \varphi$, and for $\gamma < \alpha - 1$, define $\phi_{\gamma, \alpha} = \phi_{\gamma, \alpha-1} \circ \phi_{\alpha-1, \alpha}$. If α is a limit ordinal, define $X_\alpha \in \mathcal{K}$ to be the projective limit of $(X_\gamma)_{\gamma < \alpha}$, and for $\gamma < \alpha$, define $\phi_{\gamma, \alpha} : X_\alpha \rightarrow X_\gamma$ to be the epimorphism coming from the projective limit. Clearly X_α is a G -space, and the conditions (1) and (2) are satisfied. \square

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